Acta Universitatis Apulensis No. 51/2017
ISSN: 1582-5329 pp. 89-103
http://www.uab.ro/auajournal/ doi: 10.17114/j.aua.2017.51.08

SOME CONDITION ON A POISSON DISTRIBUTION SERIES TO BE IN SUBCLASSES OF UNIVALENT FUNCTIONS

R. M. El-Ashwah, W. Y. Kota

Abstract. In this paper, we obtained some condition on Poisson distribution series and some related series to be in subclasses of analytic function. Also, we investigate some mapping properties for these subclasses.

2010 *Mathematics Subject Classification*: 30C45.

Keywords: Poisson distribution series; Analytic functions; Hadamard product; Starlike function; Convex functions.

1. Introduction

Let $\mathcal{A}$ denote the class of functions $f(z)$ of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

which are analytic in the open unit disc $\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$, and let $\mathcal{S}$ be the subclass of all functions in $\mathcal{A}$, which are univalent. For $g(z) \in \mathcal{A}$ of the form

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n,$$

the Hadamard product (or convolution) of two power series $f(z)$ and $g(z)$ is given by (see [3])

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n = (g * f)(z).$$

Definition 1. *For two functions $f(z)$ and $g(z)$ analytic in $\mathbb{U}$, we say that the function $f(z)$ is subordinate to $g(z)$ in $\mathbb{U}$ and written $f(z) \prec g(z)$, if there exists a Schwarz function $w(z)$, analytic in $\mathbb{U}$ with $w(0) = 0$ and $w(z) < 1$ such that $f(z) = g(w(z))$ $(z \in \mathbb{U})$. Furthermore, if the function $g(z)$ is univalent in $\mathbb{U}$, then we have the following equivalence (see [9]):*

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \text{ and } f(\mathbb{U}) \subset g(\mathbb{U}).$$

Let $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$ denote the subclasses of starlike and convex functions of order α, respectively. We note that $\mathcal{S}^*(0) = \mathcal{S}^*$ and $\mathcal{K}(0) = \mathcal{K}$, the subclasses of starlike and convex functions (see [8, 11, 13, 15, 16] and [19]).

Kanas and Wisniowska [6, 7] introduced the classes $k - ST$ and $k - UCV$ are uniformly starlike functions and uniformly convex functions, respectively, as following:

Definition 2. *A function $f(z)$ of the form (1) is in the class $k - ST$ if it satisfies the following condition:*

$$\Re\left\{1 + \frac{zf'(z)}{f(z)}\right\} \geq k\left|\frac{zf'(z)}{f(z)}\right| \quad (k \geq 0;\ z \in \mathbb{U}).$$

Definition 3. *A function $f(z)$ of the form (1) is in the class $k - UCV$ if it satisfies the following condition:*

$$\Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} \geq k\left|\frac{zf''(z)}{f'(z)}\right| \quad (k \geq 0;\ z \in \mathbb{U}).$$

Definition 4. *[1] with $p = 1$ For $-1 \leq A < B \leq 1$, $|\theta| < \frac{\pi}{2}$ and $0 \leq \alpha < 1$, the function $f(z) \in \mathcal{A}$ is in the class $\mathcal{L}(A, B, \theta; \alpha)$ if it satisfies the subordination condition*

$$e^{i\theta} f'(z) \prec \cos\theta\left[(1-\alpha)\frac{1+Az}{1+Bz} + \alpha\right] + i\sin\theta.$$

Using the principle of subordination, $f(z) \in \mathcal{L}(A, B, \theta; \alpha)$ if and only if there exists function $w(z)$ satisfying $w(0) = 0$ and $|w(z)| < 1$ $(z \in \mathbb{U})$ such that

$$e^{i\theta} f'(z) = \cos\theta\left[(1-\alpha)\frac{1+Aw(z)}{1+Bw(z)} + \alpha\right] + i\sin\theta.$$

or, equivalently,

$$\left|\frac{e^{i\theta}(f'(z)-1)}{Be^{i\theta}f'(z) - [Be^{i\theta} + (A-B)(1-\alpha)\cos\theta]}\right| < 1 \ (z \in \mathbb{U}).$$

For suitable choices of A, B and α, we obtain some subclasses as following:

1. Let $A = -1$ and $B = 1$, we obtain $\mathcal{L}(-1, 1, \theta; \alpha) = \mathcal{L}(\theta; \alpha)$ $(0 \leq \alpha < 1)$ the class of functions $f(z) \in \mathcal{A}$ satisfying the condition

   $$\left|\frac{e^{i\theta}(f'(z)-1)}{e^{i\theta}f'(z) - [e^{i\theta} - 2(1-\alpha)\cos\theta]}\right| < 1,$$

   which introduce by [5].

