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A SUBCLASS OF HARMONIC UNIVALENT FUNCTIONS DEFINED BY SUBORDINATION

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Abstract. In this article, we introduced and defined a new class of harmonic functions which by use of a subordination. We find necessary and sufficient conditions, distortion bounds, radii of starlikeness and convexity, compactness and extreme points for above class of harmonic functions.

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1. Introduction

Let H denote the family of continuous complex valued harmonic functions which are harmonic in the open unit disk $\mathbb{D} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$ and let A be the subclass of H consisting of functions which are analytic in $\mathbb{D}$. A function harmonic in $\mathbb{D}$ may be written as $f = h + \overline{g}$, where h and g are members of A. We call h the analytic part and g co-analytic part of f. A necessary and sufficient condition for f to be locally univalent and sense-preserving in $\mathbb{D}$ is that $|h'(z)| > |g'(z)|$ (see Clunie and Sheil-Small [1]).

Let SH denote the family of functions $f = h + \overline{g}$ which are harmonic, univalent, and sense-preserving in $\mathbb{D}$ for which $f(0) = f_z(0) - 1 = 0$. The subclass SH^0 of SH consists of all functions in SH which have the additional property $f_{\bar{z}}(0) = b_1 = 0$. To this end, without loss of generality, we may write for $f \in SH^0$

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n \text{ and } g(z) = \sum_{n=2}^{\infty} b_n z^n. \tag{1}$$

In 1984 Clunie and Sheil-Small [1] investigated the class SH as well as its geometric subclasses and obtained some coefficient bounds. Since then, there have been several related papers on SH and its subclasses. Also note that SH reduces to the class S of normalized analytic univalent functions in $\mathbb{D}$, if the co-analytic part of f is identically zero.

For $f \in S$, the differential operator D^k $(k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\})$ of f was introduced by Sălăgean [7]. For $f = h + \overline{g}$ given by (1), Jahangiri et al. [6] defined the modified Sălăgean operator of f as

$$D^k f(z) = D^k h(z) + (-1)^k \overline{D^k g(z)}, \tag{2}$$

where

$$D^k h(z) = z + \sum_{n=2}^{\infty} n^k a_n z^n, \quad D^k g(z) = \sum_{n=2}^{\infty} n^k b_n z^n.$$

We say that a function $f : \mathbb{D} \to \mathbb{C}$ is subordinate to a function $F : \mathbb{D} \to \mathbb{C}$, and write $f(z) \prec F(z)$, if there exists a complex valued function w which maps $\mathbb{D}$ into itself with $w(0) = 0$, such that

$$f(z) = F(w(z)) \quad (z \in \mathbb{D}).$$

Furthermore, if the function F is univalent in $\mathbb{D}$, then we have the following equivalence:

$$f(z) \prec F(z) \Leftrightarrow f(0) = F(0) \text{ and } f(\mathbb{D}) \subset F(\mathbb{D}).$$

The Hadamard product (or convolution) of functions f_1 and f_2 of the form

$$f_t(z) = z + \sum_{n=2}^{\infty} a_{t,n} z^n + \sum_{n=2}^{\infty} \overline{b_{t,n} z^n} \qquad (z \in \mathbb{D},\ t \in \{1, 2\})$$

is defined by

$$(f_1 * f_2)(z) = z + \sum_{n=2}^{\infty} a_{1,n}\ a_{2,n} z^n + \sum_{n=2}^{\infty} \overline{b_{1,n}\ b_{2,n} z^n} \qquad (z \in \mathbb{D}).$$

Denote by $SH_\lambda^0(k, A, B)$ the subclass of SH^0 consisting of functions f of the form (1) that satisfy the condition

$$\frac{D^{k+1} f(z)}{\lambda D^{k+1} f(z) + (1-\lambda) D^k f(z)} \prec \frac{1 + Az}{1 + Bz}, \tag{3}$$

$$(0 \leq \lambda \leq 1, k \in \mathbb{N} \cup \{0\}, \ -A \leq 0 \leq B < A \leq 1)$$

where $D^k f(z)$ is defined by (2).

