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SOME RESULTS OF SECOND ORDER DIFFERENTIAL SUBORDINATION INVOLVING GENERALIZED LINEAR OPERATOR

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Abstract. In this paper, we derive some results of second order differential subordination involving linear operator, we estimate some of the interesiting property in would provide extensions of those given in earlier works.

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1. Introduction

Let $H(U)$ denote the class of holomorphic functions in the open unit disc U =$\{z \in C : |z| < 1\}$ in C be complex plane. Let A be the class of analytic functions normalized by $\varphi(0)$= 0 and $\acute{\varphi}(0)$=1 in U of the following form

$$\varphi(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

with $A_1 = A$, where $a \in C$ and $n \in N$=$\{1, 2, ...\}$, we let

$$H[a, n] = \{\varphi \in H(U) : \varphi(z) = a + a_n z^n + a_{n+1} z^{n+1} + ...\},$$

be the subclass of $H(U)$ with $H_0 \equiv H[0, 1]$ and $H \equiv H[1, 1]$.
Next, let $\varphi(z)$ and $F(z)$ be members of $H(U)$. Then the function $\varphi(z)$ is said to be subordinate to a function $F(z)$ or $F(z)$ is said to be superordinate to $\varphi(z)$, written as

$$\varphi(z) \prec F(z) \ or \ \varphi \prec F,$$

if and only if if there exists a Schwarz function $w(z)$ holomorphic in U, with $w(0) = 0$ and $|w(z)| < 1$,$(z \in U)$, such that $\varphi(z)$=$F(w(z))$.
Furthermore, if the function F is univalent in U, then we have the following equivalence [3],[7],[9].

$$\varphi(z) \prec F(z) \Longleftrightarrow \varphi(0) = F(0) \text{ } and \text{ } \varphi(U) \subset F(U).$$

Let S denote the subclass of A consisting of functions univalent in U. If a function $\varphi \in A$ maps U onto a convex domain and φ is univalent, then φ is called a convex function. Let

$$K = \{\varphi \in A : Re(1 + \frac{z\acute{\acute{\varphi}}(z)}{\acute{\varphi}(z)} > 0), z \in U\},$$

denote the class of all convex functions defined in U and normalized by $\varphi = 0$ and $\acute{\varphi}(0) = 1$.
Let $\phi : C^3 \times U \to C$ be holomorphic in a domain D and g be univalent in U. If h is holomorphic in U and $h(0) = a$ with satisfies the second-order differential subordination

$$\phi(h(z), z\acute{h}(z), z\acute{\acute{h}}(z); z) \prec g(z), \tag{2}$$

then h is said to be solution of the differential subordination. The univalent function p is said to be dominant of the solution of the differential subordination or more simply dominant, if $h \prec p$ for all h satisfying (2). A dominant $p^{\sim}$ satisfying $p^{\sim} \prec p$ for all dominants (2) is said to be the best dominant of (2).

Now, we defined the new generalized studied by

Definition 1.1. [1] For $\varphi \in A$ the generalized derivative operator $I^{m,s}_{\lambda,\eta} : A \to A$ is defined by.

$$I^{m,s}_{\lambda,\eta}\varphi(z) = z + \sum_{n=2}^{\infty} \frac{(1 + \lambda(n-1))^m}{(1 + \eta(n-1))^{m-1}} c(s,n) a_n z^n, \tag{3}$$

where $s, m \in N_0$=$\{0, 1, ...\}$,$\eta \geq \lambda \geq 0$ and $c(s, n)$=$\frac{(s+1)_{n-1}}{(1)_{n-1}}$.

It can be easily seen that

$$I^{0,0}_{\lambda,0}\varphi(z) = I^{1,0}_{0,\eta}\varphi(z) = \varphi(z),$$

and

$$I^{1,0}_{\lambda,0}\varphi(z) = I^{1,1}_{0,\eta}\varphi(z) = z\acute{\varphi}(z).$$

Also,

$$I_{\lambda,0}^{b-1,0}\varphi(z) = I_{0,\eta}^{1,b-1}\varphi(z) \ where \ b = 1, 2, 3, ...$$

We can verify that

$$(1+s)I_{\lambda,\eta}^{m,s+1}\varphi(z) = z(I_{\lambda,\eta}^{m,s}\varphi(z))' + s(I_{\lambda,\eta}^{m,s}\varphi(z)). \tag{4}$$

Definition 1.2. [13] For $\varphi \in A$, the Dziok-Srivastava operator $H_m^i : A \to A$ is defined by

$$H_m^i(\alpha_1, \alpha_2, ..., \alpha_i; \beta_1, \beta_2..., \beta_m)\varphi(z) = z + \sum_{n=2}^{\infty} \frac{(\alpha_1)_{n-1}(\alpha_2)_{n-1}...(\alpha_i)_{n-1}}{(\beta_1)_{n-1}(\beta_2)_{n-1}...(\beta_m)_{n-1}} \frac{1}{(n-1)!} a_n z^n, \tag{5}$$

where $\alpha_i \in C$,i=1, 2, ..., i,$\beta_j \in C\backslash\{0,-1,-2,...\}$,$j$=1, 2, .., m.