2. Let $\alpha = 0$, we obtain $\mathcal{L}(A, B, \theta; 0) = \mathcal{L}(A, B, \theta)$ $(-1 \leq A < B \leq 1, |\theta| < \frac{\pi}{2})$ the class of functions $f(z) \in \mathcal{A}$ satisfying the condition

$$\left| \frac{e^{i\theta}(f'(z) - 1)}{Be^{i\theta} f'(z) - [Be^{i\theta} - (A - B)\cos\theta]} \right| < 1,$$

which introduce by [17].

3. Let $A = -\beta$, $B = \beta$ and $\theta = 0$, we obtain $\mathcal{L}(-\beta, \beta, 0; \alpha) = \mathcal{R}(\beta, \alpha)$ the class of functions $f(z) \in \mathcal{A}$ satisfying the condition

$$\left| \frac{f'(z) - 1}{f'(z) + 1 - 2\alpha} \right| < \beta \quad (0 < \beta \leq 1,\ 0 \leq \alpha < 1;\ z \in \mathbb{U}),$$

which introduce by [4].

4. Let $A = -\beta$, $B = \beta$, $\theta = 0$ and $\alpha = 0$, we obtain $\mathcal{L}(-\beta, \beta, 0; 0) = \mathcal{D}(\beta)$ the class of functions $f(z) \in \mathcal{A}$ satisfying the condition

$$\left| \frac{f'(z) - 1}{f'(z) + 1} \right| < \beta \quad (0 < \beta \leq 1;\ z \in \mathbb{U}),$$

which introduce by [2, 10].

Very recently, Porwal [12] introduce a power series whose coefficients are probabilities of Poisson distribution:

$$\mathcal{H}(m; z) = z + \sum_{n=2}^{\infty} \frac{m^{n-1}}{(n-1)!} e^{-m} z^n, \quad (z \in \mathbb{U}). \tag{2}$$

Also, we define the function

$$\begin{aligned}\psi(m, \mu; z) &= (1 - \mu)\mathcal{H}(m; z) + \mu z(\mathcal{H}(m; z))' \\ &= z + \sum_{n=2}^{\infty} [1 + \mu(n-1)] \frac{m^{n-1}}{(n-1)!} e^{-m} z^n \ (\mu \geq 0).\end{aligned}$$

and

$$\begin{aligned}\mathcal{N}(m, \mu, \lambda; z) &= (1 - \mu + \lambda)\mathcal{H}(m; z) + (\mu - \lambda) z(\mathcal{H}(m; z))' \\ &\quad + \mu\lambda z^2 (\mathcal{H}(m; z))'' \ (z \in \mathcal{U};\ \mu,\ \lambda \geq 0; \mu \geq \lambda) \\ &= z + \sum_{n=2}^{\infty} [1 + (n-1)(\mu - \lambda + n\mu\lambda)] \frac{m^{n-1}}{(n-1)!} e^{-m} z^n.\end{aligned}$$

The properties of a function $\psi(m, \mu; z)$ was studied by Shukla and Shukla [18] and Tang and Deng [20] with $p = 1$.

Now, we defined the linear operator $\mathcal{X}_m : \mathcal{A} \to \mathcal{A}$ defined by

$$\begin{aligned}[\mathcal{X}_m(f)](z) &= \mathcal{M}(m, z) * f(z) \\ &= z + \sum_{n=2}^{\infty} \frac{m^{n-1}}{(n-1)!} a_n e^{-m} z^n.\end{aligned}$$

In this paper, we obtained some condition on Poisson distribution series and some related series to be in subclasses of analytic function.

2. Main Results

Unless otherwise mentioned, we assume that $0 \leq \alpha < 1$, $k \geq 0$, $|\theta| < \frac{\pi}{2}$, $-1 \leq A < B \leq 1$, $m > 0$, $\mu, \lambda \geq 0$ and $\mu \geq \lambda$. To prove our results, we will need the following lemmas.

