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SOME NEW RESULTS ASSOCIATED WITH THE GENERALIZED LOMMEL-WRIGHT FUNCTION

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Abstract. The aim of this article is to establish a new class of unified integrals associated with the generalized Lommel-Wright functions.Some integrals involving trigonometric, generalized Bessel and Struve functions are also mentioned as special cases of the main results. Further, we establish two reduction formulas for the Wright hypergeometric function.

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1. Introduction

The Wright hypergeometric function defined by the series [15]:

$$
{}_p\psi_q\left[\begin{array}{c}(\alpha_1,A_1),\,..,(\alpha_p,A_p);\\ \\ (\beta_1,B_1)...,(\beta_q,B_q)\end{array} z\right]=\sum_{k=0}^{\infty}\frac{\prod\limits_{j=1}^{p}\Gamma(\alpha_j+A_jk)z^k}{\prod\limits_{j=1}^{q}\Gamma(\beta_j+B_jk)k!}, \tag{1}
$$

where the coefficients $A_1,....A_p$ and $B_1,....B_q$ are positive real numbers such that

$$
1+\sum_{j=1}^{q}B_j-\sum_{j=1}^{p}A_j\geq 0. \tag{2}
$$

can be slightly generalized (1) as given below.

$$
{}_p\psi_q\left[\begin{array}{c}(\alpha_1,1),\,..,(\alpha_p,1);\\ \\ (\beta_1,1)...,(\beta_q,1)\end{array} z\right]=\frac{\prod\limits_{j=1}^{p}\Gamma(\alpha_j)}{\prod\limits_{j=1}^{q}\Gamma(\beta_j)}\,{}_pF_q\left[\begin{array}{c}\alpha_1,..,\alpha_p;\\ \\ \beta_1,...,\beta_q\end{array} z\right], \tag{3}
$$

where ${}_pF_q$ is the generalized hypergeometric function defined by [15, 13]

$$ {}_pF_q\left[\begin{array}{c} \alpha_1, .., \alpha_p; \\ \\ \beta_1, ..., \beta_q \end{array} z\right] = \sum_{k=0}^{\infty} \frac{(\alpha_1)_n, ..., (\alpha_p)_n z^n}{(\beta_1)_n, ..., (\beta_q)_n n!} = {}_pF_q(\alpha_1, ..., \alpha_p; \beta_1, ..., \beta_q; z), \quad (4)$$

where $(\vartheta)_n$ is denote the Pochhammer symbol [15].

The series representation of the generalized Lommel Wright function as [5];

$$J_{\nu,\lambda}^{\mu,m}(z) = \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma(k+1)(\frac{z}{2})^{2k+\nu+2\lambda}}{\Gamma(\lambda+k+1)^m \Gamma(\nu+k\mu+\lambda+1)k!}, \quad (5)$$

$$(z \in \mathbb{N}/(-\infty, 0] \ m \in \mathbb{N}, \ \nu, \lambda \in \mathbb{C}, \mu > 0).$$

Also, we have the following relations of generalized Lommel Wright functions with trigonometric functions and the generalized Bessel function and Struve function:

$$J_{1/2,0}^{1,1}(z) = \sqrt{(}\frac{2}{\pi z}) \sin(z) \quad (6)$$

$$J_{-1/2,0}^{1,1}(z) = \sqrt{(}\frac{2}{\pi z}) \cos(z) \quad (7)$$

$$J_{\nu,\lambda}^{\mu,1}(z) = \mathbb{J}_{\nu,\lambda}^{\mu}(z) \quad (8)$$

$$J_{\nu,1/2}^{1,1}(z) = H_\nu(z). \quad (9)$$

Here, we recall the following Lavoie and Trottier integral formula[8].

$$\int_0^1 t^{\kappa-1}(1-t)^{2\vartheta-1}\left(1-\frac{t}{3}\right)^{2\kappa-1}\left(1-\frac{t}{4}\right)^{\vartheta-1} dt = \left(\frac{2}{3}\right)^{2\kappa} \frac{\Gamma(\kappa)\Gamma(\vartheta)}{\Gamma(\kappa+\vartheta)}, \quad (10)$$

where $\Re(\kappa) > 0$ and $\Re(\vartheta) > 0$.

Various generalizations and cases of Lommel-Wright function have been investigated (see, for details, [1, 14, 7]). For more details of integral involving various special functions, one may be referred to the recent research papers [4, 9, 10, 11]

Integral formulas involving Lommel-Wright functions have been developed by many authors (see, e.g., [2, 3, 6, 12]). In this sequel, here, we aim at establishing certain new generalized integral formula involving the generalized Lommel-Wright function $J_{\nu,\lambda}^{\mu,m}(z)$ interesting integral formulas which are derived as special cases.