By suitably specializing the parameters, the classes $SH_\lambda^0(n, A, B)$ reduces to the various subclasses of harmonic univalent functions. Such as,

(i) $SH_0^0(\xi, A, B) = H^{\xi}(A, B),\ \xi \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ ([4]),
(ii) $SH_0^0(0, A, B) = S_H^*(A, B) \cap SH^0$ ([2]),
(iii) $SH_0^0(k, 2\alpha - 1, 1) = H^0(k, \alpha)$ ([6]),
(iv) $SH_0^0(0, 2\alpha - 1, 1) = S_{H^0}^*(\alpha)$ ([5], [8], [9]),
(v) $SH_0^0(1, 2\alpha - 1, 1) = S_{H^0}^*(\alpha)$ ([5]),
(vi) $SH_0^0(1, 1, -1) = K_H^0$ ([8]),
(vii) $SH_0^0(0, 1, -1) = S_H^{*0}$ ([8]).

Making use of the techniques and methodology used by Dziok (see [2], [3]), Dziok et al. [4], in this paper we find necessary and sufficient conditions, distortion bounds, radii of starlikeness and convexity, compactness and extreme points for the above defined class $SH_\lambda^0(k, A, B)$.

2. Main Results

First theorem provides a necessary and sufficient convolution condition for the harmonic functions in $SH_\lambda^0(k, A, B)$.

Theorem 1. *A function f belongs to the class $SH_\lambda^0(k, A, B)$ if and only if $f \in SH^0$ and*

$$D^k f(z) * \Phi(z; \zeta) \neq 0 \qquad (\zeta \in \mathbb{C},\ |\zeta| = 1,\ z \in \mathbb{D}),$$

where

$$\begin{aligned}\Phi(z; \zeta) &= \frac{[(1-\lambda)(1 + A\zeta)]z^2 + (B - A)\zeta z}{(1 - z)^2} \\ &\quad + (-1)^k \frac{[(1 + A\zeta)(1 - \lambda)]\,\bar{z}^2 + [(1 + A\zeta)(2\lambda - 1) - (1 + B\zeta)]\,\bar{z}}{(1 - \bar{z})^2}.\end{aligned}$$

Proof. Let $f \in SH^0$. Then $f \in SH_\lambda^0(k, A, B)$ if and only if (3) holds or equivalently

$$\frac{D^{k+1} f(z)}{\lambda D^{k+1} f(z) + (1-\lambda) D^k f(z)} \neq \frac{1 + A\zeta}{1 + B\zeta} \quad (\zeta \in \mathbb{C},\ |\zeta| = 1,\ z \in \mathbb{D}). \tag{4}$$

Now for

$$D^k h(z) = D^k h(z) * \frac{z}{1 - z}, \quad D^{k+1} h(z) = D^k h(z) * \frac{z}{(1 - z)^2}$$

and

$$\overline{D^k g(z)} = \overline{D^k g(z)} * \frac{\overline{z}}{1 - \overline{z}}, \quad \overline{D^{k+1} g(z)} = \overline{D^k g(z)} * \frac{\overline{z}}{(1 - \overline{z})^2}$$

the inequality (4) yields

$$\begin{aligned}
& (1+B\zeta)D^{k+1}f(z)-(1+A\zeta)[\lambda D^{k+1}f(z)+(1-\lambda)D^{k}f\left(z\right)] \\
= \; & D^{k}h(z)*\left\{(1+B\zeta)\left[\frac{z}{(1-z)^{2}}\right]-(1+A\zeta)\left[\frac{\lambda z}{(1-z)^{2}}+\frac{(1-\lambda)z)}{1-z}\right]\right\} \\
& +(-1)^{k}\overline{D^{k}g(z)}*\left\{(1+B\zeta)\left[-\frac{\overline{z}}{(1-\overline{z})^{2}}\right]+(1+A\zeta)\left[\frac{\lambda\overline{z}}{(1-\overline{z})^{2}}-\frac{(1-\lambda)\overline{z}}{1-\overline{z}}\right]\right\} \\
= \; & D^{k}f\left(z\right)*\Phi(z;\zeta)\neq 0
\end{aligned}$$

Next we give the sufficient coefficient bound for functions in $SH_{\lambda}^{0}(k,A,B)$.