Lemma 1. *[1] [Theorem 4, with $p = 1$] A sufficient condition for $f(z)$ defined by (1) to be in the class $\mathcal{L}(A, B, \theta; \alpha)$ is:*

$$\sum_{n=2}^{\infty} n(1 + |B|)|a_n| \leq (B - A)(1 - \alpha)\cos\theta.$$

Lemma 2. *[1] [Theorem 1, with $p = 1$] A sufficient condition for $f(z)$ defined by (1) to be in the class $\mathcal{L}(A, B, \theta; \alpha)$ is:*

$$|a_n| \leq \frac{(B - A)(1 - \alpha)\cos\theta}{n} \quad (n \geq 2).$$

Lemma 3. *[7] Let $f(z) \in \mathcal{A}$. For some k, the following inequality*

$$\sum_{n=2}^{\infty} n(n-1)|a_n| \leq \frac{1}{k+2},$$

holds, then $f \in \mathcal{UCV}(k)$. The number $\frac{1}{k+2}$ cannot be increased.

Lemma 4. *[6] Let $f(z) \in \mathcal{A}$. For some k, the following inequality*

$$\sum_{n=2}^{\infty} (n + k(n-1))|a_n| \leq 1,$$

holds, then $f \in \mathcal{TS}(k)$.

Theorem 5. *The sufficient condition for $\mathcal{H}(m;z)$ to be in the class $\mathcal{L}(A,B,\theta;\alpha)$ is*

$$m - e^{-m} + 1 \leq \frac{(B-A)(1-\alpha)\cos\theta}{1+|B|}. \tag{3}$$

Proof. Since

$$\mathcal{H}(m;z) = z + \sum_{n=2}^{\infty} \frac{m^{n-1}}{(n-1)!} e^{-m} z^n, \quad (z \in \mathbb{U}).$$

By applying Lemma 1, we need to prove that

$$\sum_{n=2}^{\infty} n(1+|B|) \left| \frac{m^{n-1}}{(n-1)!} e^{-m} \right| \leq (B-A)(1-\alpha)\cos\theta.$$

Thus,

$$\begin{aligned}
I &= \sum_{n=2}^{\infty} n(1+|B|) \frac{m^{n-1}}{(n-1)!} e^{-m} \\
&= (1+|B|)e^{-m} \left[\sum_{n=2}^{\infty} (n-1) \frac{m^{n-1}}{(n-1)!} + \sum_{n=2}^{\infty} \frac{m^{n-1}}{(n-1)!} \right] \\
&= (1+|B|)e^{-m} \left[\sum_{n=2}^{\infty} \frac{m^{n-1}}{(n-2)!} + \sum_{n=2}^{\infty} \frac{m^{n-1}}{(n-1)!} \right] \\
&= (1+|B|)e^{-m} \left[m \sum_{n=0}^{\infty} \frac{m^n}{n!} + \sum_{n=1}^{\infty} \frac{m^n}{n!} \right] \\
&= (1+|B|)e^{-m} \left[m \sum_{n=0}^{\infty} \frac{m^n}{n!} + \sum_{n=0}^{\infty} \frac{m^n}{n!} - 1 \right] \\
&= (1+|B|)[m+1-e^{-m}].
\end{aligned}$$

But this last equation is bounded by $(B-A)(1-\alpha)\cos\theta$ if Eq. (3) is holds. This completes the prove of theorem 5.

Corollary 6. *Let $A = -1$ and $B = 1$ in Theorem 5, then the sufficient condition for $\mathcal{H}(m;z)$ to be in the class $\mathcal{L}(\theta;\alpha)$ is*

$$m - e^{-m} + 1 \leq (1-\alpha)\cos\theta.$$

Corollary 7. *Let $\alpha = 0$ in Theorem 5, then the sufficient condition for $\mathcal{H}(m;z)$ to be in the class $\mathcal{L}(A,B,\theta)$ is*

$$m - e^{-m} + 1 \leq \frac{(B-A)\cos\theta}{1+|B|}.$$

Corollary 8. *Let $A = -\beta$, $B = \beta$, $\theta = 0$ and $\alpha = 0$ in Theorem 5, then the sufficient condition for $\mathcal{H}(m;z)$ to be in the class $\mathcal{L}(-\beta,\beta,0;0) = \mathcal{D}(\beta)$ is*

