ON A GENERALIZED SOFT METRIC SPACE

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Abstract. In this paper our aim is to obtain new generalized fixed-point results. To do this, we introduce a new generalized soft metric space called as a soft S-metric space. We investigate some basic facts, relations and topological properties of this space. Also we define a soft S-contraction condition and study some fixed-point theorems on a complete soft S-metric space with necessary examples.

1. Introduction and Background

Metric spaces and fixed-point theory have very important role in mathematics and lead to some applications. Some mathematicians have studied new generalizations of metric spaces using various ways. Recently, it has been introduced the notion of an S-metric space as a generalization of a metric space [14]. Then some basic fixed-point theorems and their generalizations were obtained in some studies (for more details see [9], [10], [11], [14], [15] and [16]). These fixed-point theorems were used in other mathematical areas such as complex valued metric spaces, differential equations etc. (see [12] and [13]).

There are some uncertain concepts in the areas of medical science, engineering, economics etc. Hence some set theories such as fuzzy set theory, rough set theory, intuitionistic fuzzy set theory etc. can be dealt with uncertainties. Unfortunately, they are not sufficient to cope with encountered problems. Therefore, Molodtsov introduced the soft set theory as a general mathematical tool for dealing with some complicated problems [8]. Maji et al. made a theoretical study of the soft set theory [7]. Shabir and Naz studied some soft topological concepts and investigated their basic properties [17].

Das and Samanta defined the notion of a soft real number and studied their properties [4]. Therefore they introduced the concept of a soft metric space and gave some fundamental properties of this space [5]. Then some fixed-point results

were obtained using various approaches on a soft metric space (see [1], [2] and [3] for more details). Güler et al. defined the notion of a soft G-metric and proved a fixed-point theorem in soft G-metric spaces [6].

Motivated by the above studies, in this paper we introduce the concept of a soft S-metric space as a generalization of a soft metric space. We expect that our study will help to generate some new researches and applications. For example, various generalized soft contractive conditions can be given as generalizations of our results.

In Section 2, we define the notion of a soft S-metric according to a soft point and determine the relationships between the other soft metrics. In Section 3, we describe basic topological concepts. In Section 4, we present the notion of a soft fixed point on a soft S-metric space and prove a fixed point-theorem of Banach contraction principle type. Also we generalize this theorem with a counter example.

On the other hand, Abbas et al. showed that a soft metric induces a compatible metric on the collection of all soft points of the absolute soft set when the set of parameters is a finite set [3]. Therefore a cardinality of a parameter set is to be significant. The results obtained in Section 4 can be also proved using this approach on a soft S-metric space.

Before stating our main results, we recall some definitions, a proposition and an example.

Definition 1. *[8] Let U be an initial universe set and E be a set of parameters. A pair (F, E) is called a soft set over U if and only if F is a mapping from E into the set of all subsets of the universe set U. That is, $F : E \to P(U)$, where $P(U)$ is the set of all subsets of the set U.*

Definition 2. *[7] Let (F, E) be a soft set over a universe set U.*

1. *(F, E) is said to be a null soft set denoted by $\widetilde{\emptyset}$ if $F(e) = \emptyset$ for all $e \in E$.*

2. *(F, E) is said to be an absolute soft set denoted by $\widetilde{U}$ if $F(e) = U$ for all $e \in E$.*

Definition 3. *[4] Let $\mathbb{R}$ be the set of real numbers, $B(\mathbb{R})$ be the collection of all nonempty bounded subsets of $\mathbb{R}$ and E be a set of parameters. Then a mapping $F : E \to B(\mathbb{R})$ is called a soft real set. It is denoted by (F, E). If specifically (F, E) is a singleton soft set then identifying (F, E) with the corresponding soft element, it will be called a soft real number and denoted by $\widetilde{r}$, $\widetilde{s}$, $\widetilde{t}$ etc.*

$\overline{0}$, $\overline{1}$ are the soft real numbers where $\overline{0}(e) = 0$, $\overline{1}(e) = 1$ for all $e \in E$, respectively.

Definition 4. *[4] Let (F, E) and (G, E) be two soft real numbers.*

1. *$(F, E) = (G, E)$ if $F(e) = G(e)$ for each $e \in E$.*

2. $(F + G)(e) = \{x + y : x \in F(e), y \in G(e)\}$ *for each* $e \in E$.

3. $(F - G)(e) = \{x - y : x \in F(e), y \in G(e)\}$ *for each* $e \in E$.

4. $(F.G)(e) = \{x.y : x \in F(e), y \in G(e)\}$ *for each* $e \in E$.

5. $(F/G)(e) = \{x/y : x \in F(e), y \in G(e) - \{0\}\}$ *for each* $e \in E$.

Definition 5. *[4] For two soft real numbers*

1. $\widetilde{r} \, \widetilde{\leq} \, \widetilde{s}$ *if* $\widetilde{r}(e) \leq \widetilde{s}(e)$ *for all* $e \in E$,

2. $\widetilde{r} \, \widetilde{\geq} \, \widetilde{s}$ *if* $\widetilde{r}(e) \geq \widetilde{s}(e)$ *for all* $e \in E$,

3. $\widetilde{r} \, \widetilde{<} \, \widetilde{s}$ *if* $\widetilde{r}(e) < \widetilde{s}(e)$ *for all* $e \in E$,

4. $\widetilde{r} \, \widetilde{>} \, \widetilde{s}$ *if* $\widetilde{r}(e) > \widetilde{s}(e)$ *for all* $e \in E$.

Definition 6. *[5] A soft set* (P, E) *over* U *is said to be a soft point if there is exactly one* $e \in E$ *such that* $P(e) = \{x\}$ *for some* $x \in U$ *and* $P(e') = \emptyset$ *for all* $e' \in E - \{e\}$. *It will be denoted by* P_e^x.

Definition 7. *[5] A soft point* P_e^x *is said to be belongs to a soft set* (F, E) *if* $e \in E$ *and* $P(e) = \{x\} \subset F(e)$. *It is written by* $P_e^x \, \widetilde{\in} \, (F, E)$.

Definition 8. *[5] Two soft points* P_e^x, $P_{e'}^y$ *are said to be equal if* $e = e'$ *and* $P(e) = P(e')$, *that is,* $x = y$. *Thus,*

$$P_e^x \neq P_{e'}^y \Longleftrightarrow x \neq y \text{ or } e \neq e'.$$

Proposition 1. *[5] The union of any collection of soft points can be considered as a soft set and every soft set can be expressed as union of all soft points belonging to it, that is,*

$$(F, E) = \bigcup_{P_e^x \, \widetilde{\in} \, (F,E)} P_e^x.$$

Let $SP(\widetilde{U})$ be the collection of all soft points of $\widetilde{U}$ and $\mathbb{R}(E)^*$ be the set of all nonnegative soft real numbers.

Definition 9. *[5] A mapping* $\widetilde{d} : SP(\widetilde{U}) \times SP(\widetilde{U}) \to \mathbb{R}(E)^*$ *is said to be a soft metric on the soft set* $\widetilde{U}$ *if* $\widetilde{d}$ *satisfies the following conditions:*
   $(\widetilde{d}1)$ $\widetilde{d}(P_{e_1}^x, P_{e_2}^y) \, \widetilde{\geq} \, \overline{0}$ *for all* $P_{e_1}^x, P_{e_2}^y \in SP(\widetilde{U})$.
   $(\widetilde{d}2)$ $\widetilde{d}(P_{e_1}^x, P_{e_2}^y) = \overline{0}$ *if and only if* $P_{e_1}^x = P_{e_2}^y$.
   $(\widetilde{d}3)$ $\widetilde{d}(P_{e_1}^x, P_{e_2}^y) = \widetilde{d}(P_{e_2}^y, P_{e_1}^x)$ *for all* $P_{e_1}^x, P_{e_2}^y \in SP(\widetilde{U})$.

$(\widetilde{d}4)$ $\widetilde{d}(P_{e_1}^x, P_{e_3}^z) \widetilde{\leq} \widetilde{d}(P_{e_1}^x, P_{e_2}^y) + \widetilde{d}(P_{e_2}^y, P_{e_3}^z)$ *for all* $P_{e_1}^x$, $P_{e_2}^y$, $P_{e_3}^z \in SP(\widetilde{U})$.

The soft set $\widetilde{U}$ with a soft metric $\widetilde{d}$ on $\widetilde{U}$ is called a soft metric space and denoted by $\left(\widetilde{U}, \widetilde{d}, E\right)$.

Example 1. *[5] Let $U \subset \mathbb{R}$ be a nonempty set and $E \subset \mathbb{R}$ be the nonempty set of parameters. Let $\widetilde{U}$ be the absolute soft set and $\overline{x}$ denotes the soft real number such that*

$$\overline{x}(e) = x,$$

for all $e \in E$. Then the function $\widetilde{d} : SP(\widetilde{U}) \times SP(\widetilde{U}) \to \mathbb{R}(E)^*$ defined by

$$\widetilde{d}(P_{e_1}^x, P_{e_2}^y) = |\overline{x} - \overline{y}| + |\overline{e}_1 - \overline{e}_2|,$$

for all $P_{e_1}^x, P_{e_2}^y \in SP(\widetilde{U})$, where "$|.|$" denotes the modulus of soft real numbers, is a soft metric on $\widetilde{U}$.