For complex numbers a, b and c other than 0, -1, -2,..., the Gauss hypergeometric function ${}_2F_1(a, b, c; z)$is defined by

$${}_2F_1(a, b, c; z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n} \frac{z^n}{n!} = \frac{\Gamma(c)}{\Gamma(a).\Gamma(b)} \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)} \frac{z^n}{n!} \tag{6}$$

where $(x)_n$ is the Pochhammer symbol defined, in terms of the Gamma function Γ, by

$$(x)_n = \frac{\Gamma(x+n)}{\Gamma(x)} = \begin{cases} x(x+1)(x+2)...(x+n-1) \ for \ n \in N and \ x \in C, \\ 1, \ if \ n = 0 \ and \ x \in C \setminus \{0\} \ . \end{cases}$$

In order to make the notation simple, we write

$$H_m^i[\alpha_1]\varphi(z) = H_m^i(\alpha_1, \alpha_2, ..., \alpha_i; \beta_1, \beta_2, \beta_m)\varphi(z).$$

We can verify that

$$\alpha_1 H_m^i[\alpha_1+1]\varphi(z) = z(H_m^i[\alpha_1]\varphi(z))' + (\alpha_1 - 1)H_m^i[\alpha_1]\varphi(z). \tag{7}$$

Here, we defined the following.

Definition 1.3. For $\varphi \in A$ the operator IH_m^i:$A \to A$ is defined by the Hadamard product of the generalized operator $I_{\lambda,\eta}^{m,s}$ and the Dziok-Srivastava operator H_m^i.

$$IH_m^i\varphi(z) = (I_{\lambda,\eta}^{m,s} * H_m^i)\varphi(z) \ (z \in U),$$

and

$$IH_m^i\varphi(z) = z + \sum_{n=2}^{\infty} \frac{(1+\lambda(n-1))^m}{(1+\eta(n-1))^{m-1}} c(s,n) \frac{(\alpha_1)_{n-1}(\alpha_2)_{n-1}...(\alpha_i)_{n-1}}{(\beta_1)_{n-1}(\beta_2)_{n-1}...(\beta_m)_{n-1}} \frac{1}{(n-1)!} a_n^2 z^n. \tag{8}$$

Not that, the following are the special cases of operator IH_m^i.

1. When $I_{0,0}^{m,0}=1$, inclued the Dziok-Srivastava operator H_m^i [5],[13].
2. When $i=2$,$m=1$, H_1^2 which is introduced by the Gauss hypergeometric function [2].
3. When H_1^1=1, inclued the generalized derivative operator $I_{\lambda,\eta}^{m,s}$ [1].
4. When $s=0$,$\lambda=1$,$\eta=0$,$I_{\lambda,\eta}^{m,s}$ reduces to $I_{1,0}^{m,0}$ which is introduced by Salagean derivative operator [8].
5. When $s=0$,$\eta=0$,$I_{\lambda,\eta}^{m,s}$ reduces to $I_{\lambda,0}^{m,0}$ which is introduced by generalized Salagean derivative operator introduced by Al-oboudi [11].
6. When $\eta=0$,$I_{\lambda,\eta}^{m,s}$ reduces to $I_{\lambda,0}^{m,s}$ which is introduced by generalized Al-Shaqsi and Darus derivative operator [4].
7. When $\lambda=0$,$\eta=0$,$I_{\lambda,\eta}^{m,s}$ reduces to $I_{0,0}^{m,s}$ which is introduced by Srivastava-Attiya derivative operator [15].
8. When $m=1$ or $m=0$,$\lambda=0$ or $\lambda=\lambda$,$\eta=\eta$ or $\eta=0$,$I_{\lambda,\eta}^{m,s}$ reduces to $I_{0,\eta}^{1,s}?I_{\lambda,0}^{0,s}$ which is introduced by Ruscheweyh derivative operator [12].
9. When $m=0$ or $m=1$,$I_{\lambda,\eta}^{m,s}$ reduces to $I_{\lambda,\eta}^{0,s}?I_{\lambda,\eta}^{1,s}$ which is introduced by generalized Ruscheweyh derivative operator [14].

In order to prove the results, we need the following lemmas.

Lemma 1.1. *[6] If q is holomorphic, univalent and convex function in U, let $q(0)=a$ and $\gamma \in C^*=C\setminus\{0\}$ be a complex number such that $Re\{\gamma\}\geq 0$. If $p\in H[a,n]$ and*

$$p(z)+\frac{zp\acute{(}z)}{\gamma}\prec q(z), \tag{9}$$

then

$$p(z)\prec h(z)\prec q(z),$$

where

$$h(z)=\frac{\gamma}{nz^{\frac{\gamma}{n}}}\int_0^z q(t)t^{\frac{\gamma}{n}-1}dt.$$

The function h is convex and it is the best (a,n)-dominant of the subordination (9).

Lemma 1.2. *[10] If r is a convex function in U with*

$$h(z) = r(z) + n\beta z r'(z) \quad (z \in U),$$

where $\beta > 0$ and $n \in N$. If

$$p(z) = r(0) + p_n z^n + p_{n+1} z^{n+1} +, \quad (z \in U),$$

is holomorphic in U and

$$p(z) + \beta z p'(z) \prec h(z),$$

then

$$p(z) \prec r(z),$$

and this result is sharp.