$$m - e^{-m} + 1 \leq \frac{2\beta}{1+|\beta|}.$$

Corollary 9. *Let $A = -\beta$, $B = \beta$ and $\theta = 0$ in Theorem 5, then the sufficient condition for $\mathcal{H}(m;z)$ to be in the class $\mathcal{R}(\beta;\alpha)$ is*

$$m - e^{-m} + 1 \leq \frac{2\beta(1-\alpha)}{1+|\beta|}.$$

Theorem 10. *The sufficient condition for $\psi(m,\mu;z)$ to be in the class $\mathcal{L}(A,B,\theta;\alpha)$ is*

$$\mu m^2 + (1+2\mu)m - e^{-m} + 1 \leq \frac{(B-A)(1-\alpha)\cos\theta}{1+|B|}. \tag{4}$$

Proof. Since

$$\psi(m,\mu;z) = z + \sum_{n=2}^{\infty}[1+\mu(n-1)]\frac{m^{n-1}}{(n-1)!}e^{-m}z^n, \quad (z \in \mathbb{U}).$$

By applying Lemma 1, we need to prove that

$$\sum_{n=2}^{\infty} n(1+|B|)\left|[1+\mu(n-1)]\frac{m^{n-1}}{(n-1)!}e^{-m}\right| \leq (B-A)(1-\alpha)\cos\theta.$$

Thus,

$$\begin{aligned}
I_1 &= \sum_{n=2}^{\infty} n(1+|B|)[1+\mu(n-1)]\frac{m^{n-1}}{(n-1)!}e^{-m} \\
&= (1+|B|)e^{-m}\left[\sum_{n=2}^{\infty}(1+2\mu)(n-1)\frac{m^{n-1}}{(n-1)!} + \sum_{n=2}^{\infty}\frac{m^{n-1}}{(n-1)!} + \sum_{n=2}^{\infty}\mu(n-1)(n-2)\frac{m^{n-1}}{(n-1)!}\right] \\
&= (1+|B|)e^{-m}\left[\sum_{n=2}^{\infty}(1+2\mu)\frac{m^{n-1}}{(n-2)!} + \sum_{n=2}^{\infty}\frac{m^{n-1}}{(n-1)!} + \mu\sum_{n=3}^{\infty}\frac{m^{n-1}}{(n-3)!}\right] \\
&= (1+|B|)e^{-m}\left[m(1+2\mu)\sum_{n=0}^{\infty}\frac{m^n}{n!} + \sum_{n=1}^{\infty}\frac{m^n}{n!} + \mu m^2\sum_{n=0}^{\infty}\frac{m^n}{n!}\right] \\
&= (1+|B|)e^{-m}\left[m(1+2\mu)\sum_{n=0}^{\infty}\frac{m^n}{n!} + \sum_{n=0}^{\infty}\frac{m^n}{n!} - 1 + \mu m^2\sum_{n=0}^{\infty}\frac{m^n}{n!}\right] \\
&= (1+|B|)[\mu m^2 + (1+2\mu)m + 1 - e^{-m}].
\end{aligned}$$

But this last equation is bounded by $(B-A)(1-\alpha)\cos\theta$ if Eq. (4) is holds. This completes the prove of theorem 10.

Corollary 11. *Let $A = -1$ and $B = 1$ in Theorem 10, then the sufficient condition for $\mathcal{H}(m;z)$ to be in the class $\mathcal{L}(\theta;\alpha)$ is*

$$\mu m^2 + (1+2\mu)m - e^{-m} + 1 \leq (1-\alpha)\cos\theta.$$

Corollary 12. *Let $\alpha = 0$ in Theorem 10, then the sufficient condition for $\mathcal{H}(m;z)$ to be in the class $\mathcal{L}(A,B,\theta)$ is*

$$\mu m^2 + (1+2\mu)m - e^{-m} + 1 \leq \frac{(B-A)\cos\theta}{1+|B|}.$$

Corollary 13. *Let $A = -\beta$, $B = \beta$, $\theta = 0$ and $\alpha = 0$ in Theorem 10, then the sufficient condition for $\mathcal{H}(m;z)$ to be in the class $\mathcal{L}(-\beta,\beta,0;0) = \mathcal{D}(\beta)$ is*