Theorem 2. *Let $f=h+\overline{g}$ be so that h and g are given by (1). Then $f\in SH_{\lambda}^{0}(k,A,B)$, if*

$$\sum_{n=2}^{\infty}\left(\phi_{n}\left|a_{n}\right|+\psi_{n}\left|b_{n}\right|\right)\leq A-B \tag{5}$$

where

$$\phi_{n}=n^{k}[\left(1-\lambda+A\lambda-B\right)n+\left(1-\lambda\right)\left(A-1\right)] \tag{6}$$

and

$$\psi_{n}=n^{k}[\left(1-\lambda-A\lambda+B\right)n+\left(1-\lambda\right)\left(A+1\right)]. \tag{7}$$

Proof. It is easy to see that the theorem is true for $f(z)=z$. So, we assume that $a_n\neq 0$ or $b_n\neq 0$ for $n\geq 2$. Since $\phi_n\geq n\left(B-A\right)$ and $\psi_n\geq n\left(B-A\right)$ by (5), we obtain

$$\begin{aligned}
\left|h'(z)\right|-\left|g'(z)\right| & \geq 1-\sum_{n=2}^{\infty}n\left|a_{n}\right|\left|z\right|^{n-1}-\sum_{n=2}^{\infty}n\left|b_{n}\right|\left|z\right|^{n-1} \\
& \geq 1-\left|z\right|\sum_{n=2}^{\infty}\left(n\left|a_{n}\right|+n\left|b_{n}\right|\right) \\
& \geq 1-\frac{\left|z\right|}{A-B}\sum_{n=2}^{\infty}\left(\phi_{n}\left|a_{n}\right|+\psi_{n}\left|b_{n}\right|\right) \\
& \geq 1-\left|z\right|>0.
\end{aligned}$$

Therefore f is sense preserving and locally univalent in $\mathbb{D}$. For the univalence condition, consider $z_1, z_2 \in \mathbb{D}$ so that $z_1 \neq z_2$. Then

$$\left|\frac{z_1^n - z_2^n}{z_1 - z_2}\right| = \left|\sum_{m=1}^{n} z_1^{m-1} z_2^{n-m}\right| \leq \sum_{m=1}^{n} \left|z_1^{m-1}\right| \left|z_2^{n-m}\right| < n, \quad n \geq 2.$$

Hence

$$\begin{aligned}
|f(z_1) - f(z_2)| &\geq |h(z_1) - h(z_2)| - |g(z_1) - g(z_2)| \\
&\geq \left|z_1 - z_2 - \sum_{n=2}^{\infty} a_n (z_1^n - z_2^n)\right| - \left|\sum_{n=2}^{\infty} \overline{b_n (z_1^n - z_2^n)}\right| \\
&\geq |z_1 - z_2| - \sum_{n=2}^{\infty} |a_n| |z_1^n - z_2^n| - \sum_{n=2}^{\infty} |b_n| |z_1^n - z_2^n| \\
&= |z_1 - z_2| \left(1 - \sum_{n=2}^{\infty} |a_n| \left|\frac{z_1^n - z_2^n}{z_1 - z_2}\right| - \sum_{n=2}^{\infty} |b_n| \left|\frac{z_1^n - z_2^n}{z_1 - z_2}\right|\right) \\
&> |z_1 - z_2| \left(1 - \sum_{n=2}^{\infty} n |a_n| - \sum_{n=2}^{\infty} n |b_n|\right) \geq 0.
\end{aligned}$$

which proves univalence.

On the other hand, $f \in SH_{\lambda}^{0}(k, A, B)$ if and only if there exists a complex valued function w; $w(0) = 0$, $|w(z)| < 1$ $(z \in \mathbb{D})$ such that

$$\frac{D^{k+1} f(z)}{\lambda D^{k+1} f(z) + (1-\lambda) D^k f(z)} = \frac{1 + Aw(z)}{1 + Bw(z)}$$

or equivalently

$$\left|\frac{(1-\lambda) D^{k+1} f(z) - (1-\lambda) D^k f(z)}{(A\lambda - B) D^{k+1} f(z) + A(1-\lambda) D^k f(z)}\right| < 1, \tag{8}$$

The above inequality (8) holds, since for $|z| = r$ $(0 < r < 1)$ we obtain

$$\left|(1-\lambda) D^{k+1} f(z) - (1-\lambda) D^k f(z)\right| - \left|(A\lambda - B) D^{k+1} f(z) + A(1-\lambda) D^k f(z)\right|$$

$$= \left| \sum_{n=2}^{\infty} n^k (n-1)(1-\lambda) a_n z^n + (-1)^{k+1} \sum_{n=2}^{\infty} n^k (n+1)(1-\lambda) \overline{b_n z^n} \right|$$
$$- \left| (A-B)\, z + \sum_{n=2}^{\infty} n^k [(A\lambda - B)n + A(1-\lambda)] a_n z^n \right.$$
$$\left. + (-1)^k \sum_{n=2}^{\infty} n^k \left[(B - A\lambda)n + A(1-\lambda)\right] \overline{b_n z^n} \right|$$