2. Main Results

Theorem 2.1. *If h be a convex function with $h(0) = 1$ with q be the following function*

$$q(z) = h(z) + \frac{z}{\tau} h'(z), \ \ \tau, i \geq 0, m \in N.$$

If $\varphi \in A$ and the following differential subordination

$$\left(\frac{IH_m^i \varphi(z)}{z}\right)^{\tau-1} (IH_m^i \varphi(z))' \prec q(z), \tag{10}$$

holds true, then

$$\left(\frac{IH_m^i \varphi(z)}{z}\right)^{\tau} \prec h(z),$$

Proof. If $\varphi \in A$, given by (1) therefore, we get

$$IH_m^i \varphi(z) = z + \sum_{n=2}^{\infty} \frac{(1+\lambda(n-1))^m}{(1+\eta(n-1))^{m-1}} c(s,n) \frac{(\alpha_1)_{n-1}(\alpha_2)_{n-1}...(\alpha_i)_{n-1}}{(\beta_1)_{n-1}(\beta_2)_{n-1}...(\beta_m)_{n-1}} \frac{1}{(n-1)!} a_n^2 z^n, \ (z \in U)$$

let us consider

$$p(z) = \left(\frac{IH_m^i \varphi(z)}{z}\right)^{\tau} = \left(\frac{z + \sum_{n=2}^{\infty} \frac{(1+\lambda(n-1))^m}{(1+\eta(n-1))^{m-1}} c(s,n) \frac{(\alpha_1)_{n-1}(\alpha_2)_{n-1}...(\alpha_i)_{n-1}}{(\beta_1)_{n-1}(\beta_2)_{n-1}...(\beta_m)_{n-1}} \frac{1}{(n-1)!} a_n^2 z^n}{z}\right)^{\tau}$$

$$= \left(1 + \sum_{n=2}^{\infty} \frac{(1+\lambda(n-1))^m}{(1+\eta(n-1))^{m-1}} c(s,n) \frac{(\alpha_1)_{n-1}(\alpha_2)_{n-1}...(\alpha_i)_{n-1}}{(\beta_1)_{n-1}(\beta_2)_{n-1}...(\beta_m)_{n-1}} \frac{1}{(n-1)!} a_n^2 z^{n-1}\right)^{\tau},$$

$$p(z) = 1 + p_\tau z^\tau + p_{\tau+1} z^{\tau+1} + ... \, .$$

By take the derivative of both sides, we get

$$(\frac{IH_m^i \varphi(z)}{z})^{\tau-1}(IH_m^i \varphi(z))' = p(z) + \frac{zp'(z)}{\tau}.$$

Thus, by aid of the relation (10) becomes

$$p(z) + \frac{zp'(z)}{\tau} \prec q(z) = h(z) + \frac{zh'(z)}{\tau}, \ (z \in U)$$

therefore, with Lemma 1.2 we obtain

$$p(z) \prec h(z),$$

that is,

$$(\frac{IH_m^i \varphi(z)}{z})^\tau \prec h(z).$$

Theorem 2.2. *If q be a holomorphic function and satisfies in the following inequality*

$$Re\{1 + \frac{zq''(z)}{q'(z)}\} > -1/2, \ where \ q(0) = 1, \tau, i \geq 0, m \in N,$$

and $\varphi \in A$ satisfies in the following differential subordination

$$(\frac{IH_m^i \varphi(z)}{z})^{\tau-1}(IH_m^i \varphi(z))' \prec q(z).$$

Then

$$(\frac{IH_m^i \varphi(z)}{z})^\tau \prec g(z),$$

where

$$g(z) = \frac{\tau}{z^\tau} \int_0^z q(t) t^{\tau-1} dt.$$

Proof. Setting

$$p(z) = (\frac{IH_m^i\varphi(z)}{z})^\tau = (\frac{z + \sum_{n=2}^{\infty} \frac{(1+\lambda(n-1))^m}{(1+\eta(n-1))^{m-1}} c(s,n) \frac{(\alpha_1)_{n-1}(\alpha_2)_{n-1}...(\alpha_i)_{n-1}}{(\beta_1)_{n-1}(\beta_2)_{n-1}...(\beta_m)_{n-1}} \frac{1}{(n-1)!} a_n^2 z^n}{z})^\tau$$
$$= (1 + \sum_{n=2}^{\infty} \frac{(1+\lambda(n-1))^m}{(1+\eta(n-1))^{m-1}} c(s,n) \frac{(\alpha_1)_{n-1}(\alpha_2)_{n-1}...(\alpha_i)_{n-1}}{(\beta_1)_{n-1}(\beta_2)_{n-1}...(\beta_m)_{n-1}} \frac{1}{(n-1)!} a_n^2 z^{n-1})^\tau,$$

$$p(z) = 1 + p_\tau z^\tau + p_{\tau+1} z^{\tau+1} + ... = 1 + \sum_{n=\tau+1}^{\infty} p_n z^{n-1}, p \in H[1,\tau].$$

By take the derivative of both sides, we get

$$(\frac{IH_m^i\varphi(z)}{z})^{\tau-1}(IH_m^i\varphi(z))' = p(z) + \frac{zp'(z)}{\tau},$$

thus, by aid of the relation the relation (10) becomes

$$p(z) + \frac{zp'(z)}{\tau} \prec q(z),$$

therefore, with Lemma 1.2 we obtain

$$p(z) \prec g(z),$$

that is,

$$(\frac{IH_m^i\varphi(z)}{z})^\tau \prec g(z),$$

where

$$g(z) = \frac{\tau}{z^\tau} \int_0^z q(t) t^{\tau-1} dt.$$

The function $g(z)$ is convex and it is the best dominant.