2. Main Results

The integrals involving Lommel-Wright function is given in this section.

Theorem 1. *For $\eta, \theta \in \mathbb{C}$ and $t > 0$ with $\Re(\eta) > 0, \Re(\theta) > 0$, the following integral formula holds true*

$$\int_0^1 x^{\eta-1}(1-x)^{2\theta-1}\left(1-\frac{x}{3}\right)^{2\eta-1}\left(1-\frac{x}{4}\right)^{\theta-1} J_{\nu,\lambda}^{\mu,m}\left(y\left(1-\frac{t}{4}\right)(1-t)^2\right)dt$$
$$= \Gamma(\eta)\left(\frac{2}{3}\right)^{2\eta}\left(\frac{y}{2}\right)^{\nu+2\lambda}$$
$$\times\, {}_2\psi_{m+2}\left[\begin{matrix} (1,1),(\theta+\nu+2\lambda,2); \\ (\lambda+1,1),...,(\lambda+1,1),(\eta+\theta+\nu+2\lambda,2),(\nu+\lambda+1,\mu); \end{matrix} \; -\frac{y^2}{4}\right]. \tag{11}$$

Proof. On using (5) in the integrand of (11) and then interchanging the order of integral sign and summation which is verified under the given conditions, we get

$$\int_0^1 t^{\eta-1}(1-t)^{2\theta-1}\left(1-\frac{t}{3}\right)^{2\eta-1}\left(1-\frac{t}{4}\right)^{\theta-1} J_{\nu,\lambda}^{\mu,m}\left(y\left(1-\frac{t}{4}\right)(1-t)^2\right)dt$$
$$= \left(\frac{y}{2}\right)^{\nu+2\lambda}\sum_{k=0}^{\infty}\frac{\Gamma(k+1)(-y^2/4)^k}{\Gamma(\lambda+k+1)^m\Gamma(\nu+k\mu+\lambda+1)k!}$$
$$\times\int_0^1 t^{\eta-1}(1-t)^{2(\theta+2k+\nu+2\lambda)-1}\left(1-\frac{t}{3}\right)^{2\eta-1}\left(1-\frac{t}{4}\right)^{(\theta+2k+\nu+2\lambda)-1} dt. \tag{12}$$

Now using (10) in the above equation we get

$$\int_0^1 t^{\eta-1}(1-t)^{2\theta-1}\left(1-\frac{t}{3}\right)^{2\eta-1}\left(1-\frac{t}{4}\right)^{\theta-1} J_{\nu,\lambda}^{\mu,m}\left(y\left(1-\frac{t}{4}\right)(1-t)^2\right)dt = \left(\frac{2}{3}\right)^{2\eta}$$
$$\times\,\Gamma(\eta)\left(\frac{y}{2}\right)^{\nu+2\lambda}\sum_{k=0}^{\infty}\frac{\Gamma(k+1)\Gamma(\theta+\nu+2\lambda+2k)(\frac{-y^2}{4})^k}{\Gamma(\lambda+k+1)^m\Gamma(\eta+\theta+\nu+2\lambda+2k)\Gamma(\nu+k\mu+\lambda+1)k!}. \tag{13}$$

Finally, using (1) in the above equation, we get our assertion (11).

Theorem 2. *The following integral formula holds true: For $\eta, \theta \in C$ and $t > 0$ with*

$\Re(\theta) > 0, \Re(\eta) > 0,$

$$\int_0^1 t^{\eta-1}(1-t)^{2\theta-1}\left(1-\frac{t}{3}\right)^{2\eta-1}\left(1-\frac{t}{4}\right)^{\theta-1} J_{\nu,\lambda}^{\mu,m}\left(ty\left(1-\frac{t}{3}\right)^2\right)dt$$
$$= \left(\frac{2}{3}\right)^{2(\eta+\nu+2\lambda)}\left(\frac{y}{2}\right)^{\nu+2\lambda}\Gamma(\theta)$$
$$\times {}_2\psi_{m+2}\left[\begin{array}{c} (1,1),(\eta+\nu+2\lambda,2); \\ (\lambda+1,1),...,(\lambda+1,1)(1+\nu+\lambda,\mu),(\eta+\theta+\nu+2\lambda,2); \end{array} \frac{-4y^2}{81}\right]. \tag{14}$$