$$\mu m^2 + (1+2\mu)m - e^{-m} + 1 \leq \frac{2\beta}{1+|\beta|}.$$

Corollary 14. *Let $A = -\beta$, $B = \beta$ and $\theta = 0$ in Theorem 10, then the sufficient condition for $\mathcal{H}(m;z)$ to be in the class $\mathcal{R}(\beta;\alpha)$ is*

$$\mu m^2 + (1+2\mu)m - e^{-m} + 1 \leq \frac{2\beta(1-\alpha)}{1+|\beta|}.$$

Theorem 15. *The sufficient condition for $\mathcal{N}(m,\mu,\lambda;z)$ to be in the class $\mathcal{L}(A,B,\theta;\alpha)$ is*

$$\mu\lambda m^3 + (\mu-\lambda+5\mu\lambda)m^2 + (2\mu-2\lambda+1)m + 4\mu\lambda - e^{-m} + 1 \leq \frac{(B-A)(1-\alpha)\cos\theta}{1+|B|}. \quad (5)$$

Proof. Since

$$\mathcal{N}(m,\mu,\lambda;z) = z + \sum_{n=2}^{\infty}[1+(n-1)(\mu-\lambda+n\mu\lambda)]\frac{m^{n-1}}{(n-1)!}e^{-m}z^n, \quad (z \in \mathbb{U}).$$

By applying Lemma 1, we need to prove that

$$\sum_{n=2}^{\infty} n(1+|B|)\left|[1+(n-1)(\mu-\lambda+n\mu\lambda)]\frac{m^{n-1}}{(n-1)!}e^{-m}\right| \leq (B-A)(1-\alpha)\cos\theta.$$

Thus,

$$
\begin{aligned}
I_2 &= \sum_{n=2}^{\infty} n(1+|B|)[1+(n-1)(\mu-\lambda+n\mu\lambda)]\frac{m^{n-1}}{(n-1)!}e^{-m} \\
&= (1+|B|)e^{-m}\Big[\sum_{n=2}^{\infty}(1+2\mu-2\lambda)(n-1)\frac{m^{n-1}}{(n-1)!}+\sum_{n=2}^{\infty}\frac{m^{n-1}}{(n-1)!} \\
&+ \sum_{n=2}^{\infty}(\mu-\lambda+5\mu\lambda)(n-1)(n-2)\frac{m^{n-1}}{(n-1)!}+\sum_{n=2}^{\infty}\mu\lambda(n-1)(n-2)(n-3)\frac{m^{n-1}}{(n-1)!}\Big] \\
&= (1+|B|)e^{-m}\Big[\sum_{n=2}^{\infty}(1+2\mu-2\lambda)m\frac{m^{n-2}}{(n-2)!}+\sum_{n=2}^{\infty}\frac{m^{n-1}}{(n-1)!}+\sum_{n=3}^{\infty}(\mu-\lambda+5\mu\lambda)m^2\frac{m^{n-3}}{(n-3)!} \\
&+ \sum_{n=2}^{\infty}\mu\lambda m^3\frac{m^{n-4}}{(n-4)!}\Big] \\
&= (1+|B|)e^{-m}\left[(1+2\mu-2\lambda)m\sum_{n=0}^{\infty}\frac{m^n}{n!}+\sum_{n=1}^{\infty}\frac{m^n}{n!}+(\mu-\lambda+5\mu\lambda)m^2\sum_{n=0}^{\infty}\frac{m^n}{n!}+\mu\lambda m^3\sum_{n=0}^{\infty}\frac{m^n}{n!}\right] \\
&= (1+|B|)e^{-m}\left[m(1+2\mu-2\lambda)\sum_{n=0}^{\infty}\frac{m^n}{n!}+\sum_{n=0}^{\infty}\frac{m^n}{n!}-1+(\mu-\lambda+5\mu\lambda)m^2\sum_{n=0}^{\infty}\frac{m^n}{n!}+\mu\lambda m^3\sum_{n=0}^{\infty}\frac{m^n}{n!}\right] \\
&= (1+|B|)[\mu\lambda m^3+(\mu-\lambda+5\mu\lambda)m^2+(1+2\mu-2\lambda)m+1+4\mu\lambda-e^{-m}].
\end{aligned}
$$

But this last equation is bounded by $(B-A)(1-\alpha)\cos\theta$ if Eq. (5) is holds. This completes the prove of theorem 15.