Definition 10. *[5] Let $\left\{P_{e,n}^x\right\}_n$ be a sequence of soft points in a soft metric space $\left(\widetilde{U}, \widetilde{d}, E\right)$. The sequence $\left\{P_{e,n}^x\right\}_n$ is said to be convergent in $\left(\widetilde{U}, \widetilde{d}, E\right)$ if there is a soft point $P_\alpha^\beta \in SP(\widetilde{U})$ such that*

$$\widetilde{d}(P_{e,n}^x, P_\alpha^\beta) \to \overline{0} \ as \ n \to \infty.$$

That is, for every $\widetilde{\varepsilon} \widetilde{\geq} \overline{0}$, chosen arbitrarily, there exists a natural number $N = N(\widetilde{\varepsilon})$ such that $\overline{0} \widetilde{\leq} \widetilde{d}(P_{e,n}^x, P_\alpha^\beta) \widetilde{<} \widetilde{\varepsilon}$, whenever $n > N$.

Definition 11. *[5] A sequence $\left\{P_{e,n}^x\right\}_n$ of soft points in $\left(\widetilde{U}, \widetilde{d}, E\right)$ is called a Cauchy sequence if corresponding to every $\widetilde{\varepsilon} \widetilde{\geq} \overline{0}$, there exists $m \in \mathbb{N}$ such that $\widetilde{d}(P_{e,i}^x, P_{e,j}^x) \widetilde{<} \widetilde{\varepsilon}$ for every $i, j \geq m$, that is, $\widetilde{d}(P_{e,i}^x, P_{e,j}^x) \to \overline{0}$ as $i, j \to \infty$.*

Definition 12. *[5] A soft metric space $\left(\widetilde{U}, \widetilde{d}, E\right)$ is called complete if every Cauchy sequence in $\widetilde{U}$ converges to some point of $\widetilde{U}$.*

Definition 13. *[6] Let U be a nonempty set and E be the nonempty set of parameters. A mapping $\widetilde{G} : SP(\widetilde{U}) \times SP(\widetilde{U}) \times SP(\widetilde{U}) \to \mathbb{R}(E)^*$ is said to be a soft G-metric on the soft set $\widetilde{U}$ if $\widetilde{G}$ satisfies the following conditions:*

$(\widetilde{G}1)$ $\widetilde{G}\left(P_{e_1}^x, P_{e_2}^y, P_{e_3}^z\right) = \overline{0}$ *if* $P_{e_1}^x = P_{e_2}^y = P_{e_3}^z$.

$(\widetilde{G}2)$ $\overline{0} \widetilde{<} \widetilde{G}\left(P_{e_1}^x, P_{e_1}^x, P_{e_2}^y\right)$ *for all* $P_{e_1}^x, P_{e_2}^y \in SP(\widetilde{U})$ *with* $P_{e_1}^x \neq P_{e_2}^y$.

$(\widetilde{G}3)$ $\widetilde{G}\left(P_{e_1}^x, P_{e_1}^x, P_{e_2}^y\right) \widetilde{\leq} \widetilde{G}\left(P_{e_1}^x, P_{e_2}^y, P_{e_3}^z\right)$ *for all* $P_{e_1}^x$, $P_{e_2}^y$, $P_{e_3}^z \in SP(\widetilde{U})$ *with* $P_{e_2}^y \neq P_{e_3}^z$.

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$(\widetilde{G}4)$ $\widetilde{G}\left(P_{e_1}^x, P_{e_2}^y, P_{e_3}^z\right) = \widetilde{G}\left(P_{e_1}^x, P_{e_3}^z, P_{e_2}^y\right) = \widetilde{G}\left(P_{e_2}^y, P_{e_3}^z, P_{e_1}^x\right) = \cdots$ *for all possible triples* $P_{e_1}^x$, $P_{e_2}^y$, $P_{e_3}^z \in SP(\widetilde{U})$.

$(\widetilde{G}5)$ $\widetilde{G}\left(P_{e_1}^x, P_{e_2}^y, P_{e_3}^z\right) \widetilde{\leq} \widetilde{G}\left(P_{e_1}^x, P_e^a, P_e^a\right) + \widetilde{G}\left(P_e^a, P_{e_2}^y, P_{e_3}^z\right)$ *for all* $P_{e_1}^x$, $P_{e_2}^y$, $P_{e_3}^z$, $P_e^a \in SP(\widetilde{U})$.

The triplet $\left(\widetilde{U}, \widetilde{G}, E\right)$ *is said to be a soft G-metric space.*

2. Soft S-Metric spaces

In this section we define the notion of a soft S-metric space and determine its basic properties. Also we investigate some relationships between a soft metric and a soft S-metric (resp. a soft G-metric and a soft S-metric).

Let U be an initial universe set and E be the nonempty set of parameters. Let $SP(\widetilde{U})$ be a collection of all soft points of $\widetilde{U}$ and $\mathbb{R}(E)^*$ be the set of all nonnegative soft real numbers.

Definition 14. *A mapping* $\widetilde{S} : SP(\widetilde{U}) \times SP(\widetilde{U}) \times SP(\widetilde{U}) \to \mathbb{R}(E)^*$ *is said to be a soft S-metric on the soft set* $\widetilde{U}$ *if* $\widetilde{S}$ *satisfies the following conditions for each* $P_{e_1}^x$, $P_{e_2}^y$, $P_{e_3}^z$, $P_e^a \in SP(\widetilde{U})$:

$(\widetilde{S}1)$ $\widetilde{S}\left(P_{e_1}^x, P_{e_2}^y, P_{e_3}^z\right) \widetilde{\geq} \overline{0}$.

$(\widetilde{S}2)$ $\widetilde{S}\left(P_{e_1}^x, P_{e_2}^y, P_{e_3}^z\right) = \overline{0}$ *if and only if* $P_{e_1}^x = P_{e_2}^y = P_{e_3}^z$.

$(\widetilde{S}3)$ $\widetilde{S}\left(P_{e_1}^x, P_{e_2}^y, P_{e_3}^z\right) \widetilde{\leq} \widetilde{S}\left(P_{e_1}^x, P_{e_1}^x, P_e^a\right) + \widetilde{S}\left(P_{e_2}^y, P_{e_2}^y, P_e^a\right) + \widetilde{S}\left(P_{e_3}^z, P_{e_3}^z, P_e^a\right)$.

The soft set $\widetilde{U}$ *with a soft S-metric* $\widetilde{S}$ *on* $\widetilde{U}$ *is called a soft S-metric space and denoted by* $\left(\widetilde{U}, \widetilde{S}, E\right)$.

Now we give the following examples for a soft S-metric.

Example 2. *Let* $U \subset \mathbb{R}$ *be a nonempty set and* $E \subset \mathbb{R}$ *be the nonempty set of parameters. Let* $\widetilde{U}$ *be the absolute soft set, that is,* $F(e) = U$ *for all* $e \in E$, *where* $(F, E) = \widetilde{U}$. *Let* $\overline{x}$ *denote a soft real number such that* $\overline{x}(e) = x$ *for all* $e \in E$. *We define* $\widetilde{S} : SP(\widetilde{U}) \times SP(\widetilde{U}) \times SP(\widetilde{U}) \to \mathbb{R}(E)^*$ *by*

$$\widetilde{S}\left(P_{e_1}^x, P_{e_2}^y, P_{e_3}^z\right) = |\overline{y} + \overline{z} - 2\overline{x}| + |\overline{y} - \overline{z}| + |\overline{e}_2 + \overline{e}_3 - 2\overline{e}_1| + |\overline{e}_2 - \overline{e}_3|,$$

for all $P_{e_1}^x$, $P_{e_2}^y$, $P_{e_3}^z \in SP(\widetilde{U})$, *where "*$|.|$*"denotes the modulus of soft real numbers. Then* $\widetilde{S}$ *is a soft S-metric on* $\widetilde{U}$.

Example 3. *Let* U *be a nonempty set,* E *be the nonempty set of parameters and* $\widetilde{d}$ *be a soft metric on* $\widetilde{U}$. *Then the function* $\widetilde{S} : SP(\widetilde{U}) \times SP(\widetilde{U}) \times SP(\widetilde{U}) \to \mathbb{R}(E)^*$ *defined as*

$$\widetilde{S}\left(P_{e_1}^x, P_{e_2}^y, P_{e_3}^z\right) = \widetilde{d}(P_{e_1}^x, P_{e_2}^y) + \widetilde{d}(P_{e_2}^y, P_{e_3}^z) + \widetilde{d}(P_{e_1}^x, P_{e_3}^z),$$

for all $P_{e_1}^x$, $P_{e_2}^y$, $P_{e_3}^z \in SP(\widetilde{U})$, is a soft S-metric on $\widetilde{U}$.

Example 4. *Let U be a nonempty set and E be the nonempty set of parameters. Let us define the function $\widetilde{S} : SP(\widetilde{U}) \times SP(\widetilde{U}) \times SP(\widetilde{U}) \to \mathbb{R}(E)^*$ as follows:*

$$\widetilde{S}\left(P_{e_1}^x, P_{e_2}^y, P_{e_3}^z\right) = \begin{cases} \overline{0} & if \quad P_{e_1}^x = P_{e_2}^y = P_{e_3}^z \\ \overline{1} & if \quad\quad otherwise \end{cases},$$

for all $P_{e_1}^x$, $P_{e_2}^y$, $P_{e_3}^z \in SP(\widetilde{U})$. Then the function $\widetilde{S}$ is a soft S-metric. We call this soft S-metric is the soft discrete S-metric on $\widetilde{U}$. The triplet $\left(\widetilde{U}, \widetilde{S}, E\right)$ is called soft discrete S-metric space.