Proof. On using (5) in the integrand of (14) and then interchanging the order of integral sign and summation which is verified under the given conditions, we get

$$\int_0^1 t^{\eta-1}(1-t)^{2\theta-1}\left(1-\frac{t}{3}\right)^{2\eta-1}\left(1-\frac{t}{4}\right)^{\theta-1} J_{\nu,\lambda}^{\mu,m}\left(ty\left(1-\frac{t}{3}\right)^2\right)dt$$
$$= (y/2)^{\nu+2\lambda}\sum_{k=0}^{\infty}\frac{\Gamma(k+1)\left(-\frac{y^2}{4}\right)^k}{\Gamma(\lambda+k+1)^m\Gamma(\nu+k\mu+\lambda+1)k!}$$
$$\times \int_0^1 t^{\eta+\nu+2\lambda+2k-1}(1-t)^{2\theta-1}\left(1-\frac{t}{3}\right)^{2(\eta+\nu+2\lambda+2k)-1}\left(1-\frac{t}{4}\right)^{\theta-1}dt. \tag{15}$$

Now using (10) in the above equation we get

$$\int_0^1 t^{\eta-1}(1-t)^{2\theta-1}\left(1-\frac{t}{3}\right)^{2\eta-1}\left(1-\frac{t}{4}\right)^{\theta-1} J_{\nu,\lambda}^{\mu,m}\left(y\left(1-\frac{t}{4}\right)\right)dt$$
$$= \left(\frac{2}{3}\right)^{2(\eta+\nu+2\lambda)}\Gamma(\theta)\left(\frac{y}{2}\right)^{\nu+2\lambda}$$
$$\times \sum_{k=0}^{\infty}\frac{\Gamma(k+1)\Gamma(\eta+\nu+2\lambda+2k)\left(\frac{-4y^2}{81}\right)^k}{\Gamma(\lambda+k+1)^m\Gamma(\nu+k\mu+\lambda+1)\Gamma(\eta+\theta+\nu+2\lambda+2k)k!}. \tag{16}$$

Finally, using (1) in the above equation, we get our assertion (14).

Next we consider other variations of Theorem 1 and Theorem 2 in the form of corollaries.

Corollary 3. *In* (13)*, on separating the hypergeometric series into its even and odd terms,we get,*

$$\int_0^1 t^{\eta-1}(1-t)^{2\theta-1}\left(1-\frac{t}{3}\right)^{2\eta-1}\left(1-\frac{t}{4}\right)^{\theta-1} J_{\nu,\lambda}^{\mu,m}\left(y\left(1-\frac{t}{4}\right)(1-t)^2\right)dt = \left(\frac{2}{3}\right)^{2\eta}\sqrt{\pi}$$
$$\times\left(\frac{y}{2}\right)^{\nu+2\lambda}\Gamma(\eta)\ {}_2\psi_{m+3}\left[\begin{array}{c}(1,2),(\theta+\nu+2\lambda,4);\\ (\lambda+1,2),...,(\lambda+1,2)(1/2,1),(\eta+\theta+\nu+2\lambda,4),(\nu+\lambda+1,2\mu);\end{array}\ \frac{y^4}{64}\right]$$
$$-\left(\frac{2}{3}\right)^{2\eta}\frac{\sqrt{\pi}}{2}\left(\frac{y}{2}\right)^{\nu+2\lambda+2}\Gamma(\eta)$$
$$\times\ {}_2\Psi_{m+3}\left[\begin{array}{c}(2,2),(\theta+\nu+2\lambda+2,4);\\ (\lambda+2,2),...,(\lambda+2,2)(\eta+\theta+\nu+2\lambda+2,4),(3/2,1),(\mu+\nu+\lambda+1,2\mu);\end{array}\ \frac{y^4}{64}\right], \tag{17}$$

where $\Re(\eta) > 0$ *and* $\Re(\theta) > 0$.

Corollary 4. *On expanding the r.h.s of* (16) *in series form and then separating the resulting series into its even and odd terms, we obtain.*

$$\int_0^1 t^{\eta-1}(1-t)^{2\theta-1}\left(1-\frac{t}{3}\right)^{2\eta-1}\left(1-\frac{t}{4}\right)^{\theta-1} J_{\nu,\lambda}^{\mu,m}\left(ty\left(1-\frac{t}{3}\right)^2\right)dt$$
$$=\left(\frac{2}{3}\right)^{2(\eta+\theta+2\lambda)}\sqrt{\pi}\left(\frac{y}{2}\right)^{\nu+2\lambda}\Gamma(\theta)$$
$$\times\ {}_2\psi_{m+3}\left[\begin{array}{c}(1,2),(\eta+\theta+2\lambda,4);\\ (\lambda+1,2),...,(\lambda+1,2)(1/2,1),(\eta+\theta+\nu+2\lambda,4),(\nu+\lambda+1,2\mu);\end{array}\ \frac{y^4}{6561}\right]$$
$$+\left(\frac{2}{3}\right)^{2(\eta+\theta+2\lambda)}\frac{\sqrt{\pi}}{2}\left(-\frac{4y^2}{81}\right)\left(\frac{y}{2}\right)^{\nu+2\lambda}\Gamma(\theta)$$
$$\times\ {}_2\Psi_{m+3}\left[\begin{array}{c}(2,2),(\eta+\theta+2\lambda+2,4);\\ (\lambda+2,2),...,(\lambda+2,2)(3/2,1),(\eta+\theta+\nu+2\lambda+2,4),(\nu+\lambda+\mu+1,2\mu);\end{array}\ \frac{y^4}{6561}\right], \tag{18}$$

