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FEKETE-SZEGÖ PROBLEM FOR SOME SUBCLASSES OF ANALYTIC FUNCTIONS

M. Tahir, B. Khan, A. Khan

Abstract. In this paper, making use of certain linear operators, we consider Fekete-Szegö problem for some subclasses of analytic functions.

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1. Introduction

Let A denote the class of all functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \qquad z \in E \tag{1}$$

which are analytic and univalent in the open disk

$$E = \{z \in \mathbb{C} : |z| < 1\}.$$

The theory of linear operators plays an important role in Geometric Function Theory. Several differential and integral operators were introduced and studied, see for example [11, 12, 13, 14, 15, 17, 22, 23].

Jung et al. [8] introduced the following one parameter families of integral operators

$$P^{\alpha} f(z) = \frac{2^{\alpha}}{z\Gamma(\alpha)} \int_0^z \left(\log \frac{z}{t}\right)^{\alpha-1} f(t)\, dt \qquad (\alpha > 0),$$

$$Q^{\alpha}_{\beta} f(z) = \binom{\alpha+\beta}{\beta} \frac{\alpha}{z^{\beta}} \int_0^z \left(1 - \frac{t}{z}\right)^{\alpha-1} t^{\beta-1} f(t)\, dt \qquad (\alpha > 0, \beta > -1),$$

where $\Gamma(\alpha)$ is the familiar Gamma function. For

$$\alpha \in N = \{1, 2, 3, ...\},$$

the operators P^{α}, Q^{α}_{β} were considered by Bernardi [1, 2].

For $f(z) \in A$ given by (1) Jung et al. [8] showed that

$$P^{\alpha}f(z) = z + \sum_{n=2}^{\infty}\left(\frac{2}{n+1}\right)^{\alpha} a_n z^n \qquad (\alpha > 0) \tag{2}$$

and

$$Q^{\alpha}_{\beta}f(z) = z + \frac{\Gamma(\alpha+\beta+1)}{\Gamma(\beta+1)}\sum_{n=2}^{\infty}\frac{\Gamma(\beta+n)}{\Gamma(\alpha+\beta+n)}a_n z^n \qquad (\alpha > 0, \beta > -1). \tag{3}$$

It is well known that

$$(P^{\alpha}f(z))' = 2P^{\alpha-1}f(z) - P^{\alpha}f(z) \tag{4}$$

and

$$z\left(Q^{\alpha}_{\beta}f(z)\right)' = (\alpha+\beta)\, Q^{\alpha-1}_{\beta}f(z) - (\alpha+\beta-1)\, Q^{\alpha}_{\beta}f(z).$$

The above defined operators were subsequently studied by many authors, see [3, 4, 7, 20, 21]. In 1993, Fekete and Szegö [5] found the maximum values of $\left|a_3 - \mu a_2^2\right|$ as a function of real parameter μ, for functions belonging to the class A. Since then, the Fekete-Szegö problem was solved for various subclasses of the class A. See for example, [9, 10, 19] and others see for example [6, 16].

In present paper, we solve the Fekete-Szegö problem for the functional $\left|a_3 - \mu a_2^2\right|$, where μ is a complex or real number, for functions f in some subclasses of analytic functions A.

Definition 1. *Let $\mathcal{P}$ be the class of all functions $p(z)$ given by*

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n, \qquad z \in E \tag{5}$$

which are analytic and have positive real part in E.

Definition 2. *A function $f(z) \in A$, is said to be in the class $R(P,\alpha,\lambda)$, if it satisfies the condition*

$$|\arg\{(1-\lambda)(P^{\alpha}f)' + \lambda(P^{\alpha-1}f)'\}| < \frac{\pi}{2}\delta \tag{6}$$

where

$$\alpha > 1, \lambda \geq 0 \quad \textit{and} \quad 0 < \delta \leq 1.$$

Definition 3. *A function $f(z) \in A$, is said to be in the class $R(Q,\alpha,\lambda)$, if it satisfies the condition*

$$|\arg\{(1-\lambda)(Q_{\beta}^{\alpha}f)' + \lambda(Q_{\beta}^{\alpha-1}f)'\}| < \frac{\pi}{2}\delta \tag{7}$$

where

$$\alpha > 1, \beta > -1, \lambda \geq 0 \quad \text{and} \quad 0 < \delta \leq 1.$$

The following result is due to Pommerenke [18].