Lemma 1. *Let $\left(\widetilde{U}, \widetilde{S}, E\right)$ be a soft S-metric space. Then we have*

$$\widetilde{S}\left(P_{e_1}^x, P_{e_1}^x, P_{e_2}^y\right) = \widetilde{S}\left(P_{e_2}^y, P_{e_2}^y, P_{e_1}^x\right).$$

Proof. By the condition $(\widetilde{S}3)$ we obtain

$$\begin{aligned} \widetilde{S}\left(P_{e_1}^x, P_{e_1}^x, P_{e_2}^y\right) &\widetilde{\leq} 2\widetilde{S}\left(P_{e_1}^x, P_{e_1}^x, P_{e_1}^x\right) + \widetilde{S}\left(P_{e_2}^y, P_{e_2}^y, P_{e_1}^x\right) \\ &= \widetilde{S}\left(P_{e_2}^y, P_{e_2}^y, P_{e_1}^x\right) \end{aligned} \tag{1}$$

and

$$\begin{aligned} \widetilde{S}\left(P_{e_2}^y, P_{e_2}^y, P_{e_1}^x\right) &\widetilde{\leq} 2\widetilde{S}\left(P_{e_2}^y, P_{e_2}^y, P_{e_2}^y\right) + \widetilde{S}\left(P_{e_1}^x, P_{e_1}^x, P_{e_2}^y\right) \\ &= \widetilde{S}\left(P_{e_1}^x, P_{e_1}^x, P_{e_2}^y\right) \end{aligned}. \tag{2}$$

Using the inequalities (1) and (2) we get

$$\widetilde{S}\left(P_{e_1}^x, P_{e_1}^x, P_{e_2}^y\right) = \widetilde{S}\left(P_{e_2}^y, P_{e_2}^y, P_{e_1}^x\right).$$

Proposition 2. *Let U be a nonempty set, E be a nonempty set of the parameters and $\widetilde{d}$ be a soft metric on $\widetilde{U}$. Then*

$$\widetilde{S}_d(P_{e_1}^x, P_{e_2}^y, P_{e_3}^z) = \widetilde{d}(P_{e_1}^x, P_{e_3}^z) + \widetilde{d}(P_{e_2}^y, P_{e_3}^z),$$

for all $P_{e_1}^x$, $P_{e_2}^y$, $P_{e_3}^z \in SP(\widetilde{U})$, is a soft S-metric on $\widetilde{U}$.

Proof. It is obvious from Definitions 9 and 14.

We call the soft metric $\widetilde{S}_d$ as the soft S-metric generated by $\widetilde{d}$.

Example 5. *Let U and E be nonempty subsets of $\mathbb{R}$. Let us define $\widetilde{S} : SP(\widetilde{U}) \times SP(\widetilde{U}) \times SP(\widetilde{U}) \to \mathbb{R}(E)^*$ as*

$$\widetilde{S}\left(P_{e_1}^x, P_{e_2}^y, P_{e_3}^z\right) = |\overline{x} - \overline{z}| + |\overline{y} - \overline{z}| + |\overline{e}_1 - \overline{e}_3| + |\overline{e}_2 - \overline{e}_3|,$$

for all $P_{e_1}^x$, $P_{e_2}^y$, $P_{e_3}^z \in SP(\widetilde{U})$, where "$|.|$"denotes the modulus of soft real numbers and $\overline{x}$, $\overline{e}_1$ are constant real numbers defined by

$$\overline{x}(e) = x \ \ and \ \ \overline{e}_1(e) = e_1,$$

for all $e \in E$, respectively. Then $\widetilde{S}$ is a soft S-metric on $\widetilde{U}$ and the triplet $\left(\widetilde{U}, \widetilde{S}, E\right)$ is a soft S-metric space. This soft S-metric is generated by soft metric $\widetilde{d}$ defined in Example 1.

In the following example, we see that there exists a soft S-metric which is not generated by any soft metric.

Example 6. *Let $U \subseteq \mathbb{R}$ be a nonempty set and*

$$E = \{e_i : 1 \leq i \leq n\} \subseteq \mathbb{R},$$

be the nonempty set of parameters. Let us define a function $\widetilde{S} : SP(\widetilde{U}) \times SP(\widetilde{U}) \times SP(\widetilde{U}) \to \mathbb{R}(E)^*$ as

$$\widetilde{S}\left(P_{e_1}^x, P_{e_2}^y, P_{e_3}^z\right) = |\overline{x} - \overline{z}| + |\overline{x} + \overline{z} - 2\overline{y}| + |\overline{e}_1 - \overline{e}_3| + |\overline{e}_1 + \overline{e}_3 - 2\overline{e}_2|,$$

for all $P_{e_1}^x$, $P_{e_2}^y$, $P_{e_3}^z \in SP(\widetilde{U})$. Then $\widetilde{S}$ is a soft S-metric on $\widetilde{U}$ and the triplet $\left(\widetilde{U}, \widetilde{S}, E\right)$ is a soft S-metric space.

Now we show that there does not exist any soft metric $\widetilde{d}$ such that $\widetilde{S} = \widetilde{S}_d$. Conversely, assume that there exists a soft metric $\widetilde{d}$ such that

$$\widetilde{S}(P_{e_1}^x, P_{e_2}^y, P_{e_3}^z) = \widetilde{d}(P_{e_1}^x, P_{e_3}^z) + \widetilde{d}(P_{e_2}^y, P_{e_3}^z),$$

for all $P_{e_1}^x$, $P_{e_2}^y$, $P_{e_3}^z \in SP(\widetilde{U})$. Therefore we find

$$\widetilde{S}(P_{e_1}^x, P_{e_1}^x, P_{e_3}^z) = 2\widetilde{d}(P_{e_1}^x, P_{e_3}^z) = 2\left(|\overline{x} - \overline{z}| + |\overline{e}_1 - \overline{e}_3|\right)$$

and

$$\widetilde{S}(P_{e_2}^y, P_{e_2}^y, P_{e_3}^z) = 2\widetilde{d}(P_{e_2}^y, P_{e_3}^z) = 2\left(|\overline{y} - \overline{z}| + |\overline{e}_2 - \overline{e}_3|\right).$$

Hence we obtain

$$\widetilde{d}(P_{e_1}^x, P_{e_3}^z) = |\overline{x} - \overline{z}| + |\overline{e}_1 - \overline{e}_3| \ \ and \ \ \widetilde{d}(P_{e_2}^y, P_{e_3}^z) = |\overline{y} - \overline{z}| + |\overline{e}_2 - \overline{e}_3|.$$

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Therefore we have a contradiction since

$$|\overline{x} - \overline{z}| + |\overline{x} + \overline{z} - 2\overline{y}| + |\overline{e}_1 - \overline{e}_3| + |\overline{e}_1 + \overline{e}_3 - 2\overline{e}_2| = |\overline{x} - \overline{z}| + |\overline{e}_1 - \overline{e}_3| + |\overline{y} - \overline{z}| + |\overline{e}_2 - \overline{e}_3|.$$

Consequently we get $\widetilde{S} \neq \widetilde{S}_d$.

Notice that the class of all soft G-metrics and the class of all soft S-metrics are distinct as seen in the following examples.

Example 7. *Let $\left(\widetilde{U}, \widetilde{S}, E\right)$ be the soft S-metric space defined in Example 5. For $e = 0$, $x = 2$, $y = 1$, $z = 3$, we obtain*

$$\widetilde{S}(P_0^2, P_0^1, P_0^3)(e) = |2 - 3| + |1 - 3| = 1 + 2 = 3$$

and

$$\widetilde{S}(P_0^1, P_0^3, P_0^2)(e) = |1 - 2| + |3 - 2| = 1 + 1 = 2.$$

Then we get $\widetilde{S}(P_0^2, P_0^1, P_0^3) \neq \widetilde{S}(P_0^1, P_0^3, P_0^2)$. Consequently, the condition $(\widetilde{G}4)$ is not satisfied and $\widetilde{S}$ is not a soft G-metric.

Example 8. *Let $U = \{x, y\}$, $E = \{0\}$ and the function $\widetilde{G} : SP(\widetilde{U}) \times SP(\widetilde{U}) \times SP(\widetilde{U}) \to \mathbb{R}(E)^*$ be defined by*

$$\widetilde{G}(P_e^x, P_e^x, P_e^x) = \widetilde{G}(P_e^y, P_e^y, P_e^y) = \overline{0},$$

$$\widetilde{G}(P_e^x, P_e^x, P_e^y) = \widetilde{G}(P_e^x, P_e^y, P_e^x) = \widetilde{G}(P_e^y, P_e^x, P_e^x) = \overline{2}$$

and

$$\widetilde{G}(P_e^x, P_e^y, P_e^y) = \widetilde{G}(P_e^y, P_e^x, P_e^y) = \widetilde{G}(P_e^y, P_e^y, P_e^x) = \overline{3},$$

for all P_e^x, $P_e^y \in SP(\widetilde{U})$. Then $\widetilde{G}$ is a soft G-metric on $\widetilde{U}$ and the triplet $\left(\widetilde{U}, \widetilde{G}, E\right)$ is a soft G-metric space. But it is not soft S-metric space. Indeed, the condition $(\widetilde{S}3)$ is not satisfied since

$$
\begin{aligned}
\widetilde{S}(P_e^x, P_e^y, P_e^y)(e) &= \overline{3}(e) = 3 \leq \left[\widetilde{S}(P_e^x, P_e^x, P_e^y) + \widetilde{S}(P_e^y, P_e^y, P_e^y) + \widetilde{S}(P_e^y, P_e^y, P_e^y)\right](e) \\
&= \overline{2}(e) + \overline{0}(e) + \overline{0}(e) = 2 + 0 + 0 = 2.
\end{aligned}
$$

3. Some Topological Properties of Soft S-Metric Spaces

In this section we define some topological concepts on soft S-metric spaces and investigate some properties related to these notions.