Corollary 2.3. *If the function q defined by*

$$q(z) = \frac{1+(2\alpha-1)z}{1+z},\ (0 \le \alpha < 1)$$

be a convex function in U, where $\tau, i \ge 0$,$m \in N$ and $\varphi \in A$ be such that

$$(\frac{IH_m^i\varphi(z)}{z})^{\tau-1}(IH_m^i\varphi(z))' \prec q(z),$$

holds , then

$$(\frac{IH_m^i\varphi(z)}{z})^\tau \prec g(z),$$

$g(z)$ is given by $g(z)$=$(2\alpha-1)+\frac{2(1-\alpha)\tau}{z^\tau}\int_0^z \frac{t^{\tau-1}}{1+t}dt$. The function $g(z)$ is convex and it is the best dominant.

Proof. Putting

$$p(z) = (\frac{IH_m^i\varphi(z)}{z})^\tau,$$

thus, by aid of the relation (10) becomes

$$p(z) + \frac{zp'(z)}{\tau} \prec q(z) = \frac{1+(2\alpha-1)z}{1+z},\ (0 \leq \alpha < 1)$$

therefore, with Lemma 1.1 for $\gamma = \tau$, we obtain

$$(\frac{IH_m^i\varphi(z)}{z})^\tau \prec g(z) = \frac{\tau}{z^\tau}\int_0^z q(t)t^{\tau-1}dt = \frac{\tau}{z^\tau}\int_0^z t^{\tau-1}\frac{1+(2\alpha-1)t}{1+t}dt$$

$$= \frac{\tau}{z^\tau}\int_0^z [(2\alpha-1)t^{\tau-1} + 2(1-\alpha)\frac{t^{\tau-1}}{1+t}]dt = (2\alpha-1) + \frac{2(1-\alpha)\tau}{z^\tau}\int_0^z \frac{t^{\tau-1}}{1+t}dt.$$

Theorem 2.4. *If h is a convex function with $h(0) = 1$ such that q the following function*

$$q(z) = h(z) + \frac{z}{\tau}h'(z),\ (z \in U)$$

where $\tau, i \geq 0, m \in N$.If $\varphi \in A$ and satisfies the following differential subordination

$$z\frac{\tau+1}{\tau}\frac{IH_m^i\varphi(z)}{(IH_m^i[\alpha_1+1]\varphi(z))^2} + \frac{z^2}{\tau}\frac{IH_m^i\varphi(z)}{(IH_m^i[\alpha_1+1]\varphi(z))^2}[\frac{(IH_m^i\varphi(z))'}{IH_m^i\varphi(z)} - 2\frac{(IH_m^i[\alpha_1+1]\varphi(z))'}{IH_m^i\varphi(z)}] \prec q(z), \tag{11}$$

then

$$z\frac{IH_m^i\varphi(z)}{(IH_m^i[\alpha_1+1]\varphi(z))^2} \prec h(z).$$

The result is sharp.

Proof. If $\varphi \in A$, given by (1) therefore, we get

$$IH_m^i\varphi(z) = z + \sum_{n=2}^{\infty} \frac{(1+\lambda(n-1))^m}{(1+\eta(n-1))^{m-1}} c(s,n) \frac{(\alpha_1)_{n-1}(\alpha_2)_{n-1}...(\alpha_i)_{n-1}}{(\beta_1)_{n-1}(\beta_2)_{n-1}...(\beta_m)_{n-1}} \frac{1}{(n-1)!} a_n^2 z^n,$$

let us consider

$$p(z) = z\frac{IH_m^i\varphi(z)}{(IH_m^i[\alpha_1+1]\varphi(z))^2},$$

we have

$$p(z) + \frac{z}{\tau}p'(z) =$$

$$z\frac{\tau+1}{\tau}\frac{IH_m^i\varphi(z)}{(IH_m^i[\alpha_1+1]\varphi(z))^2} + \frac{z^2}{\tau}\frac{IH_m^i\varphi(z)}{(IH_m^i[\alpha_1+1]\varphi(z))^2}[\frac{(IH_m^i\varphi(z))'}{IH_m^i\varphi(z)} - 2\frac{(IH_m^i[\alpha_1+1]\varphi(z))'}{IH_m^i\varphi(z)}].$$

Thus, by aid of the relation (11) becomes

$$p(z) + \frac{z}{\tau}p'(z) \prec q(z) = h(z) + \frac{z}{\tau}h'(z), \ (z \in U)$$

therefore, with Lemma 1.2 we obtain

$$p(z) = z\frac{IH_m^i\varphi(z)}{(IH_m^i[\alpha_1+1]\varphi(z))^2} \prec h(z).$$

