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COEFFICIENT BOUNDS AND FEKETE-SZEGŐ PROBLEM FOR NEW CLASSES OF ANALYTIC FUNCTIONS DEFINED BY SĂLĂGEAN INTEGRO-DIFFERENTIAL OPERATOR

Páll-Szabó Ágnes Orsolya

Abstract. In this paper we introduce new classes containing the linear operator $\mathscr{L}^n : \mathcal{A} \to \mathcal{A}$, $\mathscr{L}^n f(z) = (1-\lambda)\mathscr{D}^n f(z) + \lambda I^n f(z)$, where $\mathscr{D}^n$ is the Sălăgean differential operator and I^n is the generalized Alexander operator. We study the characteristics and other properties of these classes. We obtain Fekete-Szegő functional for these classes.

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1. Preliminaries

Let $\mathcal{A}$ denote the class of functions of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1}$$

which are analytic and univalent in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$.

Definition 1. *[9]*

For $f \in \mathcal{A}$, $n \in \mathbb{N} = \{0, 1, 2, \ldots\}$, the Sălăgean differential operator $\mathscr{D}^n$ is defined by
$\mathscr{D}^n : \mathcal{A} \to \mathcal{A}$,

$$\mathscr{D}^0 f(z) = f(z),$$
$$\mathscr{D}^{n+1} f(z) = z\left(\mathscr{D}^n f(z)\right)', z \in U$$

Remark 1. *If $f \in \mathcal{A}$ and $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$, then*

$$\mathscr{D}^n f(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k, z \in U.$$

Definition 2. *[9]*
For $f \in \mathcal{A}, n \in \mathbb{N} \setminus \{0\}$, the Sălăgean integral operator I^n is defined by

$$I^0 f(z) = f(z),$$
$$I^1 f(z) = I f(z) = \int_0^z f(t)\, t^{-1} dt, \ldots$$
$$I^n f(z) = I\left(I^{n-1} f(z)\right), z \in U$$

I^1 is the Alexander operator used for the first time in [1], and I^n is called the generalized Alexander operator.

Remark 2. *If $f \in \mathcal{A}$ and $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$, then*

$$I^n f(z) = z + \sum_{k=2}^{\infty} \frac{a_k}{k^n} z^k, \tag{2}$$

$z \in U$, $(n \in \mathbb{N} \setminus \{0\})$ and $z\left(I^n f(z)\right)' = I^{n-1} f(z)$.

Remark 3. *We have $\mathscr{D}^n I^n f(z) = I^n \mathscr{D}^n f(z) = f(z), \quad f \in \mathcal{A}, \ z \in U$.*

Definition 3. *Let $\lambda \geq 0, n \in \mathbb{N}$. Denote by $\mathscr{L}^n$ the operator given by $\mathscr{L}^n : \mathcal{A} \to \mathcal{A}$,*

$$\mathscr{L}^n f(z) = (1 - \lambda)\, \mathscr{D}^n f(z) + \lambda I^n f(z), z \in U.$$

Remark 4. *If $f \in \mathcal{A}$ and $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$, then*

$$\mathscr{L}^n f(z) = z + \sum_{k=2}^{\infty} \left[k^n (1 - \lambda) + \lambda \frac{1}{k^n}\right] a_k z^k, z \in U. \tag{3}$$

In the following definitions, new classes of analytic functions containing the new operator (3) are introduced:

Definition 4. *Let $f \in \mathcal{A}$. Then $f(z)$ is in the class $\mathcal{S}^n(\mu)$ if and only if*

$$\Re\left(\frac{z\left(\mathscr{L}^n f(z)\right)'}{\mathscr{L}^n f(z)}\right) > \mu, \quad 0 \leq \mu < 1, z \in U. \tag{4}$$

Definition 5. *Let $f \in \mathcal{A}$. Then $f(z)$ is in the class $\mathcal{C}^n(\mu)$ if and only if*

$$\Re\left(\frac{\left[z\left(\mathscr{L}^n f(z)\right)'\right]'}{\left(\mathscr{L}^n f(z)\right)'}\right) > \mu, \quad 0 \leq \mu < 1, z \in U. \tag{5}$$