where $\Re(\eta) > 0$ *and* $\Re(\theta) > 0$.

3. Special Cases

In this section, we derive some integrals containing trigonometric function and generalized Lommel-Wright function as follows:

Corollary 5. *If we take $m = 1, \mu = 1, \lambda = 0$ and $\nu = 1/2$ in* (11) *and then by using* (6), *we derive the following integral formula:*

$$\int_0^1 t^{\eta-1}(1-t)^{2(\theta-1/2)-1}\left(1-\frac{t}{3}\right)^{2\eta-1}\left(1-\frac{t}{4}\right)^{(\theta-1/2)-1}\sin\big(y(1-t/4)(1-t)^2\big)dt$$
$$= \left(\frac{2}{3}\right)^{2\eta}\sqrt{\pi}(\frac{y}{2})\ \Gamma(\eta){}_1\psi_2\left[\begin{array}{c} (\theta+1/2,2); \\ (\eta+\theta+1/2,2),(\frac{3}{2},1); \end{array} -\frac{y^2}{4}\right] \quad (19)$$

where $\Re(\eta) > 0$ *and* $\Re(\theta) > 0$.

Corollary 6. *Again by taking $m = 1, \mu = 1, \lambda = 0$ and $\nu = 1/2$ in* (14) *and then by using* (6), *we deduce the following integral formula:*

$$\int_0^1 t^{\left(\eta-\frac{1}{2}\right)-1}(1-t)^{2\theta-1}\left(1-\frac{t}{3}\right)^{2\left(\eta-\frac{1}{2}\right)-1}\left(1-\frac{t}{4}\right)^{\theta-1}\sin\left(ty\left(1-\frac{t}{3}\right)^2\right)dt$$
$$= \left(\frac{2}{3}\right)^{2\left(\eta+\theta\right)}\sqrt{\pi}\ (\frac{y}{2})\ \Gamma(\theta){}_1\psi_2\left[\begin{array}{c} (\eta+\theta,2); \\ (\eta+\theta+\frac{1}{2},2),(\frac{3}{2},1); \end{array} -\frac{4y^2}{81}\right] \quad (20)$$

where $\Re(\eta) > 0$ *and* $\Re(\theta) > 0$.

Corollary 7. *If we put $m = 1, \mu = 1, \lambda = 0$ and $\nu = 1/2$ in* (17) *and then by using* (6), *we obtain:*

$$\int_0^1 t^{\eta-1}(1-t)^{2(\theta-\frac{1}{2})-1}\left(1-\frac{t}{3}\right)^{2\eta-1}\left(1-\frac{t}{4}\right)^{(\theta-\frac{1}{2})-1}\sin\left(y\left(1-\frac{t}{4}\right)(1-t)^2\right)dt$$
$$= \left(\frac{2}{3}\right)^{2\eta}(\frac{y\pi}{2})\Gamma(\eta){}_1\psi_3\left[\begin{array}{c} (\theta+\frac{1}{2},4); \\ (\eta+\theta+\frac{1}{2},4),(\frac{3}{2},1),(1/2,1); \end{array} \frac{y^4}{64}\right]$$
$$-\left(\frac{2}{3}\right)^{2\eta}(\frac{y^3\pi}{16})\Gamma(\eta){}_1\psi_3\left[\begin{array}{c} (\theta+\frac{5}{2},4); \\ (\eta+\theta+\frac{5}{2},4),(\frac{3}{2},1),(\frac{5}{2},2); \end{array} \frac{y^4}{64}\right], \quad (21)$$

where $\Re(\eta) > 0$ *and* $\Re(\theta) > 0$.

