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SOME ALGEBRAIC RELATIONS ON AN INTEGER SEQUENCE WITH FIXED PARAMETER

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Abstract. Let $a \geq 2$ be an integer. In this work we set an integer sequence $U_n^a = U_n(a+1, a)$ and deduced some algebraic relations on it.

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1. Preliminaries

Let p and q be two non–zero integers and let $d = p^2 - 4q \neq 0$ (to exclude a degenerate case). We set the integer sequences U_n and V_n to be

$$\begin{aligned} U_n &= U_n(p,q) = pU_{n-1} - qU_{n-2} \\ V_n &= V_n(p,q) = pV_{n-1} - qV_{n-2} \end{aligned} \tag{1}$$

for $n \geq 2$ with $U_0 = 0, U_1 = 1, V_0 = 2$ and $V_1 = p$. The characteristic equation of (1) is

$$x^2 - px + q = 0$$

and hence the roots are

$$\alpha = \frac{p+\sqrt{d}}{2} \quad \text{and} \quad \beta = \frac{p-\sqrt{d}}{2}.$$

So their Binet formulas are

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad V_n = \alpha^n + \beta^n$$

for $n \geq 1$. For the companion matrix $M = \begin{bmatrix} p & -q \\ 1 & 0 \end{bmatrix}$, we have

$$\begin{bmatrix} U_n \\ U_{n-1} \end{bmatrix} = M^{n-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} V_n \\ V_{n-1} \end{bmatrix} = M^{n-1} \begin{bmatrix} p \\ 0 \end{bmatrix}.$$

Furthermore for U_n and V_n, we have the following formal power series

$$\sum_{n=0}^{\infty} U_n x^n = \frac{x}{1-px+qx^2} \quad \text{and} \quad \sum_{n=0}^{\infty} V_n x^n = \frac{2-px}{1-px+qx^2}.$$

It is easily seen from (1) that

$$\begin{aligned}
U_n(1,-1) &= F_n \text{ Fibonacci numbers (A000045 in OEIS)},\\
V_n(1,-1) &= L_n \text{ Lucas numbers (A000032 in OEIS)},\\
U_n(2,-1) &= P_n \text{ Pell numbers (A000129 in OEIS)},\\
V_n(2,-1) &= Q_n \text{ Pell–Lucas numbers (A002203 in OEIS)}.
\end{aligned}$$

2. Main Results.

Integer sequences (such as Fibonacci numbers, Lucas numbers, Pell numbers, Pell–Lucas numbers and balancing numbers) with two or more than two parameters and their generalizations have been investigated by several authors ([1, 2, 5, 6, 7, 8, 9, 10, 11, 13]).

In [4], we derived some new results on balancing numbers and in [15], we obtained some new results on oblong and balancing numbers. Later in [14], we defined an integer sequence with four parameters and derived some algebraic relations on it.

In [12], Ribenboim set an integer sequence for $P = a+1$ and $Q = a$ for an integer $a \geq 2$, namely, $U_n(a+1, a)$. We rewrite

$$U_n^a = U_n(a+1, a). \tag{2}$$

Then from (1), we get $U_0^a = 0, U_1^a = 1$ and

$$U_n^a = (a+1)U_{n-1}^a - aU_{n-2}^a \tag{3}$$

for $n \geq 2$. The characteristic equation of (3) is

$$x^2 - (a+1)x + a = 0$$

and hence the roots of it are $\alpha = a$ and $\beta = 1$. So its Binet formula is

$$U_n^a = \frac{a^n - 1}{a - 1} \tag{4}$$

for $n \geq 1$.

For the integer sequence defined in (3), we can give the following theorems.

Theorem 1. *For the integer sequence U_n^a, we have*

1. The sum of first $n-$terms is

$$\sum_{i=1}^{n} U_i^a = \frac{U_{n+1}^a - n - 1}{a - 1}$$

for $n \geq 1$.

2. Also

(i) $U_{n+2}^a - a^2 U_n^a = a + 1$ for $n \geq 0$.

(ii) $U_{2n}^a = (a^2+1)U_{2n-2}^a - a^2 U_{2n-4}^a$ and $U_{2n+1}^a = (a^2+1)U_{2n-1}^a - a^2 U_{2n-3}^a$ for $n \geq 2$.