Definition 6. *[3] Let $\varphi(z) = 1 + B_1 z + B_2 z^2 + \cdots$ be an univalent starlike function with respect to 1 which maps the unit disk U onto a region in the right half plane which is symmetric with respect to the real axis, $\varphi(0) = 1$ and $\varphi'(0) > 0$. The class $S^*(\varphi)$ consists of all functions $f \in \mathcal{A}$ satisfying the following subordination:*

$$\frac{z\left(\mathscr{L}^n f(z)\right)'}{\mathscr{L}^n f(z)} \prec \varphi(z), \tag{6}$$

and $C(\varphi)$ be the class of functions $f \in \mathcal{A}$ for which

$$\frac{\left[z\left(\mathscr{L}^n f(z)\right)'\right]'}{\left(\mathscr{L}^n f(z)\right)'} \prec \varphi(z). \tag{7}$$

Remark 5. *If $\varphi_\mu(z) = \dfrac{1 + (1 - 2\mu)z}{1 - z}$ then $\mathcal{S}^n(\mu) = S^*(\varphi_\mu)$ and $\mathcal{C}^n(\mu) = C(\varphi_\mu)$.*

2. Main results

2.1. Coefficient estimates

Theorem 1. *Let the function $f(z)$ defined by (1) be in $\mathcal{A}$. If*

$$\sum_{k=2}^{\infty} (k - \mu)\left[k^n (1 - \lambda) + \lambda \frac{1}{k^n}\right] |a_k| \le 1 - \mu, \tag{8}$$

then $f(z) \in \mathcal{S}^n(\mu)$. The result (8) is sharp.

Proof. Suppose that (8) holds true. Since

$$\begin{aligned} 1 - \mu &\ge \sum_{k=2}^{\infty} (k - \mu)\left[k^n (1 - \lambda) + \lambda \frac{1}{k^n}\right] |a_k| \\ &\ge \sum_{k=2}^{\infty} \left[k^{n+1} (1 - \lambda) + \lambda \frac{1}{k^{n-1}}\right] |a_k| - \mu \sum_{k=2}^{\infty} \left[k^n (1 - \lambda) + \lambda \frac{1}{k^n}\right] |a_k| \end{aligned}$$

then this implies that

$$\frac{1 - \sum_{k=2}^{\infty} \left[k^{n+1} (1 - \lambda) + \lambda \frac{1}{k^{n-1}}\right] |a_k|}{1 - \sum_{k=2}^{\infty} \left[k^n (1 - \lambda) + \lambda \frac{1}{k^n}\right] |a_k|} \ge \mu, \tag{9}$$

because the denominator is positive, we have from (8)

$$\sum_{k=2}^{\infty}\left[k^n(1-\lambda)+\lambda\frac{1}{k^n}\right]|a_k|\leq\frac{1-\mu}{2-\mu}<1.$$

On the other side,

$$\left|\frac{z\left(\mathscr{L}^n f(z)\right)'}{\mathscr{L}^n f(z)}-1\right|<1-\mu\ \Rightarrow\ \Re\left[\frac{z\left(\mathscr{L}^n f(z)\right)'}{\mathscr{L}^n f(z)}\right]>\mu.$$

But

$$\left|\frac{z\left(\mathscr{L}^n f(z)\right)'}{\mathscr{L}^n f(z)}-1\right|=\left|\frac{\sum_{k=2}^{\infty}\left[k^{n+1}(1-\lambda)-k^n(1-\lambda)+\lambda\frac{k-1}{k^n}\right]a_k z^{k-1}}{1+\sum_{k=2}^{\infty}\left[k^n(1-\lambda)+\lambda\frac{1}{k^n}\right]a_k z^{k-1}}\right|$$

$$<\frac{\sum_{k=2}^{\infty}\left[k^n(1-\lambda)(k-1)+\lambda\frac{k-1}{k^n}\right]|a_k|}{1-\sum_{k=2}^{\infty}\left[k^n(1-\lambda)+\lambda\frac{1}{k^n}\right]|a_k|}\leq 1-\mu \tag{10}$$

is equivalent with (9), hence (4) holds true.