Lemma 1. *If a function $p(z) \in \mathcal{P}$ then for $n \geq 1$*

$$i) \qquad |c_n| \leq 2 \tag{8}$$

$$ii) \qquad |c_2 - \frac{c_1^2}{2}| \leq 2 - \frac{c_1^2}{2}. \tag{9}$$

2. Main Results

First we consider the functional $\left|a_3 - \mu a_2^2\right|$ for complex parameter μ.

Theorem 2. *Let $\delta \in (o,1]$, $\alpha > 1$, $\lambda \geq 0$. If $f(z) \in R(P,\alpha,\lambda)$ and $\mu \in \mathbb{C}$. Then*

$$\left|a_3 - \mu a_2^2\right| \leq \delta \frac{2^{\alpha+1}}{3(1+\lambda)} \max\left\{1, \frac{\delta\left|2^{3\alpha-1}(2+\lambda)^2 - \mu 3^{2\alpha+1}(1+\lambda)\right|}{2^{3\alpha-1}(2+\lambda)^2}\right\}.$$

Proof. Let $f(z) \in R(P,\alpha,\lambda)$ then by using equation (6), we obtain

$$(1-\lambda)(P^{\alpha}f)' + \lambda(P^{\alpha-1}f)' = p^{\delta}(z) \tag{10}$$

where $p(z) \in \mathcal{P}$. From equation (2), we have

$$(1-\lambda)(P^{\alpha}f)' + \lambda(P^{\alpha-1}f)' = 1 + \sum_{n=2}^{\infty}\left(\frac{2}{n+1}\right)^{\alpha}\left(\frac{2+\lambda(n-1)}{2}\right) n a_n z^{n-1}.$$

Then by using (5) and above equation, (10), we have

$$1 + \sum_{n=2}^{\infty}\left(\frac{2}{n+1}\right)^{\alpha}\left(\frac{2+\lambda(n-1)}{2}\right) n a_n z^{n-1} = \left(1 + c_1 z + c_2 z^2 + ...\right)^{\delta}.$$

A simple computation gives

$$\begin{aligned} 1 + \sum_{n=2}^{\infty}\left(\frac{2}{n+1}\right)^{\alpha}\left(\frac{2+\lambda(n-1)}{2}\right) n a_n z^{n-1} &= 1 + \delta\left(c_1 z + c_2 z^2 + ...\right) + \\ &\quad \frac{\delta(\delta-1)}{2}(c_1^2 z^2 + c_2^2 z^4 + ...) + ... \end{aligned}$$

Equating the coefficients of z and z^2, we obtain

$$a_2 = \left(\frac{3}{2}\right)^{\alpha} \frac{\delta}{2+\lambda} c_1. \tag{11}$$

$$a_3 = \frac{2^{\alpha}}{3(1+\lambda)} \left\{ \delta \left(c_2 - \frac{c_1^2}{2} \right) + \frac{\delta^2}{2} c_1^2 \right\}. \tag{12}$$

From (11) and (12) it results that

$$a_3 - \mu a_2^2 = \delta \frac{2^{\alpha}}{3(1+\lambda)} \left(c_2 - \frac{c_1^2}{2} \right) + \frac{\delta^2 \left(2^{3\alpha-1} (2+\lambda)^2 - 3^{2\alpha+1} \mu (1+\lambda) \right)}{3 \left(2^{2\alpha} (1+\lambda)(2+\lambda)^2 \right)} c_1^2, \tag{13}$$

therefore,

$$\left| a_3 - \mu a_2^2 \right| \leq \delta \frac{2^{\alpha}}{3(1+\lambda)} \left| c_2 - \frac{c_1^2}{2} \right| + \frac{\delta^2 \left| 2^{3\alpha-1} (2+\lambda)^2 - 3^{2\alpha+1} \mu (1+\lambda) \right|}{\left| 3 \left(2^{2\alpha} (1+\lambda)(2+\lambda)^2 \right) \right|} \left| c_1^2 \right|.$$