(iii) $\alpha^n + \beta^n = U_{n+1}^a - aU_{n-1}^a$ for $n \geq 1$ or $\alpha^n + \beta^n = U_{n+1}^a - U_n^a + 1$ for $n \geq 0$.

(iv) $U_{2n+1}^a = a^2 U_{2n-1}^a + a + 1$ and $2U_{n+1}^a - (a+1)U_n^a = a^n + 1$ for $n \geq 1$.

(v) $\frac{U_{2n-1}^a - 1}{a+1}$ and $\frac{U_{2n-2}^a}{a+1}$ are integers, in fact,

$$\frac{U_{2n-1}^a - 1}{a+1} = \sum_{i=1}^{n-1} a^{2i-1} \quad \text{and} \quad \frac{U_{2n-2}^a}{a+1} = \sum_{i=1}^{n-1} a^{2i-2}$$

for $n \geq 1$.

(vi) $U_{n+1}^a + U_{n-1}^a = (\frac{a^2+1}{a^2-a})\alpha^n - (\frac{2}{a-1})\beta^n$ and $U_n^a - U_{n-1}^a = \alpha^{n-1}$ for $n \geq 1$.

Proof. **(1)** Note that $U_n^a = (a+1)U_{n-1}^a - aU_{n-2}^a$. So $U_{n+2}^a = (a+1)U_{n+1}^a - aU_n^a$. Since $U_{n+2}^a - a^2 U_n^a = a + 1$, we get $U_{n+2}^a = a^2 U_n^a + a + 1$ and hence

$$a^2 U_n^a + a + 1 = (a+1)U_{n+1}^a - aU_n^a \Leftrightarrow (a+1)U_{n+1}^a - (a+a^2)U_n^a = a + 1. \quad (5)$$

Since $a + 1 \neq 0$, we can divide both side of (5) with $a + 1$. So

$$U_{n+1}^a - aU_n^a = 1. \quad (6)$$

Applying (6), we deduce that

$$\begin{aligned} &U_1^a - aU_0^a = 1 \\ &U_2^a - aU_1^a = 1 \\ &\cdots \\ &U_n^a - aU_{n-1}^a = 1 \\ &U_{n+1}^a - aU_n^a = 1. \end{aligned} \quad (7)$$

If we sum of both sides of (7), then we obtain $(U_1^a + U_2^a + \cdots + U_{n+1}^a) - a(U_0^a + U_1^a + \cdots + U_n^a) = n + 1$ and hence

$$(1-a)(U_1^a + U_2^a + \cdots + U_n^a) = -U_{n+1}^a + n + 1$$

since $U_0^a = 0$. Thus

$$U_1^a + U_2^a + \cdots + U_n^a = \frac{U_{n+1}^a - n - 1}{a-1}$$

as we wanted.

(2-i) Recall that $U_n^a = \frac{a^n-1}{a-1} = a^{n-1} + a^{n-2} + \cdots + a + 1$ and hence

$$a^2U_n^a = a^2(a^{n-1} + a^{n-2} + \cdots + a + 1) = a^{n+1} + a^n + \cdots + a^2. \qquad (8)$$

Adding $a+1$ both sides of (8), we get

$$a^2U_n^a + a + 1 = a^{n+1} + a^n + \cdots + a^2 + a + 1 = U_{n+2}^a$$

and hence

$$U_{n+2}^a - a^2U_n^a = a + 1.$$

The other cases can be proved similarly. ■

Applying Theorem 1, we can give the following result without giving its proof since it can be proved by induction on n.

Theorem 2. *If a is odd, then U_n^a is even if and only if n is even and U_n^a is odd if and only if n is odd. If a is even, then U_n^a is always odd.*

Now we can give the special case, namely, $a = 10$. Then we have

$$U_n^{10} = 11U_{n-1}^{10} - 10U_{n-2}^{10}.$$

Thus we see that $U_0^{10} = 0$ and

$$U_n^{10} : 1, 11, 111, 1111, 11111, 111111, \cdots$$

for $n \geq 1$. Similarly we get

$U_n^{10^2} : 1, 101, 10101, 1010101, 101010101, \cdots$
$U_n^{10^3} : 1, 1001, 1001001, 1001001001, 1001001001001, \cdots$
$U_n^{10^4} : 1, 10001, 100010001, 1000100010001, 10001000100010001, \cdots$
$U_n^{10^5} : 1, 100001, 10000100001, 1000010000100001, 100001000010000100001, \cdots$
$\cdots$

for $n \geq 1$. Thus we can give the following theorem.