Definition 15. *Let* $\left(\widetilde{U}, \widetilde{S}, E\right)$ *be a soft S-metric space and* (F, E) *be a non-null soft subset of* $\widetilde{U}$. *Then the diameter of* (F, E) *is denoted by* $\delta(F, E)$ *and*

$$\delta(F, E)(\alpha) = \sup\{\widetilde{S}(P_{e_1}^x, P_{e_1}^x, P_{e_2}^y)(\alpha) : P_{e_1}^x, P_{e_2}^y \widetilde{\in} (F, E)\},$$

for all $\alpha \in E$.

Definition 16. *Let* $\left(\widetilde{U}, \widetilde{S}, E\right)$ *be a soft S-metric space,* $P_{e_1}^x$ *be a fixed soft point of* $\widetilde{U}$ *and* (F, E) *be a non-null soft subset of* $\widetilde{U}$. *Then the distance of the soft point* $P_{e_1}^x$ *from the soft set* (F, E) *is denoted by* $\delta(P_{e_1}^x, (F, E))$ *and*

$$\delta(P_{e_1}^x, (F, E))(\alpha) = \inf\{\widetilde{S}(P_{e_1}^x, P_{e_1}^x, P_{e_2}^y)(\alpha) : P_{e_2}^y \widetilde{\in} (F, E)\},$$

for all $\alpha \in E$.

Definition 17. *Let* $\left(\widetilde{U}, \widetilde{S}, E\right)$ *be a soft S-metric space and* (F, E), (G, E) *be two non-null soft subsets of* $\widetilde{U}$. *The distance between the soft sets* (F, E), (G, E) *is denoted by* $\delta((F, E), (G, E))$ *and*

$$\delta((F, E), (G, E))(\alpha) = \inf\{\widetilde{S}(P_{e_1}^x, P_{e_1}^x, P_{e_2}^y)(\alpha) : P_{e_1}^x \widetilde{\in} (F, E), P_{e_2}^y \widetilde{\in} (G, E)\},$$

for all $\alpha \in E$.

Definition 18. *Let* $\left(\widetilde{U}, \widetilde{S}, E\right)$ *be a soft S-metric space. If there exists a positive soft real number* $\widetilde{k}$ *such that*

$$\widetilde{S}(P_{e_1}^x, P_{e_1}^x, P_{e_2}^y) \widetilde{\leq} \widetilde{k},$$

for all $P_{e_1}^x, P_{e_2}^y \widetilde{\in} \widetilde{U}$, *then* $\left(\widetilde{U}, \widetilde{S}, E\right)$ *is called a bounded soft S-metric space. Otherwise it is called unbounded.*

In the following definition we define the notion of soft open S-ball and soft closed S-ball, respectively.

Definition 19. *Let* $\left(\widetilde{U}, \widetilde{S}, E\right)$ *be a soft S-metric space and* $\widetilde{r}$ *be a nonnegative soft real number.*

1. *The soft open S-ball is defined by*

$$B_S(P_{e_1}^x, \widetilde{r}) = \{P_{e_2}^y \widetilde{\in} \widetilde{U} : \widetilde{S}(P_{e_1}^x, P_{e_1}^x, P_{e_2}^y) \widetilde{<} \widetilde{r}\},$$

   with center $P_{e_1}^x$ *and radius* $\widetilde{r}$.

2. *The soft closed S-ball is defined by*

$$B_S[P_{e_1}^x, \widetilde{r}] = \{P_{e_2}^y \widetilde{\in} \widetilde{U} : \widetilde{S}(P_{e_1}^x, P_{e_1}^x, P_{e_2}^y) \widetilde{\leq} \widetilde{r}\},$$

with center $P_{e_1}^x$ and radius $\widetilde{r}$.

Proposition 3. *Let $\left(\widetilde{U}, \widetilde{S}, E\right)$ be a soft S-metric space, $P_e^x \in SP(\widetilde{U})$ and $\widetilde{r} \widetilde{\geq} \overline{0}$. If $P_e^y \in B_S(P_e^x, \widetilde{r})$ then there exists a $\widetilde{\rho} \widetilde{\geq} \overline{0}$ such that*

$$B_S(P_e^y, \widetilde{\rho}) \subseteq B_S(P_e^x, \widetilde{r}).$$

Proof. Let $P_e^y \in B_S(P_e^x, \widetilde{r})$. Then we get

$$\widetilde{S}(P_e^y, P_e^y, P_e^x) \widetilde{<} \widetilde{r}.$$

We show that

$$B_S(P_e^y, \widetilde{\rho}) \subseteq B_S(P_e^x, \widetilde{r}).$$

Let us choose

$$\widetilde{\rho} = \frac{\widetilde{r} - \widetilde{S}(P_e^x, P_e^x, P_e^y)}{2}.$$

If $P_e^z \in B_S(P_e^y, \widetilde{\rho})$, then we have

$$\widetilde{S}(P_e^z, P_e^z, P_e^y) \widetilde{<} \widetilde{\rho}.$$

Using the condition $(\widetilde{S}3)$ we find

$$\widetilde{S}(P_e^z, P_e^z, P_e^x) \widetilde{\leq} 2\widetilde{S}(P_e^z, P_e^z, P_e^y) + \widetilde{S}(P_e^x, P_e^x, P_e^y) \widetilde{<} 2\widetilde{\rho} + \widetilde{S}(P_e^x, P_e^x, P_e^y) = \widetilde{r}$$

and so

$$B_S(P_e^y, \widetilde{\rho}) \subseteq B_S(P_e^x, \widetilde{r}).$$

Definition 20. *Let $\left(\widetilde{U}, \widetilde{S}, E\right)$ be a soft S-metric space having at least two soft points. Then $\left(\widetilde{U}, \widetilde{S}, E\right)$ is said to poses soft S-Hausdorff property if for any two soft elements $P_{e_1}^x$, $P_{e_2}^y$ such that $\widetilde{S}(P_{e_1}^x, P_{e_1}^x, P_{e_2}^y) \widetilde{>} \overline{0}$, there are two soft open S-balls $B_S(P_{e_1}^x, \widetilde{r})$ and $B_S(P_{e_2}^y, \widetilde{r})$ with radius $\widetilde{r}$ and centers $P_{e_1}^x$, $P_{e_2}^y$, respectively, such that*

$$B_S(P_{e_1}^x, \widetilde{r}) \cap B_S(P_{e_2}^y, \widetilde{r}) = \widetilde{\emptyset}.$$

Theorem 2. *Every soft S-metric space is Hausdorff.*

Proof. Let $\left(\widetilde{U}, \widetilde{S}, E\right)$ be a soft S-metric space having at least two soft elements. Let $P_{e_1}^x$, $P_{e_2}^y$ be two soft elements in $\widetilde{U}$ such that $\widetilde{S}(P_{e_1}^x, P_{e_1}^x, P_{e_2}^y) \stackrel{\sim}{>} \overline{0}$. Let us consider any nonnegative soft real number $\widetilde{r}$ such that

$$\overline{0} \stackrel{\sim}{<} \widetilde{r} \stackrel{\sim}{<} \frac{1}{3}\widetilde{S}(P_{e_1}^x, P_{e_1}^x, P_{e_2}^y),$$

and the soft open S-balls $B_S(P_{e_1}^x, \widetilde{r})$ and $B_S(P_{e_2}^y, \widetilde{r})$ with radius $\widetilde{r}$ and centers $P_{e_1}^x$, $P_{e_2}^y$, respectively.

Assume that there exists $P_{e_3}^z \in SP(\widetilde{U})$ such that

$$P_{e_3}^z \in B_S(P_{e_1}^x, \widetilde{r}) \cap B_S(P_{e_2}^y, \widetilde{r}).$$

Then we get

$$P_{e_3}^z \in B_S(P_{e_1}^x, \widetilde{r}) \Rightarrow \widetilde{S}(P_{e_1}^x, P_{e_1}^x, P_{e_3}^z) \stackrel{\sim}{<} \widetilde{r} \tag{3}$$

and

$$P_{e_3}^z \in B_S(P_{e_2}^y, \widetilde{r}) \Rightarrow \widetilde{S}(P_{e_2}^y, P_{e_2}^y, P_{e_3}^z) \stackrel{\sim}{<} \widetilde{r}. \tag{4}$$

Using the conditions $(\widetilde{S}3)$, (3) and (4) we have

$$\widetilde{S}(P_{e_1}^x, P_{e_1}^x, P_{e_2}^y) \stackrel{\sim}{\leq} 2\widetilde{S}(P_{e_1}^x, P_{e_1}^x, P_{e_3}^z) + \widetilde{S}(P_{e_2}^y, P_{e_2}^y, P_{e_3}^z) = 3\widetilde{r},$$

which is a contradiction since $\overline{0} \stackrel{\sim}{<} \widetilde{r} \stackrel{\sim}{<} \frac{1}{3}\widetilde{S}(P_{e_1}^x, P_{e_1}^x, P_{e_2}^y)$. Hence it should be

$$B_S(P_{e_1}^x, \widetilde{r}) \cap B_S(P_{e_2}^y, \widetilde{r}) = \widetilde{\emptyset}.$$

Consequently, soft S-metric spaces satisfy the soft S-Hausdorff property.