The assertion (8) is sharp and the extremal function is given by:

$$f(z)=z+\sum_{k=2}^{\infty}\frac{1-\mu}{(k-\mu)\left[k^n(1-\lambda)+\lambda\frac{1}{k^n}\right]}z^k.$$

Corollary 2. *If* (8) *holds true, then*

$$|a_k|\leq\frac{1-\mu}{(k-\mu)\left[k^n(1-\lambda)+\lambda\frac{1}{k^n}\right]},\forall k\geq 2. \tag{11}$$

Theorem 3. *Let the function* $f(z)$ *defined by* (1) *be in* $\mathcal{A}$. *If*

$$\sum_{k=2}^{\infty}(k-\mu)\left[k^{n+1}(1-\lambda)+\lambda\frac{1}{k^{n-1}}\right]|a_k|\leq 1-\mu, \tag{12}$$

then $f(z)\in\mathcal{C}^n(\mu)$. *The result* (12) *is sharp.*

Proof. The proof is similar to the proof of Theorem 1.

Corollary 4. *If* (12) *holds true, then*

$$|a_k| \le \frac{1-\mu}{(k-\mu)\left[k^{n+1}(1-\lambda)+\lambda\frac{1}{k^{n-1}}\right]}, \forall k \ge 2. \tag{13}$$

3. Distortion theorems

Theorem 5. *If* (8) *holds true, then*

$$|z| - \frac{1-\mu}{2-\mu}|z|^2 \le |\mathscr{L}^n f(z)| \le |z| + \frac{1-\mu}{2-\mu}|z|^2, \quad \forall z \in U, 0 \le \mu < 1.$$

Proof. By (8) it is easy to verify that

$$(2-\mu)\sum_{k=2}^{\infty}\left[k^n(1-\lambda)+\lambda\frac{1}{k^n}\right]|a_k| \le \sum_{k=2}^{\infty}(k-\mu)\left[k^n(1-\lambda)+\lambda\frac{1}{k^n}\right]|a_k| \le 1-\mu.$$

Hence,

$$\sum_{k=2}^{\infty}\left[k^n(1-\lambda)+\lambda\frac{1}{k^n}\right]|a_k| \le \frac{1-\mu}{2-\mu}.$$

We obtain

$$\begin{aligned}|\mathscr{L}^n f(z)| &\le |z| + \sum_{k=2}^{\infty}\left[k^n(1-\lambda)+\lambda\frac{1}{k^n}\right]|a_k||z|^k \\ &\le |z| + \sum_{k=2}^{\infty}\left[k^n(1-\lambda)+\lambda\frac{1}{k^n}\right]|a_k||z|^2 \\ &\le |z| + \frac{1-\mu}{2-\mu}|z|^2.\end{aligned}$$

The other assertion can be proved as follows

$$\begin{aligned}|\mathscr{L}^n f(z)| &\ge |z| - \sum_{k=2}^{\infty}\left[k^n(1-\lambda)+\lambda\frac{1}{k^n}\right]|a_k||z|^k \\ &\ge |z| - \sum_{k=2}^{\infty}\left[k^n(1-\lambda)+\lambda\frac{1}{k^n}\right]|a_k||z|^2 \\ &\ge |z| - \frac{1-\mu}{2-\mu}|z|^2.\end{aligned}$$

Theorem 6. *If* (12) *holds true, then*

$$|z| - \frac{1-\mu}{2(2-\mu)}|z|^2 \leq |\mathscr{L}^n f(z)| \leq |z| + \frac{1-\mu}{2(2-\mu)}|z|^2, \quad \forall z \in U, 0 \leq \mu < 1.$$

Proof. By (12) it is easy to verify that

$$2(2-\mu)\sum_{k=2}^{\infty}\left[k^n(1-\lambda)+\lambda\frac{1}{k^n}\right]|a_k| \leq \sum_{k=2}^{\infty}(k-\mu)\left[k^{n+1}(1-\lambda)+\lambda\frac{1}{k^{n-1}}\right]|a_k| \leq 1-\mu.$$

Hence,

$$\sum_{k=2}^{\infty}\left[k^n(1-\lambda)+\lambda\frac{1}{k^n}\right]|a_k| \leq \frac{1-\mu}{2(2-\mu)}.$$