Theorem 3. *If $a = 10^k$ for some integer $k \geq 1$, then the terms of $U_n^{10^k}$ are*

$$U_n^{10^k} = 1, \underbrace{(10)^{k-1}}_{n-1 \text{ times}} 1$$

for $n \geq 1$ with $U_0^{10^k} = 0$.

Proof. It can be proved by induction on n. ■

Notice that the rank of an integer N is defined to be

$$\rho(N) = \begin{cases} p & \text{if } p \text{ is the smallest prime with } p|N \\ \infty & \text{if } N \text{ is prime.} \end{cases}$$

For the rank of U_n^a, we can give the following theorem.

Theorem 4. *If a is odd, then $\rho(U_{2n}^a) = 2$. Also let $a = 10^k$ for an integer $k \geq 1$. If $k = 1$, then $\rho(U_2^{10}) = \rho(U_{19}^{10}) = \rho(U_{23}^{10}) = \infty$ and $\rho(U_{3n}^{10}) = 3$, if $k = 1$ and $3 \nmid 2n$, then $\rho(U_{2n}^{10}) = 11$, and $\rho(U_{3n}^a) = 3$ for every k and n.*

Proof. We see in Theorem 2 that if a is odd, then U_n^a is even if and only if n is even. So U_{2n}^a is always even and therefore $\rho(U_{2n}^a) = 2$.

Let $k = 1$. Then $U_2^{10} = 11, U_{19}^{10} = 1111111111111111111$ and $U_{23}^{10} = 111111111111$ 11111111111 are primes. So $\rho(U_2^a) = \rho(U_{19}^a) = \rho(U_{23}^a) = \infty$. Similarly since

$$U_{3n}^{10} : \underbrace{11 \cdots 1}_{3n \text{ times}},$$

which is divisible by 3 and so $\rho(U_{3n}^{10}) = 3$. Notice that

$$U_{2n}^{10} = \underbrace{1111 \cdots 1}_{2n \text{ times}}.$$

Hence clearly U_{2n}^{10} is divisible by 11 and therefore $\rho(U_{2n}^{10}) = 11$ (when $3|2n$, we see as below that $\rho(U_{3n}^{10}) = 3$).

Finally let $k \geq 1$ be any integer. Then

$$U_{3n}^a = \underbrace{(10)^{k-1}}_{3n-1 \text{ times}} 1$$

is divisible by 3 and hence $\rho(U_{3n}^a) = 3$. ■

Theorem 5. *Let $M = \begin{bmatrix} U_2^a & 1 - U_2^a \\ U_1^a & U_0^a \end{bmatrix}, W = \begin{bmatrix} U_2^a & U_1^a \\ U_1^a & U_0^a \end{bmatrix}$ and $A = \begin{bmatrix} U_1^a & U_0^a \end{bmatrix}$. Then*

1. $M^n = \begin{bmatrix} U^a_{n+1} & 1-U^a_{n+1} \\ \\ U^a_n & 1-U^a_n \end{bmatrix}$ *for* $n \geq 1$.

2. $U^a_n = AM^{n-1}A^t$ *for* $n \geq 1$ *and* $U^a_n = AM^{n-2}WA^t$ *for* $n \geq 2$.