$$\leq \sum_{n=2}^{\infty} n^k (n-1)(1-\lambda) |a_n| r^n + \sum_{n=2}^{\infty} n^k (n+1)(1-\lambda) |b_n| r^n$$
$$- (A-B)\, r + \sum_{n=2}^{\infty} n^k [(A\lambda - B)n + A(1-\lambda)] |a_n| r^n$$
$$+ \sum_{n=2}^{\infty} n^k \left[(B - A\lambda)n + A(1-\lambda)\right] |b_n| r^n$$
$$\leq r \left\{ \sum_{n=2}^{\infty} \phi_n |a_n| r^{n-1} + \sum_{n=2}^{\infty} \psi_n |b_n| r^{n-1} - (A-B) \right\} < 0,$$

therefore $f \in SH_\lambda^0(k, A, B)$, and so the proof is complete.

Next we show that the condition (5) is also necessary for the functions $f \in SH$ to be in the class $SHT_\lambda^0(k, A, B) = T^k \cap SH_\lambda^0(k, A, B)$ where T^k is the class of functions $f = h + \overline{g} \in SH^0$ so that

$$f = h + \overline{g} = z - \sum_{n=2}^{\infty} |a_n| z^n + (-1)^k \sum_{n=2}^{\infty} |b_n| \overline{z}^n \quad (z \in \mathbb{D}). \tag{9}$$

Theorem 3. *Let $f = h + \overline{g}$ be defined by (9). Then $f \in SHT_\lambda^0(k, A, B)$ if and only if the condition (5) holds.*

Proof. The 'if' part follows from Theorem 2. For the 'only-if' part, assume that $f \in SHT_\lambda^0(k, A, B)$, then by (8) we have

$$\left| \frac{\sum_{n=2}^{\infty} n^k \{[(n-1)(1-\lambda)]|a_n| z^n + [(n+1)(1-\lambda)]|b_n| \overline{z}^n\}}{(A-B)z - \sum_{n=2}^{\infty} n^k \{[(A\lambda - B)n + A(1-\lambda)]|a_n| z^n + [(B - A\lambda)n + A(1-\lambda)]|b_n| \overline{z}^n\}} \right| < 1.$$

For $z = r < 1$ we obtain

$$\frac{\sum\limits_{n=2}^{\infty} n^k \{(n-1)(1-\lambda)|a_n| + (n+1)(1-\lambda)|b_n|\} r^{n-1}}{A - B - \sum\limits_{n=2}^{\infty} n^k \{[(A\lambda - B)n + A(1-\lambda)]|a_n| + [(B - A\lambda)n + A(1-\lambda)]|b_n|\} r^{n-1}} < 1.$$

Thus, for ϕ_n and ψ_n as defined by (6) and (7), we have

$$\sum_{n=2}^{\infty} [\phi_n |a_n| + \psi_n |b_n|] r^{n-1} < A - B \quad (0 \leq r < 1). \tag{10}$$

Let $\{\sigma_n\}$ be the sequence of partial sums of the series

$$\sum_{n=2}^{\infty} [\phi_n |a_n| + \psi_n |b_n|].$$

Then $\{\sigma_n\}$ is a nondecreasing sequence and by (10) it is bounded above by $A - B$. Thus, it is convergent and

$$\sum_{n=2}^{\infty} [\phi_n |a_n| + \psi_n |b_n|] = \lim_{n \to \infty} \sigma_n \leq A - B.$$

This gives the condition (5).

In the following we show that the class of functions of the form (9) is convex and compact.

Theorem 4. *The class $SHT_\lambda^0(k, A, B)$ is a convex and compact subset of SH.*

Proof. Let $f_t \in SHT_\lambda^0(k, A, B)$, where

$$f_t(z) = z - \sum_{n=2}^{\infty} |a_{t,n}| z^n + (-1)^k \sum_{n=2}^{\infty} |b_{t,n}| \overline{z^n} \quad (z \in \mathbb{D},\ t \in \mathbb{N}). \tag{11}$$

