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APPLICATIONS OF BESSEL AND STRUVE FUNCTIONS ON GENERAL INTEGRAL OPERATORS

D. Breaz, C.-I. Tănase

Abstract. Using Bessel and Struve functions will find some univalent conditions for general integral operators. Also we obtain some particular cases.

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1. Introduction and preliminaries

Let

$$U(z_0, r) = \{z \in \mathbb{C} : |z - z_0| < r\}$$

be the disc with center z_0 and and of radius r, the particular case $U(0,1)$ will be denote by U. Let $H(U)$ be the set of functions which are regular in the unit disc U. Consider $A = \{f \in H(U) : f(z) = z + a_2 z^2 + a_3 z^3 + ..., z \in U\}$ be the class of analytic functions in U and $S = \{f \in A : f \text{ is univalent in } U\}$

Theorem 1.1. [5] Let α be a complex number, $\operatorname{Re}\alpha > 0$, and $f(z) = z + a_2 z^2 + ...$ be a regular function in U. If

$$\frac{1 - |z|^{\operatorname{Re}\alpha}}{\operatorname{Re}\alpha} \left| \frac{z f''(z)}{f'(z)} \right| \leq 1 \tag{1}$$

for all $z \in U$, then for any complex number $\beta, \operatorname{Re}\beta \geq \operatorname{Re}\alpha$, the function

$$F_\beta(z) = \left[\int_0^z t^{\beta - 1} f'(t) dt \right]^{\frac{1}{\beta}} \tag{2}$$

is in the class S.

Theorem 1.2. [3] If the function g is regular in U and $|g(z)| < 1$ in U, then for all $\xi \in U$and $z \in U$ the following inequalities hold

$$\left|\frac{g(\xi) - g(z)}{1 - \overline{g(z)} \cdot g(\xi)}\right| \leq \left|\frac{\xi - z}{1 - \overline{z} \cdot \xi}\right| \tag{3}$$

and

$$|g^{'}(z)| \leq \frac{1 - |g(z)|^2}{1 - z^2} \tag{4}$$

the equalities hold in case $g(z) = \varepsilon \dfrac{z + u}{1 + \overline{u}z}$ where $|\varepsilon| = 1$ and $|u| < 1$.

Remark 1.1. [2] For $z = 0$ from inequality (3) we obtain for every $\xi \in U$

$$\left|\frac{g(\xi) - g(0)}{1 - \overline{g(0)}g(\xi)}\right| \leq |\xi| \tag{5}$$

and hence

$$|g(\xi)| \leq \frac{|\xi| + |g(0)|}{1 + |g(0)||\xi|}. \tag{6}$$

Considering $g(0) = a$ and $\xi = z$, then

$$|g(z)| \leq \frac{|z| + |a|}{1 + |a||z|} \tag{7}$$

for all $z \in U$.

Let us consider the second-order inhomogeneous differential equation(([7]), p.341)

$$z^2 w^{''}(z) + zw^{'}(z) + (z^2 - v^2)w(z) = \frac{4(\frac{z}{2})^{v+1}}{\sqrt{\pi}\Gamma(v + \frac{1}{2})} \tag{8}$$

whose homogeneous part is Bessel's equation, where v is an unrestricted real(or complex) number. The function H_v, which is called the Struve function of order v, is defined as a particular solution of (8). This function has the form

$$H_v(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n + \frac{3}{2}) \cdot \Gamma(v + n + \frac{3}{2})} \cdot \left(\frac{z}{2}\right)^{2n+v+1} \quad \text{for all } z \in \mathbb{C} \tag{9}$$

We consider the transformation

$$g_v(z) = 2^v\sqrt{\pi}\Gamma(v+\frac{3}{2})\cdot z^{\frac{-v-1}{2}} H_v(\sqrt{z}) \tag{10}$$

After some calculus we obtain

$$g_v(z) = \sum_{n=0}^{\infty} \frac{(-1)^n\Gamma(\frac{3}{2})\Gamma(v+\frac{3}{2})}{4^n\cdot\Gamma(n+\frac{3}{2})\Gamma(v+n+\frac{3}{2})}\cdot z^n \tag{11}$$

Let $u_v(z) = z\cdot g_v(z) = \sum_{n=0}^{\infty} \frac{(-1)^n\Gamma(\frac{3}{2})\Gamma(v+\frac{3}{2})}{4^n\cdot\Gamma(n+\frac{3}{2})\Gamma(v+n+\frac{3}{2})}\cdot z^{n+1}$.