3. If $n \geq 3$ *is odd, then*

$$W^n = \begin{bmatrix} \sum\limits_{i=0}^{\frac{n-1}{2}} \binom{n-i}{i}(a+1)^{n-2i} & \sum\limits_{i=0}^{\frac{n-1}{2}} \binom{n-1-i}{i}(a+1)^{n-1-2i} \\ \\ \sum\limits_{i=0}^{\frac{n-1}{2}} \binom{n-1-i}{i}(a+1)^{n-1-2i} & \sum\limits_{i=0}^{\frac{n-3}{2}} \binom{n-2-i}{i}(a+1)^{n-2-2i} \end{bmatrix}$$

and if $n \geq 2$ *even, then*

$$W^n = \begin{bmatrix} \sum\limits_{i=0}^{\frac{n}{2}} \binom{n-i}{i}(a+1)^{n-2i} & \sum\limits_{i=0}^{\frac{n-2}{2}} \binom{n-1-i}{i}(a+1)^{n-1-2i} \\ \\ \sum\limits_{i=0}^{\frac{n-2}{2}} \binom{n-1-i}{i}(a+1)^{n-1-2i} & \sum\limits_{i=0}^{\frac{n-2}{2}} \binom{n-2-i}{i}(a+1)^{n-2-2i} \end{bmatrix}.$$

Proof. **(1)** We prove it by induction on n. Let $n = 1$. Then

$$M = \begin{bmatrix} \sum\limits_{i=0}^{1} a^i & 1-\sum\limits_{i=0}^{1} a^i \\ \\ \sum\limits_{i=0}^{0} a^i & 1-\sum\limits_{i=0}^{0} a^i \end{bmatrix} = \begin{bmatrix} a+1 & -a \\ \\ 1 & 0 \end{bmatrix}.$$

So it is true for $n = 1$. Let us assume that this relation is satisfied for $n-1$. Then since $M^n = M^{n-1} \cdot M$, we get

$$M^n = \begin{bmatrix} (a+1)\sum\limits_{i=0}^{n-1} a^i + (1-\sum\limits_{i=0}^{n-1} a^i) & -a\sum\limits_{i=0}^{n-1} a^i \\ \\ (a+1)\sum\limits_{i=0}^{n-2} a^i + (1-\sum\limits_{i=0}^{n-2} a^i) & -a\sum\limits_{i=0}^{n-2} a^i \end{bmatrix}. \quad (9)$$

In (9), we notice that

$$(a+1)\sum_{i=0}^{n-1} a^i + (1-\sum_{i=0}^{n-1} a^i) = (a+1)(1+a+a^2+\cdots+a^{n-1})$$
$$+1-(1+a+\cdots+a^{n-1})$$

$$
\begin{aligned}
&= 1 + a + a^2 + \cdots + a^n \\
&= \sum_{i=0}^{n} a^i, \\
-a\sum_{i=0}^{n-1} a^i &= -a - a^2 - \cdots - a^n \\
&= 1 - (1 + a + a^2 + \cdots + a^n) \\
&= 1 - \sum_{i=0}^{n} a^i, \\
(a+1)\sum_{i=0}^{n-2} a^i + (1 - \sum_{i=0}^{n-2} a^i) &= (a+1)(1 + a + \cdots + a^{n-2}) \\
&\quad + 1 - (1 + a + a^2 + \cdots + a^{n-2}) \\
&= 1 + a + a^2 + \cdots + a^{n-1} \\
&= \sum_{i=0}^{n-1} a^i
\end{aligned}
$$

and

$$
\begin{aligned}
-a\sum_{i=0}^{n-2} a^i &= -a - a^2 - \cdots - a^{n-1} \\
&= 1 - (1 + a + a^2 + \cdots + a^{n-1}) \\
&= 1 - \sum_{i=0}^{n-1} a^i.
\end{aligned}
$$

So (9) becomes

$$
M^n = \begin{bmatrix} \sum\limits_{i=0}^{n} a^i & 1 - \sum\limits_{i=0}^{n} a^i \\ \\ \sum\limits_{i=0}^{n-1} a^i & 1 - \sum\limits_{i=0}^{n-1} a^i \end{bmatrix}.
$$

Since $U_n^a = a^{n-1} + a^{n-2} + \cdots + a + 1$, we conclude that

$$
M^n = \begin{bmatrix} U_{n+1}^a & 1 - U_{n+1}^a \\ \\ U_n^a & 1 - U_n^a \end{bmatrix}.
$$