Definition 21. *Let* $\left(\widetilde{U}, \widetilde{S}, E\right)$ *be a soft S-metric space and* $P_e^x \in SP(\widetilde{U})$. *A collection* $N_S(P_e^a)$ *of soft points containing the soft point* P_e^a *is called soft S-neighbourhood of the soft point* P_e^x *if there exists a positive soft real number* $\widetilde{r}$ *such that*

$$P_e^x \in B_S(P_e^x, \widetilde{r}) \subset N_S(P_e^a).$$

Theorem 3. *Every soft open S-ball is a soft S-neighbourhood of each of its soft points.*

Proof. Let $\left(\widetilde{U}, \widetilde{S}, E\right)$ be a soft S-metric space and $B_S(P_{e_1}^x, \widetilde{r})$ be a soft open S-ball with center $P_{e_1}^x$ and radius $\widetilde{r}$. By Definition 21, $B_S(P_{e_1}^x, \widetilde{r})$ is a soft S-neighbourhood of the soft point $P_{e_1}^x$ in $\left(\widetilde{U}, \widetilde{S}, E\right)$.

Let us consider any soft point $P_{e_2}^y \in B_S(P_{e_1}^x, \widetilde{r})$ such that $P_{e_1}^x \neq P_{e_2}^y$. Then we have

$$\overline{0} \neq \widetilde{S}(P_{e_2}^y, P_{e_2}^y, P_{e_1}^x) \widetilde{<} \widetilde{r}.$$

If we choose $\widetilde{\rho}$ such that

$$\overline{0} \widetilde{<} \widetilde{\rho} \widetilde{<} \frac{\widetilde{r} - \widetilde{S}(P_{e_2}^y, P_{e_2}^y, P_{e_1}^x)}{2},$$

then $\widetilde{\rho}$ is a positive soft real number. For $P_{e_3}^z \in B_S(P_{e_2}^y, \widetilde{\rho})$, we have

$$\widetilde{S}(P_{e_3}^z, P_{e_3}^z, P_{e_2}^y) \widetilde{<} \widetilde{\rho}.$$

Using the condition $(\widetilde{S}3)$ and Lemma 1, we get

$$\widetilde{S}(P_{e_3}^z, P_{e_3}^z, P_{e_1}^x) \widetilde{<} 2\widetilde{S}(P_{e_3}^z, P_{e_3}^z, P_{e_2}^y) + \widetilde{S}(P_{e_1}^x, P_{e_1}^x, P_{e_2}^y) \widetilde{<} \widetilde{r},$$

that is

$$P_{e_3}^z \in B_S(P_{e_2}^y, \widetilde{\rho}) \subset B_S(P_{e_1}^x, \widetilde{r}).$$

Consequently, $B_S(P_{e_1}^x, \widetilde{r})$ is a soft S-neighbourhood of its soft points.

Definition 22. Let $\left(\widetilde{U}, \widetilde{S}, E\right)$ be a soft S-metric space and $\left\{P_{e,n}^x\right\}_n$ be a sequence of soft points in $\widetilde{U}$. The sequence $\left\{P_{e,n}^x\right\}_n$ is called soft S-convergent in $\widetilde{U}$ if there is a soft point $P_\alpha^\beta \in SP(\widetilde{U})$ such that

$$\widetilde{S}(P_{e,n}^x, P_{e,n}^x, P_\alpha^\beta) \to \overline{0},$$

as $n \to \infty$. That is, for every $\widetilde{\varepsilon} \widetilde{\geq} \overline{0}$, there exists a natural number $N = N(\widetilde{\varepsilon})$ such that

$$\overline{0} \widetilde{\leq} \widetilde{S}(P_{e,n}^x, P_{e,n}^x, P_\alpha^\beta) \widetilde{<} \widetilde{\varepsilon},$$

whenever $n \geq N$. Then we get

$$P_{e,n}^x \in B_S(P_\alpha^\beta, \widetilde{\varepsilon}).$$

We denote this by

$$\lim_{n \to \infty} P_{e,n}^x = P_\alpha^\beta$$

or

$$P_{e,n}^x \to P_\alpha^\beta \ as \ n \to \infty.$$

Lemma 4. Let $\left(\widetilde{U}, \widetilde{S}, E\right)$ be a soft S-metric space. If the sequence $\left\{P_{e,n}^x\right\}_n$ converges to P_α^β then P_α^β is unique.

Proof. Let $\left\{P_{e,n}^x\right\}_n$ converges to P_α^β and P_μ^λ. There exist two natural numbers n_1, n_2 for every $\widetilde{\varepsilon} \, \widetilde{\geq} \, \overline{0}$ such that

$$\widetilde{S}(P_{e,n}^x, P_{e,n}^x, P_\alpha^\beta) \, \widetilde{<} \, \frac{\widetilde{\varepsilon}}{4}$$

and

$$\widetilde{S}(P_{e,n}^x, P_{e,n}^x, P_\mu^\lambda) \, \widetilde{<} \, \frac{\widetilde{\varepsilon}}{2},$$

where $n \geq n_1$, n_2. If we choose $n_0 = \max\{n_1, n_2\}$, then for each $n \geq n_0$, using the condition ($\widetilde{S}3$) and Lemma 1, we obtain

$$\widetilde{S}(P_\alpha^\beta, P_\alpha^\beta, P_\mu^\lambda) \, \widetilde{\leq} \, 2\widetilde{S}(P_\alpha^\beta, P_\alpha^\beta, P_{e,n}^x) + \widetilde{S}(P_\mu^\lambda, P_\mu^\lambda, P_{e,n}^x) \, \widetilde{<} \, \widetilde{\varepsilon}.$$

Therefore we get

$$\widetilde{S}(P_\alpha^\beta, P_\alpha^\beta, P_\mu^\lambda) = \overline{0}$$

and using the condition ($\widetilde{S}2$),

$$P_\alpha^\beta = P_\mu^\lambda.$$

Consequently, the limit of $\left\{P_{e,n}^x\right\}_n$ is unique.

Lemma 5. *Let $\left(\widetilde{U}, \widetilde{S}, E\right)$ be a soft S-metric space. If there exist sequences $\left\{P_{e,n}^x\right\}_n$ and $\{P_{e,n}^y\}_n$ such that*

$$\lim_{n \to \infty} P_{e,n}^x = P_\alpha^\beta$$

and

$$\lim_{n \to \infty} P_{e,n}^y = P_\mu^\lambda,$$

then we get

$$\lim_{n \to \infty} \widetilde{S}(P_{e,n}^x, P_{e,n}^x, P_{e,n}^y) = \widetilde{S}(P_\alpha^\beta, P_\alpha^\beta, P_\mu^\lambda).$$

Proof. Using the hypothesis, for each $\widetilde{\varepsilon} \, \widetilde{\geq} \, \overline{0}$, there exist two natural numbers n_1, n_2 such that

$$\widetilde{S}(P_{e,n}^x, P_{e,n}^x, P_\alpha^\beta) \, \widetilde{<} \, \frac{\widetilde{\varepsilon}}{4}$$

and

$$\widetilde{S}(P_{e,n}^y, P_{e,n}^y, P_\mu^\lambda) \, \widetilde{<} \, \frac{\widetilde{\varepsilon}}{4},$$

where $n \geq n_1$, n_2. If we choose $n_0 = \max\{n_1, n_2\}$, then for each $n \geq n_0$, using the condition ($\widetilde{S}3$) we have

$$\widetilde{S}(P_{e,n}^x, P_{e,n}^x, P_{e,n}^y) \, \widetilde{\leq} \, 2\widetilde{S}(P_{e,n}^x, P_{e,n}^x, P_\alpha^\beta) + \widetilde{S}(P_{e,n}^y, P_{e,n}^y, P_\alpha^\beta)$$
$$\widetilde{\leq} \, 2\widetilde{S}(P_{e,n}^x, P_{e,n}^x, P_\alpha^\beta) + 2\widetilde{S}(P_{e,n}^y, P_{e,n}^y, P_\mu^\lambda) + \widetilde{S}(P_\alpha^\beta, P_\alpha^\beta, P_\mu^\lambda)$$
$$\widetilde{<} \, \frac{\widetilde{\varepsilon}}{2} + \frac{\widetilde{\varepsilon}}{2} + \widetilde{S}(P_\alpha^\beta, P_\alpha^\beta, P_\mu^\lambda).$$

Therefore we get

$$\widetilde{S}(P_{e,n}^x, P_{e,n}^x, P_{e,n}^y) - \widetilde{S}(P_\alpha^\beta, P_\alpha^\beta, P_\mu^\lambda) \overset{\sim}{\lesssim} \widetilde{\varepsilon}. \tag{5}$$

On the other hand, using the condition $(\widetilde{S}3)$ and Lemma 1, we find

$$\begin{aligned}
\widetilde{S}(P_\alpha^\beta, P_\alpha^\beta, P_\mu^\lambda) &\overset{\sim}{\lesssim} 2\widetilde{S}(P_\alpha^\beta, P_\alpha^\beta, P_{e,n}^x) + \widetilde{S}(P_\mu^\lambda, P_\mu^\lambda, P_{e,n}^x) \\
&\overset{\sim}{\lesssim} 2\widetilde{S}(P_\alpha^\beta, P_\alpha^\beta, P_{e,n}^x) + 2\widetilde{S}(P_\mu^\lambda, P_\mu^\lambda, P_{e,n}^y) + \widetilde{S}(P_{e,n}^x, P_{e,n}^x, P_{e,n}^y) \\
&\overset{\sim}{\lesssim} \frac{\widetilde{\varepsilon}}{2} + \frac{\widetilde{\varepsilon}}{2} + \widetilde{S}(P_{e,n}^x, P_{e,n}^x, P_{e,n}^y).
\end{aligned}$$

Hence we obtain

$$\widetilde{S}(P_\alpha^\beta, P_\alpha^\beta, P_\mu^\lambda) - \widetilde{S}(P_{e,n}^x, P_{e,n}^x, P_{e,n}^y) \overset{\sim}{\lesssim} \widetilde{\varepsilon}. \tag{6}$$

Using the inequalities (5) and (6) we have

$$\left| \widetilde{S}(P_{e,n}^x, P_{e,n}^x, P_{e,n}^y) - \widetilde{S}(P_\alpha^\beta, P_\alpha^\beta, P_\mu^\lambda) \right| \overset{\sim}{\lesssim} \widetilde{\varepsilon}$$

and so

$$\lim_{n\to\infty} \widetilde{S}(P_{e,n}^x, P_{e,n}^x, P_{e,n}^y) = \widetilde{S}(P_\alpha^\beta, P_\alpha^\beta, P_\mu^\lambda).$$

Definition 23. *Let $\left(\widetilde{U}, \widetilde{S}, E\right)$ be a soft S-metric space and $\left\{P_{e,n}^x\right\}_n$ be a sequence of soft points in $\widetilde{U}$. The sequence $\left\{P_{e,n}^x\right\}_n$ is called soft S-bounded if there exists a positive soft real number $\widetilde{R} \overset{\sim}{>} \overline{0}$ such that*

$$\widetilde{S}(P_{e,n}^x, P_{e,n}^x, P_{e,m}^x) \overset{\sim}{\lesssim} \widetilde{R},$$

for each $m, n \in \mathbb{N}$.