We obtain

$$\begin{aligned}|\mathscr{L}^n f(z)| &\leq |z| + \sum_{k=2}^{\infty}\left[k^n(1-\lambda)+\lambda\frac{1}{k^n}\right]|a_k||z|^k \\ &\leq |z| + \sum_{k=2}^{\infty}\left[k^n(1-\lambda)+\lambda\frac{1}{k^n}\right]|a_k||z|^2 \\ &\leq |z| + \frac{1-\mu}{2(2-\mu)}|z|^2.\end{aligned}$$

The other assertion can be proved as follows

$$\begin{aligned}|\mathscr{L}^n f(z)| &\geq |z| - \sum_{k=2}^{\infty}\left[k^n(1-\lambda)+\lambda\frac{1}{k^n}\right]|a_k||z|^k \\ &\geq |z| - \sum_{k=2}^{\infty}\left[k^n(1-\lambda)+\lambda\frac{1}{k^n}\right]|a_k||z|^2 \\ &\geq |z| - \frac{1-\mu}{2(2-\mu)}|z|^2.\end{aligned}$$

Theorem 7. *If* (8) *holds true, then*

$$|z| - \frac{1-\mu}{(2-\mu)\left[2^n(1-\lambda)+\lambda\frac{1}{2^n}\right]}|z|^2 \leq |f(z)| \leq |z| + \frac{1-\mu}{(2-\mu)\left[2^n(1-\lambda)+\lambda\frac{1}{2^n}\right]}|z|^2,$$

$\forall z \in U, 0 \leq \mu < 1.$

Proof. By (8) we have

$$(2-\mu)\left[2^n(1-\lambda)+\lambda\frac{1}{2^n}\right]\sum_{k=2}^{\infty}|a_k| \le \sum_{k=2}^{\infty}(k-\mu)\left[k^n(1-\lambda)+\lambda\frac{1}{k^n}\right]|a_k| \le 1-\mu,$$

thus

$$\sum_{k=2}^{\infty}|a_k| \le \frac{1-\mu}{(2-\mu)\left[2^n(1-\lambda)+\lambda\frac{1}{2^n}\right]}.$$

We obtain

$$\begin{aligned}|f(z)| &\le \left|z+\sum_{k=2}^{\infty}a_k z^k\right| \\ &\le |z|+\sum_{k=2}^{\infty}|a_k|\,|z|^2 \\ &\le |z|+\frac{1-\mu}{(2-\mu)\left[2^n(1-\lambda)+\lambda\frac{1}{2^n}\right]}|z|^2.\end{aligned}$$

The other assertion can be proved as follows

$$\begin{aligned}|f(z)| &\ge \left|z+\sum_{k=2}^{\infty}a_k z^k\right| \\ &\ge |z|-\sum_{k=2}^{\infty}|a_k|\,|z|^2 \\ &\ge |z|-\frac{1-\mu}{(2-\mu)\left[2^n(1-\lambda)+\lambda\frac{1}{2^n}\right]}|z|^2.\end{aligned}$$

Theorem 8. *If* (12) *holds true, then*

$$|z|-\frac{1-\mu}{2(2-\mu)\left[2^n(1-\lambda)+\lambda\frac{1}{2^n}\right]}|z|^2 \le |f(z)| \le |z|+\frac{1-\mu}{2(2-\mu)\left[2^n(1-\lambda)+\lambda\frac{1}{2^n}\right]}|z|^2,$$

$\forall z\in U, 0\le\mu<1.$

Proof. By (12) we have

$$2\left(2-\mu\right)\left[2^{n}\left(1-\lambda\right)+\lambda\frac{1}{2^{n}}\right]\sum_{k=2}^{\infty}\left|a_{k}\right|\leq\sum_{k=2}^{\infty}k\left(k-\mu\right)\left[k^{n}\left(1-\lambda\right)+\lambda\frac{1}{k^{n}}\right]\left|a_{k}\right|\leq1-\mu,$$

thus

$$\sum_{k=2}^{\infty}\left|a_{k}\right|\leq\frac{1-\mu}{2\left(2-\mu\right)\left[2^{n}\left(1-\lambda\right)+\lambda\frac{1}{2^{n}}\right]}.$$