Corollary 16. *Let $A=-1$ and $B=1$ in Theorem 15, then the sufficient condition for $\mathcal{H}(m;z)$ to be in the class $\mathcal{L}(\theta;\alpha)$ is*

$$\mu\lambda m^3+(\mu-\lambda+5\mu\lambda)m^2+(2\mu-2\lambda+1)m+4\mu\lambda-e^{-m}+1\leq(1-\alpha)\cos\theta.$$

Corollary 17. *Let $\alpha=0$ in Theorem 15, then the sufficient condition for $\mathcal{H}(m;z)$ to be in the class $\mathcal{L}(A,B,\theta)$ is*

$$\mu\lambda m^3+(\mu-\lambda+5\mu\lambda)m^2+(2\mu-2\lambda+1)m+4\mu\lambda-e^{-m}+1\leq\frac{(B-A)\cos\theta}{1+|B|}.$$

Corollary 18. *Let $A=-\beta$, $B=\beta$, $\theta=0$ and $\alpha=0$ in Theorem 15, then the sufficient condition for $\mathcal{H}(m;z)$ to be in the class $\mathcal{L}(-\beta,\beta,0;0)=\mathcal{D}(\beta)$ is*

$$\mu\lambda m^3+(\mu-\lambda+5\mu\lambda)m^2+(2\mu-2\lambda+1)m+4\mu\lambda-e^{-m}+1\leq\frac{2\beta}{1+|\beta|}.$$

Corollary 19. *Let $A=-\beta$, $B=\beta$ and $\theta=0$ in Theorem 15, then the sufficient condition for $\mathcal{H}(m;z)$ to be in the class $\mathcal{R}(\beta;\alpha)$ is*

$$\mu\lambda m^3+(\mu-\lambda+5\mu\lambda)m^2+(2\mu-2\lambda+1)m+4\mu\lambda-e^{-m}+1\leq\frac{2\beta(1-\alpha)}{1+|\beta|}.$$

3. Inclusion Properties

Theorem 20. *If condition*

$$m^2+3m-e^{-m}+1\leq \frac{(B-A)(1-\alpha)\cos\theta}{1+|B|}. \tag{6}$$

holds, then $[\mathcal{X}_m(f)](z)$ *maps the class* $\mathcal{S}$ *or* $(\mathcal{S}^*)$ *to the class* $\mathcal{L}(A,B,\theta;\alpha)$.

Proof. Since

$$[\mathcal{X}_m(f)](z)=z+\sum_{n=2}^{\infty}\frac{m^{n-1}}{(n-1)!}e^{-m}a_nz^n, \quad (z\in\mathbb{U}).$$

By applying Lemma 1, we need to prove that

$$\sum_{n=2}^{\infty}n(1+|B|)\left|\frac{m^{n-1}}{(n-1)!}\right||a_n|e^{-m}\leq (B-A)(1-\alpha)\cos\theta.$$

Using $f(z)\in S$, then the inequality $|a_n|\leq n$ holds, we obtain that

$$\begin{aligned}
I_3 &\leq \sum_{n=2}^{\infty}n^2(1+|B|)\frac{m^{n-1}}{(n-1)!}e^{-m}\\
&=(1+|B|)e^{-m}\left[\sum_{n=2}^{\infty}(n-1)(n-2)\frac{m^{n-1}}{(n-1)!}+\sum_{n=2}^{\infty}3(n-1)\frac{m^{n-1}}{(n-1)!}+\sum_{n=2}^{\infty}\frac{m^{n-1}}{(n-1)!}\right]\\
&=(1+|B|)e^{-m}\left[\sum_{n=2}^{\infty}\frac{m^{n-1}}{(n-3)!}+3\sum_{n=2}^{\infty}\frac{m^{n-1}}{(n-2)!}+\sum_{n=2}^{\infty}\frac{m^{n-1}}{(n-1)!}\right]\\
&=(1+|B|)e^{-m}\left[m^2\sum_{n=0}^{\infty}\frac{m^n}{n!}+3m\sum_{n=0}^{\infty}\frac{m^n}{n!}+\sum_{n=1}^{\infty}\frac{m^n}{n!}\right]\\
&=(1+|B|)e^{-m}\left[m^2\sum_{n=0}^{\infty}\frac{m^n}{n!}+3m\sum_{n=0}^{\infty}\frac{m^n}{n!}+\sum_{n=0}^{\infty}\frac{m^n}{n!}-1\right]\\
&=(1+|B|)[m^2+3m+1-e^{-m}].
\end{aligned}$$

But this last equation is bounded by $(B-A)(1-\alpha)\cos\theta$ if Eq. (6) is holds. This completes the prove of theorem 20.