Corollary 8. *Further we take $m = 1, \mu = 1, \lambda = 0$ and $\nu = 1/2$ in* (18) *and then by* *using* (6)*, we obtain:*

$$\int_0^1 t^{(\eta-\frac{1}{2})-1}(1-t)^{2\theta-1}\left(1-\frac{t}{3}\right)^{2(\eta-\frac{1}{2})-1}\left(1-\frac{t}{4}\right)^{\theta-1}\sin\left(ty\left(1-\frac{t}{3}\right)^2\right)dt$$
$$=\left(\frac{2}{3}\right)^{2(\eta+\theta)}\left(\frac{\pi y}{2}\right)\Gamma(\theta)\,{}_1\psi_3\left[\begin{array}{cc}(\eta+\theta,4); & \\ & \frac{4y^4}{6561}\\ (\eta+\theta+\frac{1}{2},4),(\frac{1}{2},1),(\frac{3}{2},2); & \end{array}\right]$$
$$+\left(\frac{2}{3}\right)^{2(\eta+\theta)}\Gamma(\theta)\left(\frac{4y^3}{81}\right)\,{}_1\psi_3\left[\begin{array}{cc}(\eta+\theta+2,4); & \\ & \frac{4y^4}{6561}\\ (\eta+\theta+\frac{5}{2},4),(\frac{3}{2},1),(\frac{5}{2},2); & \end{array}\right], \quad (22)$$

where $\Re(\eta) > 0$ *and* $\Re(\theta) > 0$.

Corollary 9. *If we take* $m = 1, \mu = 1, \lambda = 0$ *and* $\nu = -1/2$ *in* (11)*and then by using* (7)*,we get the following integral formula:*

$$\int_0^1 t^{\eta-1}(1-t)^{2(\theta-\frac{1}{2})-1}\left(1-\frac{t}{3}\right)^{2\eta-1}\left(1-\frac{t}{4}\right)^{(\theta-\frac{1}{2})-1}\cos\left(y\left(1-\frac{t}{4}\right)(1-t^2\right)dt$$
$$=\left(\frac{2}{3}\right)^{2\eta}\sqrt{\pi}\;\Gamma(\eta)\,{}_1\psi_2\left[\begin{array}{cc}(\theta-\frac{1}{2},2); & \\ & -\frac{y^2}{4}\\ (\eta+\theta-\frac{1}{2},2),(\frac{1}{2},1); & \end{array}\right], \quad (23)$$

where $\Re(\eta) > 0$ *and* $\Re(\theta) > 0$.

Corollary 10. *Again we take* $m = 1, \mu = 1, \lambda = 0$ *and* $\nu = -1/2$ *in* (14) *and then by using* (7)*, we get the following integral formula:*

$$\int_0^1 t^{\left(\eta-\frac{1}{2}\right)-1}(1-t)^{2\theta-1}\left(1-\frac{t}{3}\right)^{2\left(\eta-\frac{1}{2}\right)-1}\left(1-\frac{t}{4}\right)^{\theta-1}\cos\left(t\;y\left(1-\frac{t}{3}\right)^2\right)dt$$
$$=\left(\frac{2}{3}\right)^{2(\eta+\theta)}\sqrt{\pi}\;\Gamma(\theta)\,{}_1\psi_2\left[\begin{array}{cc}(\eta+\theta,2); & \\ & -\frac{4y^2}{81}\\ (\eta+\theta-\frac{1}{2},2),(\frac{1}{2},1); & \end{array}\right], \quad (24)$$

where $\Re(\eta) > 0$ *and* $\Re(\theta) > 0$.

Corollary 11. *If we take* $m = 1, \mu = 1, \lambda = 0$ *and* $\nu = -1/2$ *in* (17) *and then by* *using* (7)*, we obtain:*

$$\int_0^1 t^{\eta-1}(1-t)^{2(\theta-\frac{1}{2})-1}\left(1-\frac{t}{3}\right)^{2\eta-1}\left(1-\frac{t}{4}\right)^{(\theta-\frac{1}{2})-1}\cos\left(y\left(1-\frac{t}{4}\right)(1-t^2)\right)dt$$

$$= \left(\frac{2}{3}\right)^{2\eta}\pi\Gamma(\eta)\,{}_1\psi_3\left[\begin{array}{c}(\theta-\frac{1}{2},4);\\ (\eta+\theta-\frac{1}{2},4),(\frac{1}{2},2),(1/2,1);\end{array}\ \frac{y^4}{64}\right]$$

$$+\left(\frac{2}{3}\right)^{2\eta}\left(\frac{\pi y^2}{16}\right)\Gamma(\eta)\,{}_1\psi_3\left[\begin{array}{c}(\theta+\frac{3}{2},4);\\ (\eta+\theta+\frac{3}{2},4),(\frac{3}{2},1),(\frac{3}{2},2);\end{array}\ \frac{y^4}{64}\right], \quad (25)$$

where $\Re(\eta) > 0$ *and* $\Re(\theta) > 0$.