In view of Lemma $1(ii)$, we obtain

$$\left| a_3 - \mu a_2^2 \right| \leq \delta \frac{2^{\alpha+1}}{3(1+\lambda)} + \frac{\delta^2 \left| 2^{3\alpha-1} (2+\lambda)^2 - 3^{2\alpha+1} \mu (1+\lambda) \right| - \delta 2^{3\alpha-1} (2+\lambda)^2}{3 \left(2^{2\alpha} (1+\lambda)(2+\lambda)^2 \right)} \left| c_1^2 \right|. \tag{14}$$

Suppose that

$$\delta^2 \left| 2^{3\alpha-1} (2+\lambda)^2 - 3^{2\alpha+1} \mu (1+\lambda) \right| \leq \delta 2^{3\alpha-1} (2+\lambda)^2.$$

Then it immediately follows that

$$\left| a_3 - \mu a_2^2 \right| \leq \delta \frac{2^{\alpha+1}}{3(1+\lambda)}. \tag{15}$$

Now if

$$\delta^2 \left| 2^{3\alpha-1} (2+\lambda)^2 - 3^{2\alpha+1} \mu (1+\lambda) \right| \geq \delta 2^{3\alpha-1} (2+\lambda)^2,$$

then in view of Lemma $1(i)$ we obtain

$$\left| a_3 - \mu a_2^2 \right| \leq \frac{\delta 2^{3\alpha+1} (2+\lambda)^2 + 4\delta^2 \left| 2^{3\alpha-1} (2+\lambda)^2 - 3^{2\alpha+1} \mu (1+\lambda) \right| - \delta 2^{3\alpha+1} (2+\lambda)^2}{3 \left(2^{2\alpha} (1+\lambda)(2+\lambda)^2 \right)}$$

$$= \frac{\delta^2 \left| 2^{3\alpha-1} (2+\lambda)^2 - 3^{2\alpha+1} \mu (1+\lambda) \right|}{3 \left(2^{2(\alpha-1)} (1+\lambda) (2+\lambda)^2 \right)}. \tag{16}$$

The result now follows from (15) and (16).

Proceeding in a similar way as above we can prove the following result.

Theorem 3. *Let $\delta \in (0,1]$, $\alpha > 1$, $\beta > 0$, $\lambda \geq 0$. If $f(z) \in R(Q, \alpha, \lambda)$ and $\mu \in \mathbb{C}$. Then*

$$\left| a_3 - \mu a_2^2 \right| \leq \delta \frac{2(\alpha+\beta)_3}{3(\beta+1)_2 (\alpha+\beta+2\lambda)} \max \left\{ 1, \frac{\delta \left| \eta(\alpha,\beta) \right|}{2(\alpha+\beta)_3 (\beta+1)(\alpha+\beta+\lambda)^2} \right\},$$

where

$$\eta(\alpha,\beta) = 2(\alpha+\beta)_3 (\beta+1)(\alpha+\beta+\lambda)^2 - 3\mu(\alpha+\beta)_2^2 (\beta+2)(\alpha+\beta+2\lambda).$$

Next we consider the case of μ real parameter.