Then $0 \leq \eta \leq 1$, let $f_1, f_2 \in SHT_\lambda^0(k, A, B)$ be defined by (11). Then

$$\begin{aligned} \kappa(z) &= \eta f_1(z) + (1-\eta) f_2(z) \\ &= z - \sum_{n=2}^{\infty} (\eta |a_{1,n}| + (1-\eta) |a_{2,n}|) z^n \\ &\quad + (-1)^k \sum_{n=2}^{\infty} (\eta |b_{1,n}| + (1-\eta) |b_{2,n}|) \overline{z^n} \end{aligned}$$

and

$$\begin{aligned}
&\sum_{n=2}^{\infty}\left\{\phi_n\left[\eta\left|a_{1,n}\right|+(1-\eta)\left|a_{2,n}\right|\right]+\psi_n\left[\eta\left|b_{1,n}\right|+(1-\eta)\left|b_{2,n}\right|\right]\right\} \\
= \;& \eta\sum_{n=2}^{\infty}\left\{\phi_n\left|a_{1,n}\right|+\psi_n\left|b_{1,n}\right|\right\}+(1-\eta)\sum_{n=2}^{\infty}\left\{\phi_n\left|a_{2,n}\right|+\psi_n\left|b_{2,n}\right|\right\} \\
\leq \;& \eta\,(A-B)+(1-\eta)\,(A-B)=A-B.
\end{aligned}$$

Thus, the function $\kappa=\eta f_1+(1-\eta)f_2$ belongs to the class $SHT_{\lambda}^{0}(k,A,B)$. This means that the class $SHT_{\lambda}^{0}(k,A,B)$ is convex.

On the other hand, for $f_t \in SHT_{\lambda}^{0}(k,A,B)$, $t \in \mathbb{N}$ and $|z| \leq r$ $(0<r<1)$, we get

$$\begin{aligned}
\left|f_t(z)\right| &\leq r+\sum_{n=2}^{\infty}\left\{\left|a_{t,n}\right|+\left|b_{t,n}\right|\right\}r^n \\
&\leq r+\sum_{n=2}^{\infty}\left\{\phi_n\left|a_{t,n}\right|+\psi_n\left|b_{t,n}\right|\right\}r^n \\
&\leq r+(A-B)\,r^2.
\end{aligned}$$

Therefore, $SHT_{\lambda}^{0}(k,A,B)$ is locally uniformly bounded. Let

$$f_t(z)=z-\sum_{n=2}^{\infty}\left|a_{t,n}\right|z^n+(-1)^k\sum_{n=2}^{\infty}\left|b_{t,n}\right|\overline{z^n} \quad (z\in\mathbb{D},\; t\in\mathbb{N})$$

and let $f=h+\overline{g}$ be so that h and g are given by (1). Using Theorem 3 we obtain

$$\sum_{n=2}^{\infty}\left\{\phi_n\left|a_{t,n}\right|+\psi_n\left|b_{t,n}\right|\right\}\leq A-B. \tag{12}$$

If we assume that $f_t \to f$, then we conclude that $|a_{t,n}| \to |a_n|$ and $|b_{t,n}| \to |b_n|$ as $n \to \infty$ $(t \in \mathbb{N})$. Let $\{\sigma_n\}$ be the sequence of partial sums of the series $\sum\limits_{n=2}^{\infty}\{\phi_n|a_n|+\psi_n|b_n|\}$. Then $\{\sigma_n\}$ is a nondecreasing sequence and by (12) it is bounded above by $A-B$. Thus, it is convergent and

$$\sum_{n=2}^{\infty}\left\{\phi_n\left|a_n\right|+\psi_n\left|b_n\right|\right\}=\lim_{n\to\infty}\sigma_n\leq A-B.$$

Therefore $f \in SHT_\lambda^0(k, A, B)$ and therefore the class $SHT_\lambda^0(k, A, B)$ is closed. In consequence, the class $SHT_\lambda^0(k, A, B)$ is compact subset of SH, which completes the proof.

We continue with the following lemma due to Jahangiri [5].

Lemma 5. *Let $f = h + \overline{g}$ be so that h and g are given by (1). Furthermore, let*

$$\sum_{n=2}^{\infty} \left\{ \frac{n-\varrho}{1-\varrho} |a_n| + \frac{n+\varrho}{1-\varrho} |b_n| \right\} \leq 1 \quad (z \in \mathbb{D})$$

where $0 \leq \varrho < 1$. Then f is harmonic, orientation preserving, univalent in $\mathbb{D}$ and f is starlike of order ϱ.

In the following theorems we obtain the radii of starlikeness and convexity for functions in the class $SHT_\lambda^0(k, A, B)$.