(2) We easily get

$$\begin{aligned} AM^{n-1}A^t &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} U_n^a & 1-U_n^a \\ \\ U_{n-1}^a & 1-U_{n-1}^a \end{bmatrix} \begin{bmatrix} 1 \\ \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} U_n^a \\ \\ U_{n-1}^a \end{bmatrix} \\ &= U_n^a \end{aligned}$$

and since $aU_{n-1}^a + 1 = U_n^a$, we observe that

$$\begin{aligned} AM^{n-2}WA^t &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} U_{n-1}^a & 1-U_{n-1}^a \\ \\ U_{n-2}^a & 1-U_{n-2}^a \end{bmatrix} \begin{bmatrix} a+1 & 1 \\ \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} U_{n-1}^a & 1-U_{n-1}^a \\ \\ U_{n-2}^a & 1-U_{n-2}^a \end{bmatrix} \begin{bmatrix} a+1 \\ \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} (a+1)U_{n-1}^a + 1 - U_{n-1}^a \\ \\ (a+1)U_{n-2}^a + 1 - U_{n-2}^a \end{bmatrix} \\ &= aU_{n-1}^a + 1 \\ &= U_n^a. \end{aligned}$$

(3) It can be proved similarly. ■

For the simple continued fraction expansion, we can give the following result.

Theorem 6. *The simple continued fraction expansion of* $\frac{U_{n+1}^a}{U_n^a}$ *is*

$$\frac{U_{n+1}^a}{U_n^a} = \left[a; \sum_{i=0}^{n-1} a^{n-1-i}\right]$$

for $n \geq 2$. *Also*

$$\frac{U_{2n+1}^a}{U_{2n-1}^a} = \left[a^2; \sum_{i=0}^{n-2} a^{2n-3-2i}, a+1\right]$$

for $n \geq 2$ *and*

$$\frac{U^a_{2n}}{U^a_{2n-2}} = \left[a^2; \sum_{i=0}^{n-2} a^{2n-4-2i}\right]$$

for $n \geq 3$.

Proof. Recall that $U^a_n = a^{n-1} + a^{n-2} + \cdots + a + 1$. So we get

$$\frac{U^a_{n+1}}{U^a_n} = \frac{a^n + a^{n-1} + \cdots + a + 1}{a^{n-1} + a^{n-2} + \cdots + a + 1} = a + \frac{1}{a^{n-1} + a^{n-2} + \cdots + a + 1}.$$

Since $a^{n-1} + a^{n-2} + \cdots + a + 1 = \sum_{i=0}^{n-1} a^{n-1-i}$, we conclude that

$$\frac{U^a_{n+1}}{U^a_n} = \left[a; \sum_{i=0}^{n-1} a^{n-1-i}\right].$$

Similarly we obtain

$$\begin{aligned}
\frac{U^a_{2n+1}}{U^a_{2n-1}} &= \frac{a^{2n} + a^{2n-1} + \cdots + a + 1}{a^{2n-2} + a^{2n-3} + \cdots + a + 1} \\
&= a^2 + \frac{a+1}{a^{2n-2} + a^{2n-3} + \cdots + a + 1} \\
&= a^2 + \frac{1}{\frac{a^{2n-2} + a^{2n-3} + \cdots + a + 1}{a+1}} \\
&= a^2 + \frac{1}{a^{2n-3} + a^{2n-5} + \cdots + a + \frac{1}{a+1}} \\
&= a^2 + \frac{1}{\sum_{i=0}^{n-2} a^{2n-3-2i} + \frac{1}{a+1}}.
\end{aligned}$$

Thus

$$\frac{U^a_{2n+1}}{U^a_{2n-1}} = \left[a^2; \sum_{i=0}^{n-2} a^{2n-3-2i}, a+1\right].$$

The last assertion can be proved similarly. ■

For the cross–ratio of four consecutive $U^a_n, U^a_{n+1}, U^a_{n+2}$ and U^a_{n+3} numbers we can give the following result.

Theorem 7. *Let $U_n^a, U_{n+1}^a, U_{n+2}^a$ and U_{n+3}^a be four consecutive U_n^a numbers. Then*

$$[U_n^a, U_{n+1}^a; U_{n+2}^a, U_{n+3}^a] = \frac{a^2+2a+1}{a^2+a+1}$$

for $n \geq 1$.