We obtain

$$\begin{aligned}\left|f\left(z\right)\right| &\leq\left|z+\sum_{k=2}^{\infty}a_{k}z^{k}\right|\\ &\leq\left|z\right|+\sum_{k=2}^{\infty}\left|a_{k}\right|\left|z\right|^{2}\\ &\leq\left|z\right|+\frac{1-\mu}{2\left(2-\mu\right)\left[2^{n}\left(1-\lambda\right)+\lambda\frac{1}{2^{n}}\right]}\left|z\right|^{2}.\end{aligned}$$

The other assertion can be proved as follows

$$\begin{aligned}\left|f\left(z\right)\right| &\geq\left|z+\sum_{k=2}^{\infty}a_{k}z^{k}\right|\\ &\geq\left|z\right|-\sum_{k=2}^{\infty}\left|a_{k}\right|\left|z\right|^{2}\\ &\geq\left|z\right|-\frac{1-\mu}{2\left(2-\mu\right)\left[2^{n}\left(1-\lambda\right)+\lambda\frac{1}{2^{n}}\right]}\left|z\right|^{2}.\end{aligned}$$

4. Fekete-Szegő problem for the classes $\mathcal{S}^{n}\left(\mu\right)$ and $\mathcal{C}^{n}\left(\mu\right)$

Many authors obtained Fekete-Szegő inequalities for different classes of functions. [8],[3],[6],[10],[2]

Next we determine the upper bound for $\left|a_{2}\right|$ for the classes $\mathcal{S}^{n}\left(\mu\right)$ and $\mathcal{C}^{n}\left(\mu\right)$, that is sharp. Also, we calculate the Fekete-Szegő $\left|a_{3}-\xi a_{2}\right|$ functional for the above classes.

Lemma 9. *[5] Let $p \in \mathcal{P}$ (the class $\mathcal{P}$ is a Carathéodory class of functions which are analytic with positive real part in U) be of the form $p(z) = 1 + c_1 z + c_2 z^2 + \ldots$ then*

$$\left| c_2 - \frac{c_1^2}{2} \right| \leq 2 - \frac{|c_1|^2}{2} \text{ and } \quad |c_k| \leq 2, \quad \forall k \in \mathbb{N}.$$

Lemma 10. *[7] If $p(z) = 1 + c_1 z + c_2 z^2 + \ldots, z \in U$ is a function with positive real part in U and ξ is a complex number, then*

$$\left| c_2 - \xi c_1^2 \right| \leq 2 \max \{1; |2\xi - 1|\}.$$

The result is sharp for the function given by

$$p(z) = \frac{1+z^2}{1-z^2} \text{ and } p(z) = \frac{1+z}{1-z}, \; z \in U.$$

Theorem 11. *Let $0 \leq \mu < 1$ and $\varphi = \varphi_\mu$. If $f(z)$ given by (1) belongs to the class $\mathcal{S}^n(\mu)$, then*

$$|a_2| \leq \frac{B_1}{2^n (1-\lambda) + \lambda \frac{1}{2^n}}$$

and $\forall \xi \in \mathbb{C}$

$$\left| a_3 - \xi a_2^2 \right| \leq$$

$$\leq \frac{B_1}{4\left[3^n (1-\lambda) + \lambda \frac{1}{3^n}\right]} max \left\{ 1, \left| \frac{B_2}{B_1} - \frac{B_1}{2^n (1-\lambda) + \lambda \frac{1}{2^n}} \left(2 \cdot \frac{3^n (1-\lambda) + \lambda \frac{1}{3^n}}{2^n (1-\lambda) + \lambda \frac{1}{2^n}} \xi - 1 \right) \right| \right\}.$$

The result is sharp.