Theorem 4. *Let $\delta \in (0,1]$, $\alpha > 1$, $\lambda \geq 0$. If the function $f(z)$ given by (1) is in the class $R(P, \alpha, \lambda)$ and μ is real parameter then*

$$\left| a_3 - \mu a_2^2 \right| \leq \begin{cases} \dfrac{\delta^2 \left| 2^{3\alpha-1} (2+\lambda)^2 - 3^{2\alpha+1} \mu (1+\lambda) \right|}{3 \left(2^{2(\alpha-1)} (1+\lambda)(2+\lambda)^2 \right)}, & if \ \mu \leq \frac{(\delta-1) 2^{3\alpha-1} (2+\lambda)^2}{\delta 3^{2\alpha+1} (1+\lambda)} \\ \frac{\delta 2^{\alpha+1}}{3(1+\lambda)}, & if \ \frac{(\delta-1) 2^{3\alpha-1} (2+\lambda)^2}{\delta 3^{2\alpha+1} (1+\lambda)} \leq \mu \leq \frac{(\delta+1) 2^{3\alpha-1} (2+\lambda)^2}{\delta 3^{2\alpha+1} (1+\lambda)} \\ \dfrac{-\delta^2 \left| 2^{3\alpha-1} (2+\lambda)^2 - 3^{2\alpha+1} \mu (1+\lambda) \right|}{3 \left(2^{2(\alpha-1)} (1+\lambda)(2+\lambda)^2 \right)}, & if \ \mu \geq \frac{(\delta+1) 2^{3\alpha-1} (2+\lambda)^2}{\delta 3^{2\alpha+1} (1+\lambda)}. \end{cases}$$

Proof. By the virtue of (14) we have to consider the following two cases
Case I: Assume that

$$\mu \leq \frac{2^{3\alpha-1} (2+\lambda)^2}{3^{2\alpha+1} (1+\lambda)},$$

then from the inequality (14) it results

$$\left| a_3 - \mu a_2^2 \right| \leq \frac{\delta 2^{\alpha+1}}{3(1+\lambda)} + \frac{\delta \left[(\delta-1) 2^{3\alpha-1} (2+\lambda)^2 - \delta 3^{2\alpha+1} \mu (1+\lambda) \right]}{3 \left(2^{2\alpha} (1+\lambda)(2+\lambda)^2 \right)} \left| c_1 \right|^2. \tag{17}$$

From Lemma $1(i)$, we obtain

$$\left|a_3-\mu a_2^2\right| \leq \frac{\delta\left[\delta 2^{3\alpha+1}(2+\lambda)^2-\delta 4(3)^{2\alpha+1}\mu(1+\lambda)\right]}{3(2)^{2\alpha}(1+\lambda)(2+\lambda)^2}.$$

Thus

$$\left|a_3-\mu a_2^2\right| \leq \frac{\delta^2\left[2^{3\alpha-1}(2+\lambda)-3^{2\alpha+1}\mu(1+\lambda)\right]}{3(2)^{2(\alpha-1)}(1+\lambda)(2+\lambda)^2},$$

provided that

$$\mu \leq \frac{(\delta-1)2^{3\alpha-1}(2+\lambda)^2}{\delta 3^{2\alpha+1}(1+\lambda)}.$$

Now if

$$\mu \geq \frac{(\delta-1)2^{3\alpha-1}(2+\lambda)^2}{\delta 3^{2\alpha+1}(1+\lambda)}.$$

Then (17) becomes

$$\left|a_3-\mu a_2^2\right| \leq \frac{\delta 2^{\alpha+1}}{3(1+\lambda)}.$$

Case II: Assume that

$$\mu \geq \frac{2^{3\alpha-1}(2+\lambda)^2}{3^{2\alpha+1}(1+\lambda)}$$

and so, inequality (14) reduces to

$$\left|a_3-\mu a_2^2\right| \leq \frac{\delta 2^{\alpha+1}}{3(1+\lambda)}+\frac{\delta\left[\delta\mu 3^{2\alpha+1}(1+\lambda)-(\delta+1)2^{3\alpha-1}(2+\lambda)^2\right]}{3(2)^{2\alpha}(1+\lambda)(2+\lambda)^2}\left|c_1\right|^2. \quad (18)$$