Corollary 21. *Let* $A=-1$ *and* $B=1$ *in Theorem 20, then* $[\mathcal{X}_m(f)](z)$ *maps the class* $\mathcal{S}$ *or* $(\mathcal{S}^*)$ *to the class* $\mathcal{L}(\theta;\alpha)$ *if*

$$m^2+3m-e^{-m}+1\leq (1-\alpha)\cos\theta$$

is true.

Corollary 22. *Let $\alpha = 0$ in Theorem 20, then $[\mathcal{X}_m(f)](z)$ maps the class $\mathcal{S}$ or $(\mathcal{S}^*)$ to the class $\mathcal{L}(A, B, \theta)$ if*

$$m^2 + 3m - e^{-m} + 1 \leq \frac{(B - A)\cos\theta}{1 + |B|},$$

is true

Corollary 23. *Let $A = -\beta$, $B = \beta$, $\theta = 0$ and $\alpha = 0$ in Theorem 20, then $[\mathcal{X}_m(f)](z)$ maps the class $\mathcal{S}$ or $(\mathcal{S}^*)$ to the class $\mathcal{L}(-\beta, \beta, 0; 0) = \mathcal{D}(\beta)$ if*

$$m^2 + 3m - e^{-m} + 1 \leq \frac{2\beta}{1 + |\beta|},$$

is true.

Corollary 24. *Let $A = -\beta$, $B = \beta$ and $\theta = 0$ in Theorem 20, then $[\mathcal{X}_m(f)](z)$ maps the class $\mathcal{S}$ or $(\mathcal{S}^*)$ to the class $\mathcal{R}(\beta; \alpha)$ if*

$$m^2 + 3m - e^{-m} + 1 \leq \frac{2\beta(1 - \alpha)}{1 + |\beta|},$$

is true.

Theorem 25. *If condition in Eq. (3) satisfied, then $[\mathcal{X}_m(f)](z)$ maps the class $\mathcal{K}$ to the class $\mathcal{L}(A, B, \theta; \alpha)$.*

Proof. Since

$$[\mathcal{X}_m(f)](z) = z + \sum_{n=2}^{\infty} \frac{m^{n-1}}{(n-1)!} e^{-m} a_n z^n, \quad (z \in \mathbb{U}).$$

By applying Lemma 1, we need to prove that

$$\sum_{n=2}^{\infty} n(1 + |B|) \left| \frac{m^{n-1}}{(n-1)!} \right| |a_n| e^{-m} \leq (B - A)(1 - \alpha)\cos\theta.$$

Using $f(z) \in S$, then the inequality $|a_n| \leq 1$ holds, we obtain the required result. This completes the prove of theorem 25.

Theorem 26. *If condition*

$$(B - A)(1 - \alpha) m \cos\theta \leq \frac{1}{k + 2}, \tag{7}$$

holds, then $[\mathcal{X}_m(f)](z)$ maps the class $\mathcal{L}(A, B, \theta; \alpha)$ to the class $k - UCV$.

Proof. Since

$$[\mathcal{X}_m(f)](z) = z + \sum_{n=2}^{\infty} \frac{m^{n-1}}{(n-1)!} e^{-m} a_n z^n, \quad (z \in \mathbb{U}).$$

By applying Lemma 3, we need to prove that

$$\sum_{n=2}^{\infty} n(n-1) \left| \frac{m^{n-1}}{(n-1)!} \right| |a_n| e^{-m} \leq \frac{1}{k+2}.$$

Thus,

$$\begin{aligned} I_4 &\leq \sum_{n=2}^{\infty} (n-1) \frac{m^{n-1}}{(n-1)!} e^{-m} (B-A)(1-\alpha) \cos\theta \\ &= (B-A)(1-\alpha) e^{-m} \cos\theta \sum_{n=2}^{\infty} \frac{m^{n-1}}{(n-2)!} \\ &= (B-A)(1-\alpha) e^{-m} m \cos\theta \sum_{n=0}^{\infty} \frac{m^n}{n!} \\ &= (B-A)(1-\alpha) m \cos\theta. \end{aligned}$$

But this last equation is bounded by $\frac{1}{k+2}$ if Eq. (7) is holds. This completes the prove of theorem 26.