Theorem 2.5. *For $\varphi \in A$ and q be a holomorphic function and it satisfies the inequality*

$$Re\{1 + \frac{zq''(z)}{q'(z)}\} > -1/2, \ \textit{with } q(0) = 1, \tau, i \geq 0, m \in N,$$

and it satisfies the following differential subordination

$$z\frac{\tau+1}{\tau}\frac{IH_m^i\varphi(z)}{(IH_m^i[\alpha_1+1]\varphi(z))^2} + \frac{z^2}{\tau}\frac{IH_m^i\varphi(z)}{(IH_m^i[\alpha_1+1]\varphi(z))^2}[\frac{(IH_m^i\varphi(z))'}{IH_m^i\varphi(z)} - 2\frac{(IH_m^i[\alpha_1+1]\varphi(z))'}{IH_m^i\varphi(z)}] \prec q(z),$$

then

$$z\frac{IH_m^i\varphi(z)}{(IH_m^i[\alpha_1+1]\varphi(z))^2} \prec g(z),$$

where $g(z)$ is given by

$$g(z) = \frac{\tau}{z^\tau}\int_0^z q(t)t^{\tau-1}dt.$$

The function $g(z)$ is convex and it is the best dominant.

Proof. Putting

$$p(z) = z\frac{IH_m^i\varphi(z)}{(IH_m^i[\alpha_1+1]\varphi(z))^2},\ p \in H[1,1].$$

By take the derivative of both sides, we get

$$p(z) + \frac{z}{\tau}p'(z) =$$

$$z\frac{\tau+1}{\tau}\frac{IH_m^i\varphi(z)}{(IH_m^i[\alpha_1+1]\varphi(z))^2} + \frac{z^2}{\tau}\frac{IH_m^i\varphi(z)}{(IH_m^i[\alpha_1+1]\varphi(z))^2}[\frac{(IH_m^i\varphi(z))'}{IH_m^i\varphi(z)} - 2\frac{(IH_m^i[\alpha_1+1]\varphi(z))'}{IH_m^i\varphi(z)}],$$

thus, by aid of the relation (11) becomes

$$p(z) + \frac{z}{\tau}p'(z) \prec q(z).$$

Therefore, with Lemma 1.1 we obtain

$$p(z) = z\frac{IH_m^i\varphi(z)}{(IH_m^i[\alpha_1+1]\varphi(z))^2} \prec g(z).$$

The function $g(z)$ is convex and it is the best dominant.

Theorem 2.6. *If h is a convex function with $h(0) = 1$ such that q the following function*

$$q(z) = h(z) + \frac{z}{\tau}h'(z),\ (z \in U)$$

where τ,$i \geq 0$,$m \in N$.If $\varphi \in A$ and satisfies the following differential subordination

$$z^2\frac{\tau+2}{\tau}\frac{(IH_m^i\varphi(z))'}{IH_m^i\varphi(z)} + \frac{z^3}{\tau}[\frac{(IH_m^i\varphi(z))''}{IH_m^i\varphi(z)} - (\frac{(IH_m^i\varphi(z))'}{IH_m^i\varphi(z)})^2] \prec q(z), \tag{12}$$

then

$$z^2\frac{(IH_m^i\varphi(z))'}{(IH_m^i\varphi(z))} \prec h(z).$$

Proof. Putting

$$p(z) = z^2\frac{(IH_m^i\varphi(z))'}{(IH_m^i\varphi(z))},\ p \in H[0,1].$$

By take the derivative of both sides, we get

$$p(z)+\frac{z}{\tau}p'(z)=z^2\frac{\tau+2}{\tau}\frac{(IH_m^i\varphi(z))'}{IH_m^i\varphi(z)}+\frac{z^3}{\tau}[\frac{(IH_m^i\varphi(z))''}{IH_m^i\varphi(z)}-(\frac{(IH_m^i\varphi(z))'}{IH_m^i\varphi(z)})^2],$$

thus, by aid of the relation (12) becomes

$$p(z)+\frac{z}{\tau}p'(z)\prec q(z)=h(z)+\frac{z}{\tau}h'(z).\ (z\in U)$$

Therefore, with Lemma 2 we obtain

$$p(z)=z^2\frac{(IH_m^i\varphi(z))'}{(IH_m^i\varphi(z))}\prec h(z).\ (z\in U)$$

The result is sharp.

Theorem 2.7. *If q be a holomorphic function and it satisfies the inequality*

$$Re\{1+\frac{zq''(z)}{q'(z)}\}>-1/2,\ with\ q(0)=1,$$

where τ,$i\geq 0$,$m\in N$.If $\varphi\in A$ and satisfies the following differential subordination

$$z^2\frac{\tau+2}{\tau}\frac{(IH_m^i\varphi(z))'}{IH_m^i\varphi(z)}+\frac{z^3}{\tau}[\frac{(IH_m^i\varphi(z))''}{IH_m^i\varphi(z)}-(\frac{(IH_m^i\varphi(z))'}{IH_m^i\varphi(z)})^2]\prec q(z),$$

holds true,then

$$z^2\frac{(IH_m^i\varphi(z))'}{(IH_m^i\varphi(z))}\prec g(z),$$

where $g(z)$ is given by

$$g(z)=\frac{\tau}{z^\tau}\int_0^z q(t)t^{\tau-1}dt.$$

The function $g(z)$ is convex and it is the best dominant.