Corollary 12. *Further if we take* $m = 1, \mu = 1, \lambda = 0$ *and* $\nu = -1/2$ *in* (18) *and then by using* (7)*, we obtain:*

$$\int_0^1 t^{(\eta-\frac{1}{2})-1}(1-t)^{2\theta-1}\left(1-\frac{t}{3}\right)^{2(\eta-\frac{1}{2})-1}\left(1-\frac{t}{4}\right)^{\theta-1}\cos\left(ty\left(1-\frac{t}{3}\right)^2\right)dt$$

$$= \left(\frac{2}{3}\right)^{2(\eta+\theta)}\pi\Gamma(\theta)\,{}_1\psi_3\left[\begin{array}{c}(\eta+\theta,4);\\ (\eta+\theta-\frac{1}{2},4),(\frac{1}{2},1),(\frac{1}{2},2);\end{array}\ \frac{y^4}{6561}\right]$$

$$+\left(\frac{2}{3}\right)^{2(\eta+\theta)}\left(-\frac{8\pi y^2}{81}\right)\Gamma(\theta)\,{}_1\psi_3\left[\begin{array}{c}(\eta+\theta+2,4);\\ (\eta+\theta+\frac{3}{2},4),(\frac{3}{2},1),(\frac{3}{2},2);\end{array}\ \frac{y^4}{6561}\right], \quad (26)$$

where $\Re(\eta) > 0$ *and* $\Re(\theta) > 0$.

Corollary 13. *If we take* $m = 1$ *in* (11) *and then by using* (8)*, we obtain:*

$$\int_0^1 t^{\eta-1}(1-t)^{2\theta-1}\left(1-\frac{t}{3}\right)^{2\eta-1}\left(1-\frac{t}{4}\right)^{\theta-1}\mathbb{J}^{\mu}_{\nu,\lambda}(y(1-t/4)(1-t)^2)dt = \left(\frac{2}{3}\right)^{2\eta}\left(\frac{y}{2}\right)^{\nu+2\lambda}\Gamma(\eta)$$

$$\times\,{}_2\psi_3\left[\begin{array}{c}(\theta+\nu+2\lambda,2),(1,1);\\ (\lambda+1,1),(\eta+\theta+\nu+2\lambda,2),(\nu+\lambda+1,\mu);\end{array}\ -\frac{y^2}{4}\right], \quad (27)$$

where $\Re(\eta) > 0$ *and* $\Re(\theta) > 0$.

Corollary 14. *Further if we take $m = 1$ in (13) and then by using (8), we obtain:*

$$\int_0^1 t^{\eta-1}(1-t)^{2\theta-1}\left(1-\frac{t}{3}\right)^{2\eta-1}\left(1-\frac{t}{4}\right)^{\theta-1}\mathbb{J}^{\mu}_{\nu,\lambda}ty(1-t/3)^2dt$$
$$=\left(\frac{2}{3}\right)^{2(\eta+\theta+2\lambda)}(\frac{y}{2})^{\nu+2\lambda}\Gamma(\theta)_2\psi_3\left[\begin{array}{c}(\eta+\theta+2\lambda,2),(1,1);\\(\lambda+1,1),(\eta+\theta+\nu+2\lambda,2),(1+\nu+\lambda,\mu);\end{array}-\frac{4y^2}{81}\right], \tag{28}$$

where $\Re(\eta) > 0$ *and* $\Re(\theta) > 0$.

Corollary 15. *If we take $m = 1$ in (17) and then by using (8), we obtain:*

$$\int_0^1 t^{\eta-1}(1-t)^{2\theta-1}\left(1-\frac{t}{3}\right)^{2\eta-1}\left(1-\frac{t}{4}\right)^{\theta-1}\mathbb{J}^{\mu}_{\nu,\lambda}y(1-t/4)(1-t)^2dt$$
$$=\left(\frac{2}{3}\right)^{2\eta}\sqrt{\pi}\,(\frac{y}{2})^{\nu+2\lambda}\Gamma(\eta)_2\psi_4\left[\begin{array}{c}(1,2),(\theta+\nu+2\lambda,4);\\(\lambda+1,2),(1/2,1),(\eta+\theta+\nu+2\lambda,4),(\nu+\lambda+1,2\mu);\end{array}\frac{y^4}{64}\right]$$
$$+\left(\frac{2}{3}\right)^{2\eta}\sqrt{\frac{\pi}{2}}(\frac{y}{2})^{\nu+2\lambda+2}\Gamma(\eta)$$
$$\times\,{}_2\psi_4\left[\begin{array}{c}(2,2),(\theta+\nu+2\lambda+2,4);\\(\lambda+2,2),(\eta+\theta+\nu+2\lambda+2,4),(3/2,1),(\mu+\theta+\lambda+1,2\mu);\end{array}\frac{y^4}{64}\right], \tag{29}$$

where $\Re(\eta) > 0$ *and* $\Re(\theta) > 0$.