Proof. Notice that the cross–ratio of a quadruple of distinct points on the real line with coordinates z_1, z_2, z_3, z_4 is given by

$$[z_1, z_2; z_3, z_4] = \frac{(z_3-z_1)(z_4-z_2)}{(z_3-z_2)(z_4-z_1)}. \tag{10}$$

Let $U_n^a, U_{n+1}^a, U_{n+2}^a$ and U_{n+3}^a be four consecutive U_n^a numbers. Then by (10), we get

$$[U_n^a, U_{n+1}^a; U_{n+2}^a, U_{n+3}^a] = \frac{(U_{n+2}^a - U_n^a)(U_{n+3}^a - U_{n+1}^a)}{(U_{n+2}^a - U_{n+1}^a)(U_{n+3}^a - U_n^a)}. \tag{11}$$

In (11), we notice that

$$\begin{aligned} &U_{n+2}^a - U_n^a = a^{n+1} + a^n, \ U_{n+3}^a - U_{n+1}^a = a^{n+2} + a^{n+1}, \\ &U_{n+2}^a - U_{n+1}^a = a^{n+1} \ \text{ and } \ U_{n+3}^a - U_n^a = a^{n+2} + a^{n+1} + a^n. \end{aligned}$$

So (11) becomes

$$\begin{aligned} [U_n^a, U_{n+1}^a; U_{n+2}^a, U_{n+3}^a] &= \frac{(U_{n+2}^a - U_n^a)(U_{n+3}^a - U_{n+1}^a)}{(U_{n+2}^a - U_{n+1}^a)(U_{n+3}^a - U_n^a)} \\ &= \frac{(a^{n+1}+a^n)(a^{n+2}+a^{n+1})}{(a^{n+1})(a^{n+2}+a^{n+1}+a^n)} \\ &= \frac{a^{2n+1}(a^2+2a+1)}{a^{2n+1}(a^2+a+1)} \\ &= \frac{a^2+2a+1}{a^2+a+1} \end{aligned}$$

as we claimed. ■

A circulant matrix (see [3]) is a matrix M defined as

$$M = \begin{bmatrix} m_1 & m_2 & m_3 & \cdots & m_{n-1} & m_n \\ m_n & m_1 & m_2 & \cdots & m_{n-2} & m_{n-1} \\ m_{n-1} & m_n & m_1 & \cdots & m_{n-3} & m_{n-2} \\ . & . & . & \cdots & . & . \\ . & . & . & \cdots & . & . \\ . & . & . & \cdots & . & . \\ m_3 & m_4 & m_5 & \cdots & m_1 & m_2 \\ m_2 & m_3 & m_4 & \cdots & m_n & m_1 \end{bmatrix},$$

where m_i are constant. The eigenvalues of M are

$$\lambda_j(M) = \sum_{k=0}^{n-1} m_k w^{-jk}, \tag{12}$$

where $w = e^{\frac{2\pi i}{n}}$, $i = \sqrt{-1}$ and $j = 0, 1, \cdots, n-1$. The spectral norm for a matrix $M = [m_{ij}]_{n\times n}$ is defined to be

$$||M||_{spec} = \max\{\sqrt{\lambda_i} : \lambda_i \text{ are the eigenvalues of } M^H M \text{ for } 0 \leq j \leq n-1\},$$

where M^H denotes the conjugate transpose of M. Then we can give the following theorem.

Theorem 8. *Let U_n^a denote the circulant matrix for U_n^a numbers. Then the eigenvalues of U_n^a are*

$$\lambda_j(U_n^a) = \frac{w^{-j}(aU_{n-1}^a + 1) - U_n^a}{aw^{-2j} - (a+1)w^{-j} + 1}$$