Proof. Putting

$$p(z)=z^2\frac{(IH_m^i\varphi(z))'}{(IH_m^i\varphi(z))},\ p\in H[0,1].\ (z\in U)$$

By take the derivative of both sides, we have

$$p(z) + \frac{z}{\tau}p'(z) = z^2\frac{\tau+2}{\tau}\frac{(IH_m^i\varphi(z))'}{IH_m^i\varphi(z)} + \frac{z^3}{\tau}[\frac{(IH_m^i\varphi(z))''}{IH_m^i\varphi(z)} - (\frac{(IH_m^i\varphi(z))'}{IH_m^i\varphi(z)})^2],$$

thus, by aid of the relation (12) becomes

$$p(z) + \frac{z}{\tau}p'(z) \prec q(z) = h(z) + \frac{z}{\tau}h'(z).\ (z \in U)$$

Therefore, with Lemma 1.1 we obtain

$$p(z) = z^2\frac{(IH_m^i\varphi(z))'}{(IH_m^i\varphi(z))} \prec g(z),\ \ (z \in U)$$

where

$$g(z) = \frac{\tau}{z^\tau}\int_0^z q(t)t^{\tau-1}dt.$$

The function $g(z)$ is convex and it is the best dominant.

Theorem 2.8. *If h is a convex function with $h(0) = 1$ such that q the following function*

$$q(z) = h(z) + zh'(z),\ \ (z \in U)$$

where τ,$i \geq 0$,$m \in N$.If $\varphi \in A$ and satisfies the following differential subordination

$$1 - \frac{IH_m^i\varphi(z)(IH_m^i\varphi(z))''}{[(IH_m^i\varphi(z))']^2} \prec q(z), \tag{13}$$

then

$$\frac{(IH_m^i\varphi(z))}{z(IH_m^i\varphi(z))'} \prec h(z).$$

The result is sharp.

Proof. Putting

$$p(z) = \frac{(IH_m^i\varphi(z))}{z(IH_m^i\varphi(z))'}\ ,\ p \in H[1,1].$$

By take the derivative of both sides, we have

$$p(z) + zp\acute{(}z) = 1 - \frac{IH_m^i\varphi(z)(IH_m^i\varphi(z))\acute{}}{[(IH_m^i\varphi(z))\acute{}]^2},$$

thus, by aid of the relation (13) becomes

$$p(z) + zp\acute{(}z) \prec q(z) = h(z) + zh\acute{(}z). \ (z \in U)$$

Therefore, with Lemma 1.2 we obtain

$$p(z) = \frac{(IH_m^i\varphi(z))}{z(IH_m^i\varphi(z))\acute{}} \prec h(z).$$

Proof. If q be a holomorphic function and it satisfies the inequality

$$Re\{1 + \frac{z\acute{q}\acute{(}z)}{q\acute{(}z)}\} > -1/2, \ with \ q(0) = 1,$$

where τ,$i \geq 0$,$m \in N$.If $\varphi \in A$ and satisfies the following differential subordination

$$1 - \frac{IH_m^i\varphi(z)(IH_m^i\varphi(z))\acute{}}{[(IH_m^i\varphi(z))\acute{}]^2} \prec q(z), \ (z \in U)$$

holds true,then

$$\frac{(IH_m^i\varphi(z))}{z(IH_m^i\varphi(z))\acute{}} \prec g(z).$$

where $g(z)$ is given by

$$g(z) = \frac{1}{z}\int_0^z q(t)dt.$$

The function $g(z)$ is convex and it is the best dominant.

Putting

$$p(z) = \frac{(IH_m^i\varphi(z))}{z(IH_m^i\varphi(z))\acute{}} \ , \ p \in H[0,1]. \ (z \in U)$$

By take the derivative of both sides, we get

$$p(z) + zp\acute{(}z) = 1 - \frac{IH_m^i\varphi(z)(IH_m^i\varphi(z))\acute{}}{[(IH_m^i\varphi(z))\acute{}]^2},$$

thus, by aid of the relation (12) becomes

$$p(z) + zp\acute{(}z) \prec q(z) = h(z) + zh\acute{(}z).\ (z \in U)$$

Therefore, with Lemma 1 we obtain

$$p(z) = \frac{(IH^i_m\varphi(z))}{z(IH^i_m\varphi(z))\acute{}} \prec g(z),\ \ (z \in U)$$

where

$$g(z) = \frac{1}{z}\int_0^z q(t)dt.$$

The function $g(z)$ is convex and it is the best dominant.

Theorem 2.9. *If the function q defined by*

$$q(z) = \frac{1+(2\alpha-1)z}{1+z},\ \ (0 \leq \alpha < 1)$$

is a convex function in U, where $i \geq 0$,$m \in N$.If $\varphi \in A$ and the following differential subordination

$$1 - \frac{IH^i_{m,\beta}\varphi(z)(IH^i_{m,\beta}\varphi(z))\acute{}\acute{}}{[(IH^i_{m,\beta}\varphi(z))\acute{}]^2} \prec q(z), \tag{14}$$

holds true, then

$$\frac{(IH^i_{m,\beta}\varphi(z))}{z(IH^i_{m,\beta}\varphi(z))\acute{}} \prec g(z),$$

where

$$g(z) = (2\alpha-1) + 2(1-\alpha)\frac{ln(1+z)}{z}.$$

The function $g(z)$ is convex and it is the best dominant.