Definition 24. *Let $\left(\widetilde{U}, \widetilde{S}, E\right)$ be a soft S-metric space and $\left\{P_{e,n}^x\right\}_n$ be a sequence of soft points in $\widetilde{U}$. The sequence $\left\{P_{e,n}^x\right\}_n$ is called soft S-Cauchy sequence in $\widetilde{U}$ if*

$$\widetilde{S}(P_{e,n}^x, P_{e,n}^x, P_{e,m}^x) \to \overline{0} \ as \ m, n \to \infty,$$

that is, for every $\widetilde{\varepsilon} \overset{\sim}{\geq} \overline{0}$, there exists a natural number n_0 such that

$$\widetilde{S}(P_{e,n}^x, P_{e,n}^x, P_{e,m}^x) \overset{\sim}{\lesssim} \widetilde{\varepsilon},$$

whenever $n, m \geq n_0$.

Definition 25. *A soft S-metric space $\left(\widetilde{U}, \widetilde{S}, E\right)$ is called complete if every soft S-Cauchy sequence in $\widetilde{U}$ converges to some soft points of $\widetilde{U}$.*

Lemma 6. Let $\left(\widetilde{U}, \widetilde{S}, E\right)$ be a soft S-metric space and $\left\{P^x_{e,n}\right\}_n$ be a sequence of soft points in $\widetilde{U}$. If the sequence $\left\{P^x_{e,n}\right\}_n$ converges to P^β_α, then it is a soft S-Cauchy sequence.

Proof. Using the hypothesis, for each $\widetilde{\varepsilon} \widetilde{\geq} \overline{0}$, there exist two natural numbers n_1, n_2 such that

$$\widetilde{S}(P^x_{e,n}, P^x_{e,n}, P^\beta_\alpha) \widetilde{<} \frac{\widetilde{\varepsilon}}{4}$$

and

$$\widetilde{S}(P^x_{e,m}, P^x_{e,m}, P^\beta_\alpha) \widetilde{<} \frac{\widetilde{\varepsilon}}{2},$$

where $n \geq n_1$, $m \geq n_2$. If we choose $n_0 = \max\{n_1, n_2\}$, then for each $n, m \geq n_0$, using the condition $(\widetilde{S}3)$ we find

$$\widetilde{S}(P^x_{e,n}, P^x_{e,n}, P^x_{e,m}) \widetilde{\leq} 2\widetilde{S}(P^x_{e,n}, P^x_{e,n}, P^\beta_\alpha) + \widetilde{S}(P^x_{e,m}, P^x_{e,m}, P^\beta_\alpha) \widetilde{<} \widetilde{\varepsilon}.$$

Therefore $\left\{P^x_{e,n}\right\}_n$ is a soft S-Cauchy sequence.

Corollary 7. *Every soft S-Cauchy sequence is soft S-bounded.*

Corollary 8. Let $\left(\widetilde{U}, \widetilde{d}, E\right)$ be a soft metric space and $\left(\widetilde{U}, \widetilde{S_d}, E\right)$ be a soft S-metric space which is generated by soft metric $\widetilde{d}$. Then we have

1. $\left\{P^x_{e,n}\right\}_n \to P^\beta_\alpha$ in $\left(\widetilde{U}, \widetilde{d}, E\right)$ if and only if $\left\{P^x_{e,n}\right\}_n \to P^\beta_\alpha$ in $\left(\widetilde{U}, \widetilde{S_d}, E\right)$.

2. $\left\{P^x_{e,n}\right\}_n$ is Cauchy in $\left(\widetilde{U}, \widetilde{d}, E\right)$ if and only if $\left\{P^x_{e,n}\right\}_n$ is soft S-Cauchy in $\left(\widetilde{U}, \widetilde{S_d}, E\right)$.

3. $\left(\widetilde{U}, \widetilde{d}, E\right)$ is complete if and only if $\left(\widetilde{U}, \widetilde{S_d}, E\right)$ is complete.

4. SOME FIXED-POINT RESULTS

In this section we study some fixed-point results.

Definition 26. Let $\left(\widetilde{U}, \widetilde{S}, E\right)$ be a soft S-metric space and $T : \widetilde{U} \to \widetilde{U}$ be a soft mapping. If there exists a soft point $P^\beta_\alpha \in SP(\widetilde{U})$ such that

$$T(P^\beta_\alpha) = P^\beta_\alpha,$$

then P^β_α is called a soft fixed point of T.

Definition 27. Let $\left(\widetilde{U}, \widetilde{S}, E \right)$ be a soft S-metric space and $T : \widetilde{U} \to \widetilde{U}$ be a soft mapping. Then T is called a soft S-contraction if

$$\widetilde{S}(T(P_{e_1}^x), T(P_{e_1}^x), T(P_{e_2}^y)) \widetilde{\leq} \overline{h}\widetilde{S}(P_{e_1}^x, P_{e_1}^x, P_{e_2}^y),$$

for all $P_{e_1}^x, P_{e_2}^y \in SP(\widetilde{U})$, where $\overline{0} \widetilde{\leq} \overline{h} \widetilde{<} \overline{1}$ which is called a soft S-contraction constant.

Theorem 9. Let $\left(\widetilde{U}, \widetilde{S}, E \right)$ be a complete soft S-metric space where E is nonempty finite set and T be a soft S-contraction with soft S-contraction constant $\overline{h}$. Then T has a unique soft fixed point P_α^β.

Proof. Let P_e^x be a soft point and $P_{e,0}^x = P_e^x$. We define the sequence $\left\{ P_{e,n}^x \right\}_n$ by $P_{e,n}^x = T^n(P_e^x)$. Using the soft S-contraction hypothesis, we have

$$
\begin{aligned}
\widetilde{S}(P_{e,n+1}^x, P_{e,n+1}^x, P_{e,n}^x) &= \widetilde{S}(T(P_{e,n}^x), T(P_{e,n}^x), T(P_{e,n-1}^x)) \\
&\widetilde{\leq} \overline{h}\widetilde{S}(P_{e,n}^x, P_{e,n}^x, P_{e,n-1}^x) \widetilde{\leq} \overline{h}^2 \widetilde{S}(P_{e,n-1}^x, P_{e,n-1}^x, P_{e,n-2}^x) \\
&\widetilde{\leq} \cdots \\
&\widetilde{\leq} \overline{h}^n \widetilde{S}(P_{e,1}^x, P_{e,1}^x, P_{e,0}^x) = \overline{h}^n \widetilde{S}(P_{e,1}^x, P_{e,1}^x, P_e^x).
\end{aligned}
\tag{7}
$$

For $n > m$, using the conditions $(\widetilde{S}3)$, (7) and Lemma 1, we get

$$
\begin{aligned}
\widetilde{S}(P_{e,n}^x, P_{e,n}^x, P_{e,m}^x) &\widetilde{\leq} 2\widetilde{S}(P_{e,n}^x, P_{e,n}^x, P_{e,n-1}^x) + 2\widetilde{S}(P_{e,n-1}^x, P_{e,n-1}^x, P_{e,n-2}^x) \\
&\qquad + \cdots + \widetilde{S}(P_{e,m+1}^x, P_{e,m+1}^x, P_{e,m}^x) \\
&\widetilde{\leq} 2\widetilde{S}(P_{e,n}^x, P_{e,n}^x, P_{e,n-1}^x) + 2\widetilde{S}(P_{e,n-1}^x, P_{e,n-1}^x, P_{e,n-2}^x) \\
&\qquad + \cdots + 2\widetilde{S}(P_{e,m+1}^x, P_{e,m+1}^x, P_{e,m}^x) \\
&\widetilde{\leq} 2(\overline{h}^{n-1} + \overline{h}^{n-2} + \cdots + \overline{h}^m)\widetilde{S}(P_{e,1}^x, P_{e,1}^x, P_e^x) \\
&\widetilde{\leq} \frac{2\overline{h}^m}{1-\overline{h}}\widetilde{S}(P_{e,1}^x, P_{e,1}^x, P_e^x).
\end{aligned}
$$