Using Theorem 2.1 ([4]) for our case with $b = c = 1, \kappa = v + \frac{3}{2}$ we obtain that:

Theorem 1.3. [4] If $v > \frac{\sqrt{3}-7}{8}$ then the function g_v is univalent in U.

The Bessel function of the first kind is defined by

$$J_v(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(n+v+1)}\left(\frac{z}{2}\right)^{2n+v}. \tag{12}$$

We consider the transformation

$$f_v(z) = 2^v\Gamma(1+v)z^{-\frac{v}{2}} J_v(\sqrt{z}) \tag{13}$$

After some calculus we obtain

$$f_v(z) = \sum_{n=0}^{\infty} \frac{(-1)^n\Gamma(1+v)}{n!\Gamma(n+v+1)\cdot 4^n}\cdot z^n. \tag{14}$$

Let $h_v(z) = z\cdot f_v(z) = \sum_{n=0}^{\infty} \frac{(-1)^n\Gamma(1+v)}{n!\Gamma(n+v+1)\cdot 4^n}\cdot z^{n+1}$.

Theorem 1.4. [6] If $v > -2$ then $Ref_v'(z) < 0$ for $z \in U_1(0, 4(v+2))$ and f_v is univalent in $U_1(0, 4(v+2))$.

2. Main results

Theorem 2.1. Let $\alpha, \gamma_i \in \mathbb{C}, \text{Re}\alpha = b > 0, u_{v_i}$ $u_{v_i}(z) = z + a_2^1 z^2 + ...n \in \mathbb{N}^*, i \in \{1, 2, ..., n\}$. If

$$\left|\frac{u''_{v_i}(z)}{u'_{v_i}(z)}\right| < 1, \ (\forall)\ z \in U, \ (\forall)\ i \in \{1, 2, ...\} \tag{15}$$

and

$$\frac{|\gamma_1| + |\gamma_2| + ... + |\gamma_n|}{|\gamma_1 \cdot \gamma_2 ... \cdot \gamma_n|} \leq 1, \tag{16}$$

$$|\gamma_1 \cdot \gamma_2 \cdot ... \cdot \gamma_n| \leq \frac{1}{\max\limits_{|z| \leq 1} \left[\frac{1 - |z|^{2b}}{b} \cdot |z| \cdot \frac{|z| + 2|c|}{1 + 2|c| \cdot |z|}\right]} \tag{17}$$

where $c = \dfrac{\gamma_1 \cdot a_2^1 + ...\gamma_n a_2^n}{|\gamma_1 \cdot \gamma_2 \cdot ...\gamma_n|}$

$= \dfrac{-1}{3|\gamma_1 \cdot \gamma_2 \cdot ...\gamma_n|} \left[\dfrac{\gamma_1}{2v_1 + 3} + \dfrac{\gamma_2}{2v_2 + 3} + ... + \dfrac{\gamma_n}{2v_n + 3}\right]$

then for every $\beta \in \mathbb{C}, \text{Re}\beta \geq b$ the function

$$H(z) = \left\{\beta \int_0^z t^{\beta - 1} \cdot [u'_{v_1}(t)]^{\gamma_1} \cdot ...[u'_{v_n}(t)]^{\gamma_n}\right\}^{\frac{1}{\beta}} dt$$

is univalent

Proof:
Let the function h_1 defined by

$$h_1(z) = \int_0^z [u'_{v_1}(t)]^{\gamma_1} \cdot [u'_{v_2}(t)]^{\gamma_2} \cdot ... \cdot [u'_{v_n}(t)]^{\gamma_n} dt$$

and the function p defined by

$$p(z) = \frac{1}{|\gamma_1 \cdot \gamma_2 \cdot ... \cdot \gamma_n|} \cdot \frac{h''_1(z)}{h'_1(z)}$$

After some calculus we have that:

$$p(z) = \frac{\gamma_1}{|\gamma_1 \cdot \gamma_2 \cdot ...\gamma_n|} \cdot \frac{u''_{v_1}(z)}{u'_{v_1}(z)} + ... + \frac{\gamma_1}{|\gamma_1 \cdot \gamma_2 \cdot ...\gamma_n|} \cdot \frac{u''_{v_n}(z)}{u'_{v_n}(z)}$$