for $j = 0, 1, 2, \cdots, n-1$ and the spectral norm is

$$||U_n^a||_{spec} = \sum_{k=1}^{n} k a^{n-k}.$$

Proof. Applying (12), we get

$$\begin{aligned}
\lambda_j(U_n^a) &= \sum_{k=0}^{n-1} U_k^a w^{-jk} \\
&= \frac{1}{\alpha-\beta}\left[\sum_{k=0}^{n-1}(\alpha w^{-j})^k - \sum_{k=0}^{n-1}(\beta w^{-j})^k\right] \\
&= \frac{1}{\alpha-\beta}\left[\frac{\alpha^n - 1}{\alpha w^{-j} - 1} - \frac{\beta^n - 1}{\beta w^{-j} - 1}\right] \\
&= \frac{1}{\alpha-\beta}\left[\frac{(\alpha^n - 1)(\beta w^{-j} - 1) - (\beta^n - 1)(\alpha w^{-j} - 1)}{(\alpha w^{-j} - 1)(\beta w^{-j} - 1)}\right] \\
&= \frac{1}{\alpha-\beta}\left[\frac{w^{-j}(\alpha^n\beta - \alpha\beta^n - \beta + \alpha) - \alpha^n + \beta^n}{\alpha\beta w^{-2j} - w^{-j}(\alpha+\beta) + 1}\right] \\
&= \frac{1}{\alpha-\beta}\left[\frac{w^{-j}[a(\alpha^{n-1} - \beta^{n-1}) + \alpha - \beta) - (\alpha^n - \beta^n)}{\alpha\beta w^{-2j} - w^{-j}(\alpha+\beta) + 1}\right] \\
&= \frac{w^{-j}(aU_{n-1}^a + 1) - U_n^a}{aw^{-2j} - (a+1)w^{-j} + 1}.
\end{aligned}$$

Similarly we find that

$$||U_n^a||_{spec} = \sum_{k=1}^{n} k a^{n-k}$$

as we wanted. ■

From above theorem, we can give the following result.

Corollary 9. *If n is odd then λ_0 is a square and*

$$\lambda_0 = \left(\sum_{k=1}^{n} k a^{n-k} \right)^2 .$$

If n is even, then λ_0 and λ_1 are squares and

$$\lambda_0 = \left(\sum_{k=0}^{\frac{n-2}{2}} a^{n-1-2k} \right)^2 \quad and \quad \lambda_1 = \left(\sum_{k=1}^{n} k a^{n-k} \right)^2 .$$

Example 1. **i)** *Let $n = 5$. Then the eigenvalues of $(U_5^a)^H U_5^a$ are*

$$\begin{aligned} \lambda_0 &= a^8 + 4a^7 + 10a^6 + 20a^5 + 35a^4 + 44a^3 + 46a^2 + 40a + 25 \\ \lambda_1 = \lambda_3 &= (a^2 + \frac{a}{2} + 1 + \frac{a\sqrt{5}}{2})(a^4 + a^3 + a^2 + a + 1)a^2 \\ \lambda_2 = \lambda_4 &= (a^2 + \frac{a}{2} + 1 - \frac{a\sqrt{5}}{2})(a^4 + a^3 + a^2 + a + 1)a^2. \end{aligned}$$

Here

$$\lambda_0 = (a^4 + 2a^3 + 3a^2 + 4a + 5)^2 = \left(\sum_{k=1}^{5} k a^{5-k} \right)^2$$

is maximum and so the spectral norm of U_5^a is

$$||U_5^a||_{spec} = \sqrt{\lambda_0} = a^4 + 2a^3 + 3a^2 + 4a + 5.$$

ii) *Let $n = 6$. Then the eigenvalues of $(U_6^a)^H U_6^a$ are*

$$\begin{aligned} \lambda_0 &= a^{10} + 2a^8 + 3a^6 + 2a^4 + a^2 \\ \lambda_1 &= a^{10} + 4a^9 + 10a^8 + 20a^7 + 35a^6 + 56a^5 + 70a^4 + 76a^3 + 73a^2 + 60a + 36 \\ \lambda_2 = \lambda_4 &= a^{10} + 3a^9 + 5a^8 + 6a^7 + 6a^6 + 6a^5 + 5a^4 + 3a^3 + a^2 \\ \lambda_3 = \lambda_5 &= a^{10} + a^9 + a^8 + 2a^7 + 2a^6 + 2a^5 + a^4 + a^3 + a^2. \end{aligned}$$

Here

$$\lambda_0 = \left(\sum_{k=0}^{2} a^{5-2k}\right)^2 \quad and \quad \lambda_1 = \left(\sum_{k=1}^{6} ka^{6-k}\right)^2.$$

Since λ_1 *is maximum, the spectral norm of* U_6^a *is*

$$||U_6^a||_{spec} = \sqrt{\lambda_1} = a^5 + 2a^4 + 3a^3 + 4a^2 + 5a + 6.$$

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