Now we show that the sequence $\left\{ P_{e,n}^x \right\}_n$ is a soft S-Cauchy sequence. Let us choose $\widetilde{\varepsilon} \widetilde{\geq} \overline{0}$. We can construct the parameter set

$$E = \{e_i : 1 \leq i \leq k\},$$

since E is a nonempty finite set. Therefore for each $i \in \{1, \ldots, k\}$, there exists a natural number n_i such that

$$\left[\frac{2\overline{h}^{n_i}}{1 - \overline{h}}\widetilde{S}(P_{e,1}^x, P_{e,1}^x, P_e^x) \right](e_i) < \widetilde{\varepsilon}(e_i).$$

If we take $n_0 = \max\{n_i : 1 \leq i \leq k\}$, then for any $n > m \geq n_0$ and $i \in \{1, \ldots, k\}$, we obtain

$$\widetilde{S}(P_{e,n}^x, P_{e,n}^x, P_{e,m}^x)(e_i) \leq \left[\tfrac{2\overline{h}^m}{1-\overline{h}}\widetilde{S}(P_{e,1}^x, P_{e,1}^x, P_e^x)\right](e_i)$$
$$\leq \left[\tfrac{2\overline{h}^{n_0}}{1-\overline{h}}\widetilde{S}(P_{e,1}^x, P_{e,1}^x, P_e^x)\right](e_i)$$
$$< \widetilde{\varepsilon}(e_i)$$

and so

$$\widetilde{S}(P_{e,n}^x, P_{e,n}^x, P_{e,m}^x) \precsim \widetilde{\varepsilon}.$$

Hence the sequence $\left\{P_{e,n}^x\right\}_n$ is a soft S-Cauchy sequence. Using the completeness hypothesis, there exists a soft point $P_\alpha^\beta \in SP(\widetilde{U})$ such that

$$\widetilde{S}(P_{e,n}^x, P_{e,n}^x, P_\alpha^\beta) \to \overline{0},$$

as $n \to \infty$.

Suppose that $T(P_\alpha^\beta) \neq P_\alpha^\beta$. Using the soft S-contraction hypothesis, we have

$$\widetilde{S}(T(P_\alpha^\beta), T(P_\alpha^\beta), P_{e,n+1}^x) \precsim \overline{h}\widetilde{S}(P_\alpha^\beta, P_\alpha^\beta, P_{e,n}^x).$$

If we take limit for $n \to \infty$, we get

$$\widetilde{S}(T(P_\alpha^\beta), T(P_\alpha^\beta), P_\alpha^\beta) \precsim \overline{h}\widetilde{S}(P_\alpha^\beta, P_\alpha^\beta, P_\alpha^\beta),$$

which is a contradiction. So $T(P_\alpha^\beta) = P_\alpha^\beta$.

Finally we show that the soft fixed point P_α^β is unique. Assume that P_μ^λ is another soft fixed point of T. Using the hypothesis, we have

$$\widetilde{S}(P_\alpha^\beta, P_\alpha^\beta, P_\mu^\lambda) = \widetilde{S}(T(P_\alpha^\beta), T(P_\alpha^\beta), T(P_\mu^\lambda)) \precsim \overline{h}\widetilde{S}(P_\alpha^\beta, P_\alpha^\beta, P_\mu^\lambda)$$

and so

$$P_\alpha^\beta = P_\mu^\lambda,$$

since $\overline{0} \precsim \overline{h} \precsim \overline{1}$. Consequently, P_α^β is a unique soft fixed point of T.

In the following example, we see that the parameter set E must be finite in Theorem 9.

Example 9. *Let us consider the universe set U and the parameter set E defined as in $[[1], Example\ 4.2.22,\ page\ 64]$:*

$$U = E = \left\{\frac{1}{n} : n \in \mathbb{N}\right\}.$$

Let $\widetilde{S}$ be the soft S-metric on $\widetilde{U}$ defined as Example 5. Then $\left(\widetilde{U}, \widetilde{S}, E\right)$ is a complete soft S-metric space. Let us define a soft mapping $T : \widetilde{U} \to \widetilde{U}$ as

$$T(P_e^x) = P_1^{\frac{x}{4}},$$

for all $x \in U$, $e \in E$. Now we show that T satisfies the soft S-contraction condition. If we take $x, y \in U$ and $e_1, e_2 \in E$, then for every $e \in E$ we get

$$
\begin{aligned}
\widetilde{S}(T(P_{e_1}^x), T(P_{e_1}^x), T(P_{e_2}^y))(e) &= \widetilde{S}(P_1^{\frac{x}{4}}, P_1^{\frac{x}{4}}, P_1^{\frac{y}{4}})(e) = 2\left|\frac{x}{4} - \frac{y}{4}\right| \\
&\leq \overline{h}\widetilde{S}(P_{e_1}^x, P_{e_1}^x, P_{e_2}^y)(e) = |x - y| + |e_1 - e_2|,
\end{aligned}
$$

where $\overline{h}$ is a soft S-contraction constant with $\overline{h}(e) = \frac{1}{2}$ for all $e \in E$. Consequently, the soft S-contraction condition is satisfied, but T has no soft fixed point.

Example 10. Let $U = [0, 1]$, $E = \{0, 1\}$ and $\widetilde{S}$ be the soft S-metric on $\widetilde{U}$ defined as Example 5. Then $\left(\widetilde{U}, \widetilde{S}, E\right)$ is a complete soft S-metric space. Let us define a soft mapping $T : \widetilde{U} \to \widetilde{U}$ as

$$T(P_0^x) = P_0^{\frac{x}{4}} \text{ and } T(P_1^x) = P_0^0,$$

for all $x \in U$. Now we show that T satisfies the soft S-contraction condition under the following four cases:

Case 1: Let $x, y \in U$ and $e_1 = e_2 = 0$. Then we have

$$
\begin{aligned}
\widetilde{S}(T(P_0^x), T(P_0^x), T(P_0^y)) &= \widetilde{S}(P_0^{\frac{x}{4}}, P_0^{\frac{x}{4}}, P_0^{\frac{y}{4}}) = 2\left|\frac{\overline{x}}{4} - \frac{\overline{y}}{4}\right| \\
&\widetilde{\leq} \overline{h}\widetilde{S}(P_0^x, P_0^x, P_0^y) = 2\overline{h}\left|\overline{x} - \overline{y}\right|,
\end{aligned}
$$

where $\overline{h}$ is a soft S-contraction constant with $\overline{h}(0) = \overline{h}(1) = \frac{1}{2}$.

Case 2: Let $x, y \in U$ and $e_1 = e_2 = 1$. Then we have

$$\widetilde{S}(T(P_1^x), T(P_1^x), T(P_1^y)) = \widetilde{S}(P_0^0, P_0^0, P_0^0) = \overline{0} \widetilde{\leq} \overline{h}\widetilde{S}(P_1^x, P_1^x, P_1^y),$$

where $\overline{h}$ is a soft S-contraction constant.

Case 3: Let $x, y \in U$ and $e_1 = 0$, $e_2 = 1$. Then we have

$$\widetilde{S}(T(P_0^x), T(P_0^x), T(P_1^y)) = \widetilde{S}(P_0^{\frac{x}{4}}, P_0^{\frac{x}{4}}, P_0^0) = 2\left|\frac{\overline{x}}{4} - \overline{0}\right|$$

and

$$\overline{h}\widetilde{S}(P_0^x, P_0^x, P_1^y) = 2\overline{h}\left(|\overline{x} - \overline{y}| + |\overline{0} - \overline{1}|\right).$$

For $e \in E$, we get

$$2\left|\frac{\overline{x}}{4} - \overline{0}\right|(e) = \frac{x}{2} \leq 2\overline{h}\left(|\overline{x} - \overline{y}| + |\overline{0} - \overline{1}|\right)(e) = |x - y| + 1,$$

where $\overline{h}$ is a soft S-contraction constant with $\overline{h}(0) = \overline{h}(1) = \frac{1}{2}$.

Case 4: *Let $x, y \in U$ and $e_1 = 1$, $e_2 = 0$. Then by similar arguments used in Case 3 we can see that the soft S-contraction condition is satisfied.*

Consequently, the soft S-contraction condition is satisfied for $\overline{h} = \frac{1}{2}$ in all cases and P_0^0 is the unique soft fixed point of T.

Now we give a generalization of Theorem 9.

Theorem 10. *Let $\left(\widetilde{U}, \widetilde{S}, E\right)$ be a complete soft S-metric space where E is nonempty finite set and T be a soft mapping satisfying the following condition:*

There exist soft real numbers $\overline{a}, \overline{b}$ satisfying $\overline{a} + 3\overline{b} \overset{\sim}{<} \overline{1}$ with $\overline{a}, \overline{b} \overset{\sim}{\geq} \overline{0}$ such that

$$\widetilde{S}(T(P_{e_1}^x), T(P_{e_1}^x), T(P_{e_2}^y)) \overset{\sim}{\leq} \overline{a}\widetilde{S}(P_{e_1}^x, P_{e_1}^x, P_{e_2}^y) + \overline{b}\max\{\widetilde{S}(T(P_{e_1}^x), T(P_{e_1}^x), P_{e_1}^x), \tag{8}$$
$$\widetilde{S}(T(P_{e_1}^x), T(P_{e_1}^x), P_{e_2}^y), \widetilde{S}(T(P_{e_2}^y), T(P_{e_2}^y), P_{e_2}^y), \widetilde{S}(T(P_{e_2}^y), T(P_{e_2}^y), P_{e_1}^x)\},$$

for all $P_{e_1}^x, P_{e_2}^y \in SP(\widetilde{U})$. Then T has a unique soft fixed point P_α^β.