Using relation (15) and (16) we obtain that

$$|p(z)| < 1,\ (\forall)z \in U$$

and

$$p(0) = \frac{\gamma_1 \cdot a_2^1 + ...\gamma_n \cdot a_2^n}{|\gamma_1 \cdot ... \cdot \gamma_n|} = c$$

where

$$a_2^i = \frac{-1}{3(2v_i + 3)}$$

for all $i \in \{1, 2, ..., n\}$.
Applying Remark 1.1 for the function p we obtain

$$|p(z)| \leq \frac{|z| + 2|c|}{1 + 2|c| \cdot |z|},\ (\forall)z \in U.$$

Then

$$\frac{1}{|\gamma_1 \cdot \gamma_2 \cdot ... \cdot \gamma_n|} \cdot \frac{h_1''(z)}{h_1'(z)} \leq \frac{|z| + 2|c|}{1 + 2|c| \cdot |z|},\ \ (\forall)z \in U$$

$$\Rightarrow \frac{1}{|\gamma_1 \cdot \gamma_2 \cdot ... \cdot \gamma_n|} \cdot \frac{1 - |z|^{2b}}{b} \cdot \left|\frac{zh_1''(z)}{h_1'(z)}\right| \leq \left[\frac{1 - |z|^{2b}}{b} \cdot |z| \cdot \frac{|z| + 2|c|}{1 + 2|c| \cdot |z|}\right]\ \ (\forall)z \in U.$$

$$\Rightarrow \frac{1 - |z|^{2b}}{b} \cdot \left|\frac{zh_1''(z)}{h_1'(z)}\right| \leq |\gamma_1 \cdot \gamma_2 \cdot ... \cdot \gamma_n| \cdot \max_{|z| \leq 1}\left[\frac{1 - |z|^{2b}}{b} \cdot |z| \cdot \frac{|z| + 2|c|}{1 + 2|c| \cdot |z|}\right]$$

Using relation (17) we obtain that

$$\frac{1 - |z|^{2b}}{b} \cdot \left|\frac{zh_1''(z)}{h_1'(z)}\right| \leq 1\ (\forall)z \in U$$

From Theorem 1.1 results that H is univalent.

Corollary 2.1. Let $\alpha, \gamma \in \mathbb{C}, \mathrm{Re}\alpha = b > 0, u_v$ function.
If

$$\left|\sum_{n=0}^{\infty} \frac{n(n+1) \cdot z^{n-1}}{4^n \Gamma(n + \frac{3}{2}) \cdot \Gamma(v + n + \frac{3}{2})}\right| \leq \left|\sum_{n=0}^{\infty} \frac{(n+1) \cdot z^{n}}{4^n \Gamma(n + \frac{3}{2}) \cdot \Gamma(v + n + \frac{3}{2})}\right|, (\forall)z \in U. \tag{18}$$

and

$$|\gamma| \leq \frac{1}{\max\limits_{|z|\leq 1}\left[\frac{1-|z|^{2b}}{b}\cdot|z|\cdot\frac{|z|+2|c|}{1+2|c|\cdot|z|}\right]} \tag{19}$$

where $c = \frac{-\gamma_1}{|\gamma_1|\cdot 3\cdot(2v+3)}$

then for every $\beta\in\mathbb{C}, \mathrm{Re}\geq b$ the function

$$H(z) = \left\{\beta\int_0^z t^{\beta-1}\cdot[f_v'(t)]^{\gamma}\right\}^{\frac{1}{\beta}} dt$$

is univalent.

For $v=0$ the Corollary 2.1 will become:
Let $\alpha,\gamma\in\mathbb{C}, \mathrm{Re}\alpha = b>0, u_0$ Struve function.
If

$$\left|\sum_{n=0}^{\infty}\frac{(n!)^2\cdot n\cdot(n+1)\cdot 2^{2n+2}\cdot z^{n-1}}{((2n+1)!)^2}\right| \leq \left|\sum_{n=0}^{\infty}\frac{(n!)^2\cdot(n+1)\cdot 2^{2n+2}\cdot z^{n}}{((2n+1)!)^2}\right|, (\forall) z\in U. \tag{20}$$

and

$$|\gamma| \leq \frac{1}{\max\limits_{|z|\leq 1}\left[\frac{1-|z|^{2b}}{b}\cdot|z|\cdot\frac{9\cdot|z|+2}{9+2\cdot|z|}\right]} \tag{21}$$

then for every $\beta\in\mathbb{C}, \mathrm{Re}\geq b$ the function

$$H(z) = \left\{\beta\int_0^z t^{\beta-1}\cdot[u_0'(t)]^{\gamma}\right\}^{\frac{1}{\beta}} dt$$

is univalent.