Corollary 16. *Further if we take $m = 1$, in (18) and then by using (8), we obtain:*

$$\int_0^1 t^{\eta-1}(1-t)^{2\theta-1}\left(1-\frac{t}{3}\right)^{2\eta-1}\left(1-\frac{t}{4}\right)^{\theta-1}\mathbb{J}^{\mu}_{\nu,\lambda}ty(1-t/3)^2dt$$
$$=\left(\frac{2}{3}\right)^{2(\eta+\theta+2\lambda)}\sqrt{\pi}\,(\frac{y}{2})^{\nu+2\lambda}\Gamma(\theta)_2\psi_4\left[\begin{array}{c}(1,2),(\eta+\theta+2\lambda,4);\\(\lambda+1,2),(\frac{1}{2},1),(\eta+\theta+\nu+2\lambda,4),(\nu+\lambda+1,2\mu);\end{array}\frac{y^4}{6561}\right]$$
$$+\left(\frac{2}{3}\right)^{2(\eta+\theta+2\lambda)}\sqrt{\frac{\pi}{2}}\left(-\frac{16y^2}{81}\right)(\frac{y}{2})^{\theta+2\lambda}\Gamma(\theta)$$
$$\times\,{}_2\psi_4\left[\begin{array}{c}(2,2),(\eta+\theta+2\lambda+2,4);\\(\lambda+2,2),(\frac{3}{2},1),(\eta+\theta+\nu+2\lambda+2,4),(\nu+\lambda+\mu+1,2\mu);\end{array}\frac{y^4}{6561}\right], \tag{30}$$

where $\Re(\eta) > 0$ *and* $\Re(\theta) > 0$.

Corollary 17. *If we take $m = 1, \mu = 1$ and $\lambda = 1/2$ in (11) and then by using (9),* *we obtain:*

$$\int_0^1 t^{\eta-1}(1-t)^{2\theta-1}\left(1-\frac{t}{3}\right)^{2\eta-1}\left(1-\frac{t}{4}\right)^{\theta-1}\mathbb{H}_\nu(y(1-t/4)(1-t)^2)dt$$
$$=\left(\frac{2}{3}\right)^{2\eta}(\frac{y}{2})^{\nu+1}\Gamma(\eta)_2\psi_3\left[\begin{array}{c}(\theta+\nu+1,2),(1,1);\\ (\eta+\theta+\nu+1,2),(\nu+3/2,1),(3/2,1);\end{array}-\frac{y^2}{4}\right], \quad (31)$$

where $\Re(\eta)>0$ *and* $\Re(\theta)>0$.

Corollary 18. *Further if we take* $m=1,\mu=1$ *and* $\lambda=1/2$ *in* (13) *and then by using* (9)*, we obtain:*

$$\int_0^1 t^{\eta-1}(1-t)^{2\theta-1}\left(1-\frac{t}{3}\right)^{2\eta-1}\left(1-\frac{t}{4}\right)^{\theta-1}\mathbb{H}_\nu ty(1-t/3)^2dt=\left(\frac{2}{3}\right)^{2(\eta+\theta+1)}(\frac{y}{2})^{\nu+1}\Gamma(\theta)$$
$$\times\,{}_2\psi_3\left[\begin{array}{c}(\eta+\theta+1,2),(1,1);\\ (\eta+\theta+\nu+1,2),(\nu+3/2,1),(3/2,1);\end{array}-\frac{4y^2}{81}\right], \quad (32)$$

where $\Re(\eta)>0$ *and* $\Re(\theta)>0$.

Corollary 19. *If we take* $m=1,\mu=1$ *and* $\lambda=1/2$ *in* (17) *and then by using* (9)*, we obtain:*

$$\int_0^1 t^{\eta-1}(1-t)^{2\theta-1}\left(1-\frac{t}{3}\right)^{2\eta-1}\left(1-\frac{t}{4}\right)^{\theta-1}\mathbb{H}_\nu y(1-t/4)(1-t)^2dt$$
$$=\left(\frac{2}{3}\right)^{2\eta}\sqrt{\pi}\,(\frac{y}{2})^{\nu+1}\Gamma(\eta)_2\psi_4\left[\begin{array}{c}(1,2),(\theta+\nu+1,4);\\ (\eta+\theta+\nu+1,4),(\nu+3/2,2),(3/2,2),(1/2,1);\end{array}\frac{y^4}{64}\right]$$
$$+\left(\frac{2}{3}\right)^{2\eta}\sqrt{\frac{\pi}{2}}(\frac{y}{2})^{\nu+3}\Gamma(\eta)_2\psi_4\left[\begin{array}{c}(2,2),(\theta+\nu+3,4);\\ (\eta+\theta+\nu+3,4),(\nu+5/2,2),(5/2,2),(3/2,1);\end{array}\frac{y^4}{64}\right], \quad (33)$$

where $\Re(\eta)>0$ *and* $\Re(\theta)>0$.