Proof. Let P_e^x be a soft point and $P_{e,0}^x = P_e^x$. We define the sequence $\left\{P_{e,n}^x\right\}_n$ by $P_{e,n}^x = T^n(P_e^x)$. Suppose that $\left\{P_{e,n}^x\right\}_n \neq \left\{P_{e,n+1}^x\right\}_n$ for all n. Using the condition (8), we have

$$
\begin{aligned}
\widetilde{S}(P_{e,n}^x, P_{e,n}^x, P_{e,n+1}^x) &= \widetilde{S}(T(P_{e,n-1}^x), T(P_{e,n-1}^x), T(P_{e,n}^x)) \overset{\sim}{\leq} \overline{a}\widetilde{S}(P_{e,n-1}^x, P_{e,n-1}^x, P_{e,n}^x) \tag{9}\\
&+ \overline{b}\max\{\widetilde{S}(P_{e,n}^x, P_{e,n}^x, P_{e,n-1}^x), \widetilde{S}(P_{e,n}^x, P_{e,n}^x, P_{e,n}^x), \\
&\quad \widetilde{S}(P_{e,n+1}^x, P_{e,n+1}^x, P_{e,n}^x), \widetilde{S}(P_{e,n+1}^x, P_{e,n+1}^x, P_{e,n-1}^x)\} \\
&= \overline{a}\widetilde{S}(P_{e,n-1}^x, P_{e,n-1}^x, P_{e,n}^x) + \overline{b}\max\{\widetilde{S}(P_{e,n}^x, P_{e,n}^x, P_{e,n-1}^x), \\
&\quad \widetilde{S}(P_{e,n+1}^x, P_{e,n+1}^x, P_{e,n}^x), \widetilde{S}(P_{e,n+1}^x, P_{e,n+1}^x, P_{e,n-1}^x)\}
\end{aligned}
$$

By the condition $(\widetilde{S}3)$, we obtain

$$\widetilde{S}(P_{e,n+1}^x, P_{e,n+1}^x, P_{e,n-1}^x) \overset{\sim}{\leq} 2\widetilde{S}(P_{e,n+1}^x, P_{e,n+1}^x, P_{e,n}^x) + \widetilde{S}(P_{e,n-1}^x, P_{e,n-1}^x, P_{e,n}^x). \tag{10}$$

Using the conditions (9) and (10), we get

$$
\begin{aligned}
\widetilde{S}(P_{e,n}^x, P_{e,n}^x, P_{e,n+1}^x) &\overset{\sim}{\leq} \overline{a}\widetilde{S}(P_{e,n-1}^x, P_{e,n-1}^x, P_{e,n}^x) \\
&+ 2\overline{b}\widetilde{S}(P_{e,n+1}^x, P_{e,n+1}^x, P_{e,n}^x) + \overline{b}\widetilde{S}(P_{e,n-1}^x, P_{e,n-1}^x, P_{e,n}^x)
\end{aligned}
$$

and so
$$(1 - 2\bar{b})\widetilde{S}(P^x_{e,n}, P^x_{e,n}, P^x_{e,n+1}) \stackrel{\sim}{\leq} (\bar{a} + \bar{b})\widetilde{S}(P^x_{e,n-1}, P^x_{e,n-1}, P^x_{e,n}),$$

which implies

$$\widetilde{S}(P^x_{e,n}, P^x_{e,n}, P^x_{e,n+1}) \stackrel{\sim}{\leq} \frac{\bar{a} + \bar{b}}{1 - 2\bar{b}}\widetilde{S}(P^x_{e,n-1}, P^x_{e,n-1}, P^x_{e,n}). \tag{11}$$

If we take $\bar{h} = \frac{\bar{a}+\bar{b}}{1-2\bar{b}}$, then we get $\bar{h} \stackrel{\sim}{<} \bar{1}$ since $\bar{a} + 3\bar{b} \stackrel{\sim}{<} \bar{1}$. Using the inequality (11), we have

$$\widetilde{S}(P^x_{e,n}, P^x_{e,n}, P^x_{e,n+1}) \stackrel{\sim}{\leq} \bar{h}^n \widetilde{S}(P^x_e, P^x_e, P^x_{e,1}). \tag{12}$$

Now we show that the sequence $\left\{P^x_{e,n}\right\}_n$ is a soft S-Cauchy sequence. For $n > m$, using the conditions $(\widetilde{S}3)$ and (12), we obtain

$$\widetilde{S}(P^x_{e,n}, P^x_{e,n}, P^x_{e,m}) \stackrel{\sim}{\leq} \frac{2\bar{h}^m}{1 - \bar{h}}\widetilde{S}(P^x_{e,1}, P^x_{e,1}, P^x_e).$$

By the similar arguments used in the proof of Theorem 9, we see that the sequence $\left\{P^x_{e,n}\right\}_n$ is a soft S-Cauchy sequence. Since $\left(\widetilde{U}, \widetilde{S}, E\right)$ is complete, there exists a soft point $P^\beta_\alpha \in SP(\widetilde{U})$ such that

$$\widetilde{S}(P^x_{e,n}, P^x_{e,n}, P^\beta_\alpha) \to \bar{0},$$

as $n \to \infty$.

Suppose that $T(P^\beta_\alpha) \neq P^\beta_\alpha$. Using the inequality (8), we get

$$
\begin{aligned}
\widetilde{S}(P^x_{e,n}, P^x_{e,n}, T(P^\beta_\alpha)) &= \widetilde{S}(T(P^x_{e,n-1}), T(P^x_{e,n-1}), T(P^\beta_\alpha)) \stackrel{\sim}{\leq} \bar{a}\widetilde{S}(P^x_{e,n-1}, P^x_{e,n-1}, P^\beta_\alpha) \\
&+ \bar{b}\max\{\widetilde{S}(P^x_{e,n}, P^x_{e,n}, P^x_{e,n-1}), \widetilde{S}(P^x_{e,n}, P^x_{e,n}, P^\beta_\alpha), \\
&\quad \widetilde{S}(T(P^\beta_\alpha), T(P^\beta_\alpha), P^\beta_\alpha), \widetilde{S}(T(P^\beta_\alpha), T(P^\beta_\alpha), P^x_{e,n-1})\}.
\end{aligned}
$$

If we take limit for $n \to \infty$, then using Lemma 1, we have

$$\widetilde{S}(P^\beta_\alpha, P^\beta_\alpha, T(P^\beta_\alpha)) \stackrel{\sim}{\leq} \bar{b}\widetilde{S}(T(P^\beta_\alpha), T(P^\beta_\alpha), P^\beta_\alpha),$$

which is a contradiction. So $T(P^\beta_\alpha) = P^\beta_\alpha$.

Finally we show that the soft fixed point P^β_α is unique. Assume that P^λ_μ is another soft fixed point of T. Using the inequality (8), we obtain

$$
\begin{aligned}
\widetilde{S}(P^\beta_\alpha, P^\beta_\alpha, P^\lambda_\mu) &= \widetilde{S}(T(P^\beta_\alpha), T(P^\beta_\alpha), T(P^\lambda_\mu)) \stackrel{\sim}{\leq} \bar{a}\widetilde{S}(P^\beta_\alpha, P^\beta_\alpha, P^\lambda_\mu) \\
&+ \bar{b}\max\{\widetilde{S}(P^\beta_\alpha, P^\beta_\alpha, P^\beta_\alpha), \widetilde{S}(P^\beta_\alpha, P^\beta_\alpha, P^\lambda_\mu), \\
&\quad \widetilde{S}(P^\lambda_\mu, P^\lambda_\mu, P^\lambda_\mu), \widetilde{S}(P^\lambda_\mu, P^\lambda_\mu, P^\beta_\alpha)\},
\end{aligned}
$$

and so $P^\beta_\alpha = P^\lambda_\mu$ since $\bar{a} + \bar{b} \stackrel{\sim}{<} \bar{1}$. Consequently, P^β_α is a unique soft fixed point of T.

Notice that Theorem 10 is a generalization of Theorem 9 on a complete soft S-metric space. Indeed, if we choose $\overline{0} \;\widetilde{\leq}\; \overline{a} = \overline{h} \;\widetilde{<}\; \overline{1}$ and $\overline{b} = \overline{0}$ in Theorem 10, then we obtain Theorem 9.

Example 11. *Let $U = \mathbb{R}$, $E = \{0,1\}$ and $\widetilde{S}$ be the soft S-metric on $\widetilde{U}$ defined as Example 5. Then $\left(\widetilde{U}, \widetilde{S}, E \right)$ is a complete soft S-metric space. Let us define a soft mapping $T : \widetilde{U} \to \widetilde{U}$ as*

$$T(P_0^x) = P_0^{x+50} \ \text{if } x \in \{0,2\},$$

$$T(P_0^x) = P_0^{45} \ \text{if } x \in \mathbb{R} - \{0,2\}$$

and

$$T(P_1^x) = P_0^{45},$$

for all $x \in \mathbb{R}$. Then T satisfies the condition (8) for $\overline{a} = \overline{0}$ and $\overline{b} = \overline{\dfrac{1}{5}}$. Therefore P_0^{45} is the unique soft fixed point of T. But the soft S-contraction condition is not satisfied by T. Indeed, for $x = 1$, $y = 0$, $e_1 = e_2 = 0$, we find

$$
\begin{aligned}
\widetilde{S}(T(P_0^1), T(P_0^1), T(P_0^0))(e) &= \widetilde{S}(P_0^{45}, P_0^{45}, P_0^{50})(e) = 10 \\
&\leq \overline{h}\widetilde{S}(P_0^1, P_0^1, P_0^0)(e) = 2h,
\end{aligned}
$$

for all $e \in E$ and $\overline{h}$ with $\overline{h}(e) = h$, which is a contradiction since $\overline{0} \;\widetilde{\leq}\; \overline{h} \;\widetilde{<}\; \overline{1}$.

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