Again, by using Lemma $1(i)$ we obtain

$$\left|a_3-\mu a_2^2\right| \leq \frac{\delta\left[\delta 4(3)^{2\alpha+1}\mu(1+\lambda)-\delta 2^{3\alpha-1}(2+\lambda)^2\right]}{3(2)^{2\alpha}(1+\lambda)(2+\lambda)^2}.$$

Thus

$$\left|a_3-\mu a_2^2\right| \leq \frac{-\delta^2\left|2^{3\alpha-1}(2+\lambda)^2-3^{2\alpha+1}\mu(1+\lambda)\right|}{3\left(2^{2(\alpha-1)}(1+\lambda)(2+\lambda)^2\right)},$$

provided that

$$\mu \geq \frac{(\delta+1)2^{3\alpha-1}(2+\lambda)^2}{\delta 3^{2\alpha+1}(1+\lambda)}.$$

On the other hand, if

$$\mu \leq \frac{(\delta+1)\,2^{3\alpha-1}\,(2+\lambda)^2}{\delta\mu 3^{2\alpha+1}\,(1+\lambda)},$$

then, from (18) it fallows

$$\left|a_3-\mu a_2^2\right| \leq \frac{\delta 2^{\alpha-1}}{3\,(1+\lambda)}.$$

Finally, we observe that

$$\frac{(\delta-1)\,2^{3\alpha-1}\,(2+\lambda)^2}{\delta 3^{2\alpha+1}\,(1+\lambda)} \leq \mu \leq \frac{2^{3\alpha-1}\,(2+\lambda)^2}{3^{2\alpha+1}\,(1+\lambda)} \leq \frac{(\delta+1)\,2^{3\alpha-1}\,(2+\lambda)^2}{\delta 3^{2\alpha+1}\,(1+\lambda)}.$$

Thus the proof of the theorem is completed.

Similarly we can prove the following.

Theorem 5. *Let $\delta \in (0,1]$, $\alpha > 1$, $\beta > 0$, $\lambda \geq 0$. If the function $f(z)$ given by* (1) *is in the class $R(Q,\alpha,\lambda)$ and μ is real then*

$$\left|a_3-\mu a_2^2\right| \leq \begin{cases} \frac{\delta^2|\eta(\alpha,\beta)|}{3(\beta+1)^2(\beta+2)(\alpha+\beta+\lambda)^2(\alpha+\beta+2\lambda)}, & if \quad \mu \leq \gamma_1 \\ \frac{\delta 2(\alpha+\beta)_3}{3(\beta+1)_2(\alpha+\beta+2\lambda)}, & if \quad \gamma_1 \leq \mu \leq \gamma_2 \\ \frac{-\delta^2|\eta(\alpha,\beta)|}{3(\beta+1)^2(\beta+2)(\alpha+\beta+\lambda)^2(\alpha+\beta+2\lambda)}, & if \quad \mu \geq \gamma_2, \end{cases}$$

where

$$\gamma_1 = \frac{2\,(\delta-1)\,(\alpha+\beta)_3\,(\beta+1)\,(\alpha+\beta+\lambda)^2}{3\delta\,(\alpha+\beta)_2^2\,(\beta+2)\,(\alpha+\beta+2\lambda)}$$

$$\gamma_2 = \frac{2\,(\delta+1)\,(\alpha+\beta)_3\,(\beta+1)\,(\alpha+\beta+\lambda)^2}{3\delta\,(\alpha+\beta)_2^2\,(\beta+2)\,(\alpha+\beta+2\lambda)}.$$

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Muhammad Tahir
Department of Mathematics,
Abbottabad University of Science and Technology,
Abbottabad, Pakistan
email: *tahirmuhammad778@gmail.com*

Bilal Khan
Department of Mathematics,
Abbottabad University of Science and Technology,
Abbottabad, Pakistan
email: *bilalmaths789@gmail.com*

Ajmal Khan
Department of Mathematics,
Abbottabad University of Science and Technology,
Abbottabad, Pakistan