Theorem 6. *Let $0 \leq \varrho < 1$, ϕ_n and ψ_n be defined by (6) and (7). Then*

$$r_\varrho^*(SHT_\lambda^0(k, A, B)) = \inf_{n \geq 2} \left[\frac{1-\varrho}{A-B} \min \left\{ \frac{\phi_n}{n-\varrho}, \frac{\psi_n}{n+\varrho} \right\} \right]^{\frac{1}{n-1}}. \tag{13}$$

Proof. Let $f \in SHT_\lambda^0(k, A, B)$ be of the form (9). Then, for $|z| = r < 1$, we get

$$\left| \frac{Df(z) - (1+\varrho)f(z)}{Df(z) + (1-\varrho)f(z)} \right| \leq \frac{\varrho + \sum_{n=2}^{\infty} \{(n-1-\varrho)|a_n| + (n+1+\varrho)|b_n|\} r^{n-1}}{2 - \varrho - \sum_{n=2}^{\infty} \{(n+1-\varrho)|a_n| + (n-1+\varrho)|b_n|\} r^{n-1}}.$$

Note (see Lemma 5) that f is starlike of order ϱ in $\mathbb{D}_r$ if and only if

$$\left| \frac{Df(z) - (1+\varrho)f(z)}{Df(z) + (1-\varrho)f(z)} \right| < 1, \; z \in \mathbb{D}_r$$

or

$$\sum_{n=2}^{\infty} \left\{ \frac{n-\varrho}{1-\varrho} |a_n| + \frac{n+\varrho}{1-\varrho} |b_n| \right\} r^{n-1} \leq 1. \tag{14}$$

Moreover, by Theorem 2, we have

$$\sum_{n=2}^{\infty} \left\{ \frac{\phi_n}{A-B} |a_n| + \frac{\psi_n}{A-B} |b_n| \right\} r^{n-1} \leq 1.$$

Since ϕ_n and ψ_n be defined by (6) and (7).

The condition (14) is true if

$$\frac{n-\varrho}{1-\varrho}r^{n-1} \le \frac{\phi_n}{A-B}r^{n-1}$$

and

$$\frac{n+\varrho}{1-\varrho}r^{n-1} \le \frac{\psi_n}{A-B}r^{n-1} \quad (n = 2, 3, \dots)$$

or if

$$r \le \frac{1-\varrho}{A-B} \min\left\{\frac{\phi_n}{n-\varrho}, \frac{\psi_n}{n+\varrho}\right\}^{\frac{1}{n-1}} \qquad (n = 2, 3, \dots).$$

It follows that the function f is starlike of order ϱ in the disk $\mathbb{D}_{r_\varrho^*}$ where

$$r_\varrho^* := \inf_{n\ge 2}\left[\frac{1-\varrho}{A-B} \min\left\{\frac{\phi_n}{n-\varrho}, \frac{\psi_n}{n+\varrho}\right\}\right]^{\frac{1}{n-1}}.$$

The function

$$f_n(z) = h_n(z) + \overline{g_n(z)} = z - \frac{A-B}{\phi_n}z^n + (-1)^k\frac{A-B}{\psi_n}\overline{z}^n$$

proves that the radius r^* cannot be any larger. Thus we have (13).

Using a similar argument as above we obtain the following.

Theorem 7. *Let $0 \le \varrho < 1$ and ϕ_n and ψ_n be defined by (6) and (7). Then*

$$r_\varrho^c(SHT_\lambda^0(k, A, B)) = \inf_{n\ge 2}\left[\frac{1-\varrho}{A-B} \min\left\{\frac{\phi_n}{n(n-\varrho)}, \frac{\psi_n}{n(n+\varrho)}\right\}\right]^{\frac{1}{n-1}}.$$

Our next theorem is on the extreme points of $SHT_\lambda^0(k, A, B)$.