Theorem 2.2. Let $\alpha,\gamma_i\in\mathbb{C}, \mathrm{Re}\alpha = b>0, h_{v_i}$ functions $h_{v_i}(z) = z + b_2^1 z^2 + ... n \in \mathbb{N}^*, i\in\{1,2,...,n\}$. If

$$\left|\frac{h_{v_i}''(z)}{h_{v_i}'(z)}\right| < 1,\ (\forall)\ z\in U,\ (\forall)\ i\in\{1,2,...\} \tag{22}$$

and

$$\frac{|\gamma_1|+|\gamma_2|+...+|\gamma_n|}{|\gamma_1\cdot\gamma_2\cdot...\cdot\gamma_n|} \leq 1, \tag{23}$$

$$|\gamma_1 \cdot \gamma_2 \cdot \ldots \cdot \gamma_n| \leq \frac{1}{\max\limits_{|z|\leq 1}\left[\frac{1-|z|^{2b}}{b} \cdot |z| \cdot \frac{|z|+2|c|}{1+2|c|\cdot|z|}\right]} \tag{24}$$

where $c = \dfrac{\gamma_1 \cdot b_2^1 + \ldots \gamma_n b_2^n}{|\gamma_1 \cdot \gamma_2 \cdot \ldots \gamma_n|}$
$= \dfrac{-1}{4|\gamma_1 \cdot \gamma_2 \cdot \ldots \gamma_n|}\left[\dfrac{\gamma_1}{1+v_1} + \dfrac{\gamma_2}{1+v_2} + \ldots + \dfrac{\gamma_n}{1+v_n}\right]$
then for every $\beta \in \mathbb{C}, \mathrm{Re}\beta \geq b$ the function

$$T(z) = \left\{\beta \int_0^z t^{\beta-1} \cdot [h'_{v_1}(t)]^{\gamma_1} \cdot \ldots \cdot [h'_{v_n}(t)]^{\gamma_n}\right\}^{\frac{1}{\beta}} dt$$

is univalent.

Proof:
Let the function t defined by

$$t(z) = \int_0^z [h'_{v_1}(t)]^{\gamma_1} \cdot [h'_{v_2}(t)]^{\gamma_2} \cdot \ldots \cdot [h'_{v_n}(t)]^{\gamma_n} dt$$

and the function p defined by

$$p(z) = \frac{1}{|\gamma_1 \cdot \gamma_2 \cdot \ldots \cdot \gamma_n|} \cdot \frac{t''(z)}{t'(z)}$$

After some calculus we have that:

$$p(z) = \frac{\gamma_1}{|\gamma_1 \cdot \gamma_2 \cdot \ldots \cdot \gamma_n|} \cdot \frac{h''_{v_1}(z)}{h'_{v_1}(z)} + \ldots + \frac{\gamma_1}{|\gamma_1 \cdot \gamma_2 \cdot \ldots \cdot \gamma_n|} \cdot \frac{h''_{v_n}(z)}{h'_{v_n}(z)}$$

Using relation (22) and (23) we obtain that

$$|p(z)| < 1, \ (\forall) z \in U$$

and

$$p(0) = \frac{\gamma_1 \cdot b_2^1 + \ldots + \gamma_n \cdot b_2^n}{|\gamma_1 \cdot \ldots \cdot \gamma_n|} = c$$

where

$$b_2^i = \frac{-1}{4(1+v_i)}$$

for all $i \in \{1, 2, ..., n\}$
Applying Remark 1.1 for the function p we obtain

$$|p(z)| \leq \frac{|z| + 2|c|}{1 + 2|c| \cdot |z|}, \ (\forall) z \in U.$$

Then

$$\frac{1}{|\gamma_1 \cdot \gamma_2 \cdot ... \cdot \gamma_n|} \cdot \frac{t''(z)}{t'(z)} \leq \frac{|z| + 2|c|}{1 + 2|c| \cdot |z|}, \ (\forall) z \in U$$

$$\Rightarrow \frac{1}{|\gamma_1 \cdot \gamma_2 \cdot ... \cdot \gamma_n|} \cdot \frac{1 - |z|^{2b}}{b} \cdot \left| \frac{zt''(z)}{t'(z)} \right| \leq \left[\frac{1 - |z|^{2b}}{b} \cdot |z| \cdot \frac{|z| + 2|c|}{1 + 2|c| \cdot |z|} \right] \ (\forall) z \in U.$$

$$\Rightarrow \frac{1 - |z|^{2b}}{b} \cdot \left| \frac{zt''(z)}{t'(z)} \right| \leq |\gamma_1 \cdot \gamma_2 \cdot ... \cdot \gamma_n| \cdot \max_{|z| \leq 1} \left[\frac{1 - |z|^{2b}}{b} \cdot |z| \cdot \frac{|z| + 2|c|}{1 + 2|c| \cdot |z|} \right]$$

Using relation (24) we obtain that

$$\frac{1 - |z|^{2b}}{b} \cdot \left| \frac{zt''(z)}{t'(z)} \right| \leq 1 \ (\forall) z \in U$$

From Theorem 1.1 results that T is univalent.