Proof. If $f \in \mathcal{S}^n(\mu)$, then there exists a Schwarz function $w(z)$ which is analytic in U with $w(0) = 0$; $|w(z)| < 1$, and such that

$$\frac{z\left(\mathscr{L}^n f(z)\right)'}{\mathscr{L}^n f(z)} = \varphi(w(z)). \tag{14}$$

For the Schwarz function $w(z)$; let $p \in \mathcal{P}$ be defined by

$$p(z) = \frac{1+w(z)}{1-w(z)} = 1 + c_1 z + c_2 z^2 + \cdots, \quad \Re p(z) > 0, \; p(0) = 1. \tag{15}$$

Therefore,

$$\varphi(w(z)) = \varphi\left(\frac{p(z)-1}{p(z)+1}\right) = \varphi\left(\frac{1}{2} c_1 z + \frac{1}{2}\left(c_2 - \frac{c_1^2}{2}\right) z^2 + \frac{1}{2}\left(c_3 - c_1 c_2 + \frac{c_1^3}{4}\right) z^3 + \cdots\right) = \tag{16}$$

$$= 1 + \frac{1}{2}B_1c_1z + \left[\frac{1}{2}B_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{1}{4}B_2c_1^2\right]z^2 + \cdots$$

Define the function $p_1(z)$ by :

$$p_1(z) = \frac{z\left(\mathscr{L}^n f(z)\right)'}{\mathscr{L}^n f(z)} = 1 + b_1z + b_2z^2 + \cdots \tag{17}$$

so,

$$p_1(z) = 1 + \frac{1}{2}B_1c_1z + \left[\frac{1}{2}B_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{1}{4}B_2c_1^2\right]z^2 + \cdots \tag{18}$$

From (17) and (18) we obtain:

$$b_1 = \frac{1}{2}B_1c_1 \text{ and } b_2 = \frac{1}{2}B_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{1}{4}B_2c_1^2 \tag{19}$$

Since

$$\frac{z\left(\mathscr{L}^n f(z)\right)'}{\mathscr{L}^n f(z)} = 1 + \left\{a_2\left[2^n(1-\lambda) + \lambda\frac{1}{2^n}\right]\right\}z +$$

$$+\left\{\left[2\cdot 3^n(1-\lambda) + \lambda\frac{2}{3^n}\right]a_3 - \left[2^n(1-\lambda) + \lambda\frac{1}{2^n}\right]^2 a_2^2\right\}z^2 + \cdots \tag{20}$$

then we get:

$$a_2 = \frac{B_1c_1}{2\left[2^n(1-\lambda) + \lambda\frac{1}{2^n}\right]}$$

and

$$a_3 = \frac{B_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{1}{2}B_2c_1^2}{8\left[3^n(1-\lambda) + \lambda\frac{1}{3^n}\right]} + \frac{B_1^2c_1^2}{8\left[3^n(1-\lambda) + \lambda\frac{1}{3^n}\right]\left[2^n(1-\lambda) + \lambda\frac{1}{2^n}\right]}.$$

We obtain

$$a_3 - \xi a_2^2 = \frac{B_1}{4\left[3^n(1-\lambda) + \lambda\frac{1}{3^n}\right]}\left\{c_2 - \frac{1}{2}c_1^2\left[1 - \frac{B_2}{B_1} + \right.\right.$$

$$\left.\left. +B_1\frac{1}{2^n(1-\lambda) + \lambda\frac{1}{2^n}}\left(2\frac{3^n(1-\lambda) + \lambda\frac{1}{3^n}}{2^n(1-\lambda) + \lambda\frac{1}{2^n}}\xi - 1\right)\right]\right\}$$

By Lemma 9, it follows that

$$\left|a_3 - \xi a_2^2\right| \leq H\left(|c_1|\right) = C + CD\frac{|c_1|^2}{4}$$

where

$$H\left(x\right) = C + CDx^2, \quad x = |c_1| \leq 2,$$

$$C = \frac{B_1}{4\left[3^n\left(1-\lambda\right) + \lambda\frac{1}{3^n}\right]} > 0, \quad D = |E| - 1,$$

$$E = -\frac{B_2}{B_1} + B_1\frac{1}{2^n\left(1-\lambda\right) + \lambda\frac{1}{2^n}}\left(2\frac{3^n\left(1-\lambda\right) + \lambda\frac{1}{3^n}}{2^n\left(1-\lambda\right) + \lambda\frac{1}{2^n}}\xi - 1\right).$$