Proof. Putting

$$p(z) = \frac{(IH^i_{m,\beta}\varphi(z))}{z(IH^i_{m,\beta}\varphi(z))\acute{}},$$

thus, by aid of the relation (14) becomes

$$p(z) + zp'(z) \prec q(z) = \frac{1+(2\alpha-1)z}{1+z},\ (0 \leq \alpha < 1)$$

therefore, with Lemma 1.1 for γ=1, we get

$$\frac{(IH^i_{m,\beta}\varphi(z))}{z(IH^i_{m,\beta}\varphi(z))'} \prec g(z) = \frac{1}{z}\int_0^z q(t)dt = \frac{1}{z}\int_0^z \frac{1+(2\alpha-1)t}{1+t}dt$$

$$= \frac{1}{z}\int_0^z [(2\alpha-1) + \frac{2(1-\alpha)}{1+t}]dt = (2\alpha-1) + 2(1-\alpha)\frac{ln(1+z}{z}.$$

Theorem 2.10. *If h is a convex function with $h(0) = 1$ such that q the following function*

$$q(z) = h(z) + zh'(z),\ (z \in U)$$

where $i \geq 0$,$m \in N$.If $\varphi \in A$ and satisfies the following differential subordination

$$[(IH^i_m\varphi(z))']^2 + IH^i_m\varphi(z)(IH^i_m\varphi(z))'' \prec q(z), \tag{15}$$

then

$$\frac{(IH^i_m\varphi(z))(IH^i_m\varphi(z))'}{z} \prec h(z).$$

Proof. Putting

$$p(z) = \frac{(IH^i_m\varphi(z))(IH^i_m\varphi(z))'}{z},\ p \in H[0,1].$$

By take the derivative of both sides, we have

$$p(z) + zp'(z) = [(IH^i_m\varphi(z))']^2 + IH^i_m\varphi(z)(IH^i_m\varphi(z))'',$$

thus, by aid of the relation (15) becomes

$$p(z) + zp'(z) \prec q(z) = h(z) + zh'(z).\ (z \in U)$$

Therefore, with Lemma 2 we get

$$p(z) = \frac{(IH^i_m\varphi(z))(IH^i_m\varphi(z))'}{z} \prec h(z).$$

The result is sharp.

Theorem 2.11. *If q be a holomorphic function and it satisfies the inequality*

$$Re\{1+\frac{zq''(z)}{q'(z)}\} > -1/2, \ with \ q(0)=1,$$

where $i \geq 0, m \in N$.If $\varphi \in A$ and satisfies the following differential subordination

$$[(IH_m^i\varphi(z))']^2 + IH_m^i\varphi(z)(IH_m^i\varphi(z))'' \prec q(z), \ (z \in U)$$

holds true,then

$$\frac{(IH_m^i\varphi(z))(IH_m^i\varphi(z))'}{z} \prec g(z).$$

where $g(z)$ is given by

$$g(z) = \frac{1}{z}\int_0^z q(t)dt.$$

The function $g(z)$ is convex and it is the best dominant.

Proof. Putting

$$p(z) = \frac{(IH_m^i\varphi(z))(IH_m^i\varphi(z))'}{z} \ , \ p \in H[0,1]. \ (z \in U)$$

By take the derivative of both sides, we have

$$p(z) + zp'(z) = [(IH_m^i\varphi(z))']^2 + IH_m^i\varphi(z)(IH_m^i\varphi(z))'',$$

thus, by aid of the relation (15) becomes

$$p(z) + zp'(z) \prec q(z) = h(z) + zh'(z). \ (z \in U)$$

Therefore, with Lemma 1.1 we obtain

$$p(z) = \frac{(IH_m^i\varphi(z))(IH_m^i\varphi(z))'}{z} \prec g(z), \ (z \in U)$$

where

$$g(z) = \frac{1}{z}\int_0^z q(t)dt.$$

The function $g(z)$ is convex and it is the best dominant.

Theorem 2.12. *If the function q defined by*

$$q(z) = \frac{1+(2\alpha-1)z}{1+z},\ (0 \leq \alpha < 1)$$

is a convex function in U, where $i \geq 0, m \in N$. If $\varphi \in A$ and the following differential subordination

$$[(IH_m^i \varphi(z))']^2 + IH_m^i \varphi(z)(IH_m^i \varphi(z))'' \prec q(z),$$

holds true, then

$$\frac{(IH_m^i \varphi(z))(IH_m^i \varphi(z))'}{z} \prec g(z),$$

where

$$g(z) = (2\alpha-1) + 2(1-\alpha)\frac{ln(1+z)}{z}.$$

The function $g(z)$ is convex and it is the best dominant.