Corollary 20. *Further if we take* $m=1,\mu=1$ *and* $\lambda=1/2$ *in* (18) *and then by* *using* (9), *we obtain:*

$$\int_0^1 t^{\eta-1}(1-t)^{2\theta-1}\left(1-\frac{t}{3}\right)^{2\eta-1}\left(1-\frac{t}{4}\right)^{\theta-1}\mathbb{H}_\nu ty(1-t/3)^2dt$$

$$= \left(\frac{2}{3}\right)^{2(\eta+\theta+1)}\sqrt{\pi}\ (\frac{y}{2})^{\nu+1}\Gamma(\theta)_2\psi_4\left[\begin{array}{c} (1,2),(\eta+\theta+1,4); \\ (\eta+\theta+\nu+1,4),(\nu+3/2,2),(3/2,2),(1/2,1); \end{array} \frac{y^4}{6561}\right]$$

$$+\left(\frac{2}{3}\right)^{2(\eta+\theta+1)}\sqrt{\frac{\pi}{2}}\left(-\frac{16y^2}{81}\right)(\frac{y}{2})^{\theta+1}\Gamma(\theta)$$

$$\times\ {}_2\psi_4\left[\begin{array}{c} (2,2),(\eta+\theta+3,4); \\ (\eta+\theta+\nu+3,4),(\nu+5/2,2),(5/2,2),(3/2,1); \end{array} \frac{y^4}{6561}\right], \quad (34)$$

where $\Re(\eta) > 0$ *and* $\Re(\theta) > 0$.

4. Reducibility of the Wright Hypergeometic Function

Now we state two reduction formulas for the Wright hypergeometric function as follows:

$${}_2\psi_{m+2}\left[\begin{array}{c} (\theta+\nu+2\lambda,2)(1,1); \\ (\lambda+1,1),...(\lambda+1,1)(\eta+\theta+\nu+2\lambda,2)(\nu+\lambda+1,\mu); \end{array} -\frac{y^2}{4}\right]$$

$$=\sqrt{\pi}{}_2\psi_{m+3}\left[\begin{array}{c} (\theta+\nu+2\lambda,4)(1,2); \\ (\lambda+1,2),...(\lambda+1,2)(1/2,1)(\eta+\theta+\nu+2\lambda,4),(\nu+\lambda+1,2\mu); \end{array} \frac{y^4}{64}\right]+\sqrt{\frac{\pi}{8}}y^2$$

$${}_2\psi_{m+3}\left[\begin{array}{c} (\theta+\nu+2\lambda+2,4)(2,2); \\ (\lambda+2,2),...(\lambda+2,2)(\eta+\theta+\nu+2\lambda+2,4)(\frac{3}{2},1)(\mu+\nu+\lambda+1,2\mu); \end{array} \frac{y^4}{64}\right] \quad (35)$$

$${}_2\psi_{m+2}\left[\begin{array}{c} (\eta+\theta+2\lambda,2)(1,1); \\ (\lambda+1,1),...(\lambda+1,1),(\eta+\theta+\nu+2\lambda,2),(1+\nu+\lambda,\mu); \end{array} \frac{-4y^2}{81}\right]$$

$$=\sqrt{\pi}\ {}_2\psi_{m+3}\left[\begin{array}{c} (\eta+\theta+2\lambda,4)(1,2); \\ (\lambda+1,2),...(\lambda+1,2)(1/2,1),(\eta+\theta+\nu+2\lambda,4)(\nu+\lambda+1,2\mu); \end{array} \frac{y^4}{6561}\right]$$

$$+\sqrt{\pi}\left(-\frac{4y^2}{81}\right)$$

$$\times\ {}_2\psi_{m+3}\left[\begin{array}{c} (\eta+\theta+2\lambda+2,4)(2,2); \\ (\lambda+2,2),...(\lambda+2,2)(3/2,1)(\eta+\theta+\nu+2\lambda+2,4)(\nu+\lambda+\mu+1,2\mu); \end{array} \frac{y^4}{6561}\right] \quad (36)$$

By comparing (11) and (17),results (35) can be established and by comparing (14) and (18),result (36) can be established.

Declaration

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