Theorem 8. *Extreme points of the class $SHT_\lambda^0(k, A, B)$ are the functions f of the form (1) where $h = h_n$ and $g = g_n$ are of the form*

$$\begin{gathered} h_1(z) = z, \quad h_n(z) = z - \tfrac{A-B}{\phi_n}z^n, \\ g_n(z) = (-1)^k\tfrac{A-B}{\psi_n}\overline{z^n}, \quad (z \in \mathbb{D},\ n \ge 2). \end{gathered} \tag{15}$$

Proof. Let $g_n = \eta f_1 + (1-\eta) f_2$ where $0 < \eta < 1$ and $f_1, f_2 \in SHT_\lambda^0(k, A, B)$ are functions of the form

$$f_t(z) = z - \sum_{n=2}^{\infty} |a_{t,n}|\, z^n + (-1)^k \sum_{n=2}^{\infty} |b_{t,n}|\, \overline{z^n} \quad (z \in \mathbb{D},\ t \in \{1, 2\}).$$

Then, by (5), we have

$$|b_{1,n}| = |b_{2,n}| = \frac{A-B}{\psi_n},$$

and therefore $a_{1,t} = a_{2,t} = 0$ for $t \in \{2,3,\ldots\}$ and $b_{1,t} = b_{2,t} = 0$ for $t \in \{2,3,\ldots\} \setminus \{n\}$. It follows that $g_n(z) = f_1(z) = f_2(z)$ and g_n are in the class of extreme points of the function class $SHT_{\lambda}^{0}(n, A, B)$. Similarly, we can verify that the functions $h_n(z)$ are the extreme points of the class $SHT_{\lambda}^{0}(k, A, B)$. Now, suppose that a function f of the form (1) is in the family of extreme points of the class $SHT_{\lambda}^{0}(k, A, B)$ and f is not of the form (15). Then there exists $m \in \{2,3,\ldots\}$ such that

$$0 < |a_m| < \frac{A-B}{m^k[(1-\lambda+A\lambda-B)\,m+(1-\lambda)\,(A-1)]}$$

or

$$0 < |b_m| < \frac{A-B}{m^k[(1-\lambda-A\lambda+B)\,m+(1-\lambda)\,(A+1)]}.$$

If

$$0 < |a_m| < \frac{A-B}{m^k\left\{m^k[(1-\lambda+A\lambda-B)\,m+(1-\lambda)\,(A-1)]\right\}},$$

then putting

$$\eta = \frac{|a_m|\, m^k\left\{m^k[(1-\lambda+A\lambda-B)\,m+(1-\lambda)\,(A-1)]\right\}}{A-B}$$

and

$$\varphi = \frac{f-\eta h_m}{1-\eta},$$

we have $0<\eta<1$, $h_m \neq \varphi$, and

$$f = \eta h_m + (1-\eta)\,\varphi.$$

Therefore, f is not in the family of extreme points of the class $SHT_{\lambda}^{0}(k, A, B)$. Similarly, if

$$0 < |b_m| < \frac{A-B}{m^k[(1-\lambda-A\lambda+B)\,m+(1-\lambda)\,(A+1)]},$$

then putting

$$\eta = \frac{|b_m|\, m^k[(1-\lambda-A\lambda+B)\,m+(1-\lambda)\,(A+1)]}{A-B}$$

and

$$\varphi = \frac{f-\eta g_m}{1-\eta},$$

we have $0 < \eta < 1$, $g_m \neq \varphi$, and

$$f = \eta g_m + (1 - \eta)\,\varphi.$$

It follows that f is not in the family of extreme points of the class $SHT_\lambda^0(k, A, B)$ and so the proof is completed.

Therefore, by Theorem 8, we have the following corollary.

Corollary 9. *Let $f \in SHT_\lambda^0(k, A, B)$, be a function of the form (9). Then*

$$|a_n| \leq \frac{A - B}{n^k[(1 - \lambda + A\lambda - B)\,n + (1 - \lambda)\,(A - 1)]}$$

and

$$|b_n| \leq \frac{A - B}{n^k[(1 - \lambda - A\lambda + B)\,n + (1 - \lambda)\,(A + 1)]}$$

The result is sharp for the extremal functions h_n, g_n of the form (15).

Corollary 10. *Let $f \in SHT_\lambda^0(k, A, B)$ and $|z| = r < 1$. Then*

$$r - \frac{A - B}{2^k\,[(\lambda + 1)(A - 1) - 2(B - 1)]} r^2 \leq |f(z)| \leq r + \frac{A - B}{2^k\,[(\lambda + 1)(A - 1) - 2(B - 1)]} r^2.$$

The following covering result follows from Corollary 10.

Corollary 11. *If $f \in SHT_\lambda^0(k, A, B)$ then $\mathbb{D}_r \subset f(\mathbb{D})$ where*

$$r = 1 - \frac{A - B}{2^k\,[(\lambda + 1)(A - 1) - 2(B - 1)]}.$$

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