As $|c_1| \leq 2$ we infer that

$$\left|a_3 - \xi a_2^2\right| \leq \begin{cases} H\left(0\right) = C, & |E| \leq 1 \\ H\left(2\right) = C\left|E\right|, & |E| > 1 \end{cases}.$$

The result is sharp for the function $f(z)$ given by

$$\frac{z\left(\mathscr{L}^n f\left(z\right)\right)'}{\mathscr{L}^n f\left(z\right)} = \varphi\left(z^2\right)$$

or

$$\frac{z\left(\mathscr{L}^n f\left(z\right)\right)'}{\mathscr{L}^n f\left(z\right)} = \varphi\left(z\right).$$

Theorem 12. *Let $0 \leq \mu < 1$ and $\varphi = \varphi_\mu$. If $f(z)$ given by (1) belongs to the class $\mathcal{C}^n\left(\mu\right)$, then*

$$|a_2| \leq \frac{B_1}{2^{n+2}\left(1-\lambda\right) + \lambda\frac{1}{2^{n-2}}}$$

and $\forall \xi \in \mathbb{C}$

$$\left|a_3 - \xi a_2^2\right| \leq$$

$$\leq \frac{B_1}{4\left[3^{n+1}\left(1-\lambda\right) + \lambda\frac{1}{3^{n-1}}\right]} max\left\{1, \left|\frac{B_2}{B_1} - \frac{B_1}{2^{n+1}\left(1-\lambda\right) + \lambda\frac{1}{2^{n-1}}}\left(2 \cdot \frac{3^{n+1}\left(1-\lambda\right) + \lambda\frac{1}{3^{n-1}}}{2^{n+1}\left(1-\lambda\right) + \lambda\frac{1}{2^{n-1}}}\xi - 1\right)\right|\right\}.$$

The result is sharp.

Proof. The proof is similar to the proof of Theorem 11.

References

[1] J.W. Alexander, *Functions which map the interior of the unit circle upon simple regions,* Ann. of Math. 17(1915), 12-22.

[2] M. K. Aouf, R. M. El-Ashwah, A. A. M. Hassan, A. H. Hassan, *Fekete-Szegő problem for a new class of analytic functions defined by using a generalized differential operator,* Acta Univ. Palacki. Olomuc., Fac. rer. nat., Mathematica 52, 1 (2013) 21-34.

[3] R. Bucur, L. Andrei, D. Breaz, *Coefficient bounds and Fekete-Szegő problem for a class of analytic functions defined by using a new differential operator,* Appl. Math. Sci, Vol. 9, 28(2015), 1355-1368.

[4] M. Darus, R. Ibrahim, *New classes containing generalization of differential operator,* Applied Mathematical Sciences, Vol. 3, 51(2009), 2507-2515.

[5] P. L. Duren, *Univalent functions,* Springer-Verlag, New York, 1983.

[6] J. Dziok, *A general solution of the Fekete-Szegő problem,* Bound Value Probl, (2013) 2013:98.

[7] W. Ma, D. Minda, *A unified treatment of some special classes of univalent functions,* In: Li Z., Ren F., Lang L., Zhang S.(eds.): Proceedings of the conference on complex analysis, Int. Press. Conf. Proc. Lect. Notes Anal. Tianjin, China, 1(1994), 157-169.

[8] R. K. Raina, Poonam Sharma, G. S. Sălăgean, *Some characteristic properties of analytic functions,* An. Şt. Univ. Ovidius Constanţa, 24(1)(2016), 353-369.

[9] G. S. Sălăgean, *Subclasses of univalent functions,* Lecture Notes in Math. (Springer Verlag), 1013(1983), 362-372.

[10] T. M. Seoudy, *Fekete-Szegő problems for certain class of non-Bazilevič functions involving the Dziok- Srivastava operator,* ROMAI J., 10(1)(2014), 175-186.

[11] H. Silverman, *Univalent functions with varying arguments,* Houston J. Math., 17(1981), 283-287.

Páll-Szabó Ágnes Orsolya,
Babeş-Bolyai University,
Cluj-Napoca, Romania,
email: *pallszaboagnes@math.ubbcluj.ro*