Corollary 2.2. Let $\alpha, \gamma_1 \in \mathbb{C}, \text{Re}\alpha = b > 0, h_v$ function. If

$$\left| \sum_{n=0}^{\infty} \frac{n(n+1) \cdot z^{n-1}}{n! \cdot 4^n \cdot \Gamma(n + v + 1)} \right| \leq \left| \sum_{n=0}^{\infty} \frac{(n+1) \cdot z^n}{n! \cdot 4^n \cdot \Gamma(n + v + 1)} \right|, (\forall) z \in U. \tag{25}$$

and

$$|\gamma_1| \leq \frac{1}{\max\limits_{|z| \leq 1} \left[\frac{1 - |z|^{2b}}{b} \cdot |z| \cdot \frac{|z| + 2|c|}{1 + 2|c| \cdot |z|} \right]} \tag{26}$$

where $c = \dfrac{-\gamma_1}{4|\gamma_1|(1 + v)}$ then for every $\beta \in \mathbb{C}, \text{Re}\beta \geq b$ the function

$$T(z) = \left\{ \beta \int_0^z t^{\beta - 1} \cdot [h_v'(t)]^{\gamma_1} \right\}^{\frac{1}{\beta}} dt$$

is univalent

For $v = 0$ the Corollary 2.2 will become:
Let $\alpha, \gamma_1 \in \mathbb{C}, \mathrm{Re}\alpha = b > 0, h_0$ function. If

$$\left|\sum_{n=0}^{\infty} \frac{n(n+1)\cdot z^{n-1}}{(n!)^2\cdot 4^n}\right| \leq \left|\sum_{n=0}^{\infty} \frac{(n+1)\cdot z^{n}}{(n!)^2\cdot 4^n}\right|, (\forall) z \in U. \tag{27}$$

and

$$|\gamma_1| \leq \frac{1}{\max\limits_{|z|\leq 1}\left[\frac{1-|z|^{2b}}{b}\cdot |z| \cdot \frac{2|z|+1}{|z|+2}\right]} \tag{28}$$

then for every $\beta \in \mathbb{C}, \mathrm{Re}\beta \geq b$ the function

$$T(z) = \left\{\beta \int_0^z t^{\beta-1}\cdot [h_0'(t)]^{\gamma_1}\right\}^{\frac{1}{\beta}} dt$$

is univalent.
For $v = 1$ the Corollary 2.2 will become:
Let $\alpha, \gamma_1 \in \mathbb{C}, \mathrm{Re}\alpha = b > 0, h_1$ function. If

$$\left|\sum_{n=0}^{\infty} \frac{n(n+1)\cdot z^{n-1}}{n!\cdot (n+1)!\cdot 4^n}\right| \leq \left|\sum_{n=0}^{\infty} \frac{(n+1)\cdot z^{n}}{n!\cdot (n+1)!\cdot 4^n}\right|, (\forall) z \in U. \tag{29}$$

and

$$|\gamma_1| \leq \frac{1}{\max\limits_{|z|\leq 1}\left[\frac{1-|z|^{2b}}{b}\cdot |z| \cdot \frac{4|z|+1}{|z|+4}\right]} \tag{30}$$

then for every $\beta \in \mathbb{C}, \mathrm{Re}\beta \geq b$ the function

$$T(z) = \left\{\beta \int_0^z t^{\beta-1}\cdot [h_1'(t)]^{\gamma_1}\right\}^{\frac{1}{\beta}} dt$$

is univalent.

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Daniel Breaz,
"1 Decembrie 1918" University of Alba Iulia,
N.Iorga Str., No. 11-13,
510009 Alba Iulia, Alba, Romania email: *dbrea@uab.ro*

Carmen-Ioana Tănase,
University of Piteşti,
Târgul din Vale Str., No. 1,
110040 Piteşti, Argeş, Romania