Proof. Let

$$p(z) = \frac{(IH_m^i \varphi(z))(IH_m^i \varphi(z))'}{z},$$

thus, by aid of the relation (15) becomes

$$p(z) + zp'(z) \prec q(z) = \frac{1+(2\alpha-1)z}{1+z},\ (0 \leq \alpha < 1)$$

therefore, with Lemma 1.1 for $\gamma = 1$, we obtain

$$p(z) = \frac{(IH_m^i \varphi(z))(IH_m^i \varphi(z))'}{z} \prec g(z) = \frac{1}{z}\int_0^z q(t)dt = \frac{1}{z}\int_0^z \frac{1+(2\alpha-1)t}{1+t}dt$$

$$= \frac{1}{z}\int_0^z [(2\alpha-1) + \frac{2(1-\alpha)}{1+t}]dt = (2\alpha-1) + 2(1-\alpha)\frac{ln(1+z}{z}.$$

Theorem 2.13. *If h is a convex function with $h(0) = 1$ such that q the following function*

$$q(z) = h(z) + \frac{z}{1-\tau}h'(z),\ (z \in U)$$

where $\tau \in (0,1)$ *,* $i \geq 0, m \in N$*.If* $\varphi \in A$ *and satisfies the following differential subordination*

$$(\frac{z}{IH_m^i\varphi(z)})^{\tau}\frac{IH_m^i[\alpha_1+1]\varphi(z)}{1-\tau}[\frac{(IH_m^i[\alpha_1+1]\varphi(z))^{\acute{}}}{IH_m^i[\alpha_1+1]\varphi(z)}-\tau\frac{(IH_m^i\varphi(z))^{\acute{}}}{IH_m^i\varphi(z)}]\prec q(z), \quad (16)$$

then

$$\frac{(IH_m^i[\alpha_1+1]\varphi(z)}{z}(\frac{z}{IH_m^i\varphi(z)})^{\tau}\prec h(z).$$

Proof. Putting

$$p(z)=\frac{(IH_m^i[\alpha_1+1]\varphi(z)}{z}(\frac{z}{IH_m^i\varphi(z)})^{\tau}\ ,\ p\in H[1,1].$$

By take the derivative of both sides, we have

$$p(z)+\frac{z}{1-\tau}p\acute{}(z)=(\frac{z}{IH_m^i\varphi(z)})^{\tau}\frac{IH_m^i[\alpha_1+1]\varphi(z)}{1-\tau}[\frac{(IH_m^i[\alpha_1+1]\varphi(z))^{\acute{}}}{IH_m^i[\alpha_1+1]\varphi(z)}-\tau\frac{(IH_m^i\varphi(z))^{\acute{}}}{IH_m^i\varphi(z)}],$$

thus, by aid of the relation (16) becomes

$$p(z)+\frac{z}{1-\tau}p\acute{}(z)\prec q(z)=h(z)+\frac{z}{1-\tau}h\acute{}(z).\ (z\in U)$$

Therefore, with Lemma 1.2 we obtain

$$p(z)=\frac{(IH_m^i[\alpha_1+1]\varphi(z)}{z}(\frac{z}{IH_m^i\varphi(z)})^{\tau}\prec h(z).$$

The result is sharp.

Theorem 2.14. *If q be a holomorphic function and it satisfies the inequality*

$$Re\{1+\frac{zq\acute{\acute{}}(z)}{q\acute{}(z)}\}>-1/2,\ \textit{with}\ q(0)=1,$$

where $\tau \in (0,1), i \geq 0$*,*$m \in N$*.If* $\varphi \in A$ *and satisfies the following differential subordination*

$$(\frac{z}{IH_m^i\varphi(z)})^{\tau}\frac{IH_m^i[\alpha_1+1]\varphi(z)}{1-\tau}[\frac{(IH_m^i[\alpha_1+1]\varphi(z))^{\acute{}}}{IH_m^i[\alpha_1+1]\varphi(z)}-\tau\frac{(IH_m^i\varphi(z))^{\acute{}}}{IH_m^i\varphi(z)}]\prec q(z),\ (z\in U)$$

holds true,then

$$\frac{(IH_m^i[\alpha_1+1]\varphi(z)}{z}(\frac{z}{IH_m^i\varphi(z)})^{\tau}\prec g(z),$$

where $g(z)$ is given by

$$g(z) = \frac{1-\tau}{z^{1-\tau}} \int_0^z q(t) t^{-\tau} dt.$$

The function $g(z)$ is convex and it is the best dominant.

Proof. Putting

$$p(z) = \frac{(IH_m^i[\alpha_1+1]\varphi(z)}{z}\left(\frac{z}{IH_m^i\varphi(z)}\right)^{\tau}, \ p \in H[0,1]. \ (z \in U)$$

By take the derivative of both sides, we have

$$p(z) + \frac{z}{1-\tau} p\acute{(}z) = \left(\frac{z}{IH_m^i\varphi(z)}\right)^{\tau} \frac{IH_m^i[\alpha_1+1]\varphi(z)}{1-\tau}\left[\frac{(IH_m^i[\alpha_1+1]\varphi(z))'}{IH_m^i[\alpha_1+1]\varphi(z)} - \tau \frac{(IH_m^i\varphi(z))'}{IH_m^i\varphi(z)}\right],$$

thus, by aid of the relation (16) becomes

$$p(z) + \frac{z}{1-\tau} p\acute{(}z) \prec q(z) = h(z) + \frac{z}{1-\tau} h\acute{(}z). \ (z \in U)$$

Therefore, with Lemma 1 we obtain

$$p(z) = \frac{(IH_m^i[\alpha_1+1]\varphi(z)}{z}\left(\frac{z}{IH_m^i\varphi(z)}\right)^{\tau} \prec g(z), \ (z \in U)$$

where

$$g(z) = \frac{1-\tau}{z^{1-\tau}} \int_0^z q(t) t^{-\tau} dt.$$

The function $g(z)$ is convex and it is the best dominant

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