Corollary 27. *Let $A = -1$ and $B = 1$ in Theorem 26, then $[\mathcal{X}_m(f)](z)$ maps the class $\mathcal{L}(\theta; \alpha)$ to the class $k - UCV$ if*

$$2(1-\alpha) m \cos\theta \leq \frac{1}{k+2},$$

is true.

Corollary 28. *Let $\alpha = 0$ in Theorem 26, then $[\mathcal{X}_m(f)](z)$ maps the class $\mathcal{L}(A, B, \theta)$ to the class $k - UCV$ if*

$$(B-A) m \cos\theta \leq \frac{1}{k+2},$$

is true

Corollary 29. *Let $A = -\beta$, $B = \beta$, $\theta = 0$ and $\alpha = 0$ in Theorem 26, then $[\mathcal{X}_m(f)](z)$ maps the class $\mathcal{L}(-\beta, \beta, 0; 0) = \mathcal{D}(\beta)$ to the class $k - UCV$ if*

$$2\beta m \leq \frac{1}{k+2},$$

is true.

Corollary 30. *Let $A = -\beta$, $B = \beta$ and $\theta = 0$ in Theorem 26, then $[\mathcal{X}_m(f)](z)$ maps the class $\mathcal{R}(\beta;\alpha)$ to the class $k - UCV$ if*

$$2\beta(1-\alpha)m \leq \frac{1}{k+2},$$

is true.

Theorem 31. *If condition*

$$(B-A)(1-\alpha)\cos\theta\left[(k+1-\frac{k}{m})(1-e^{-m})+ke^{-m}\right] \leq 1, \tag{8}$$

holds, then $[\mathcal{X}_m(f)](z)$ maps the class $\mathcal{L}(A, B, \theta; \alpha)$ to the class $k - ST$.

Proof. Since

$$[\mathcal{X}_m(f)](z) = z + \sum_{n=2}^{\infty} \frac{m^{n-1}}{(n-1)!} e^{-m} a_n z^n, \quad (z \in \mathbb{U}).$$

By applying Lemma 4, we need to prove that

$$\sum_{n=2}^{\infty}(n(1+k)-k)\left|\frac{m^{n-1}}{(n-1)!}\right| |a_n| e^{-m} \leq 1.$$

Thus,

$$\begin{aligned}
I_5 &\leq \sum_{n=2}^{\infty}(n(1+k)-k)\frac{m^{n-1}}{(n-1)!}e^{-m}\frac{(B-A)(1-\alpha)\cos\theta}{n}\\
&= (B-A)(1-\alpha)e^{-m}\cos\theta\left[\sum_{n=2}^{\infty}(1+k)\frac{m^{n-1}}{(n-1)!} - \sum_{n=2}^{\infty}k\frac{m^{n-1}}{n!}\right]\\
&= (B-A)(1-\alpha)e^{-m}\cos\theta\left[(k+1)\sum_{n=1}^{\infty}\frac{m^n}{n!} - \frac{k}{m}\sum_{n=2}^{\infty}\frac{m^n}{n!}\right]\\
&= (B-A)(1-\alpha)e^{-m}\cos\theta\left[(k+1)\left(\sum_{n=0}^{\infty}\frac{m^n}{n!}-1\right) - \frac{k}{m}\left(\sum_{n=0}^{\infty}\frac{m^n}{n!}-1-m\right)\right]\\
&= (B-A)(1-\alpha)\cos\theta\left[(k+1-\frac{k}{m})(1-e^{-m})+ke^{-m}\right].
\end{aligned}$$

But this last equation is bounded by 1 if Eq. (8) is holds. This completes the prove of theorem 31.

Corollary 32. *Let $A = -1$ and $B = 1$ in Theorem 31, then $[\mathcal{X}_m(f)](z)$ maps the class $\mathcal{L}(\theta;\alpha)$ to the class $k-ST$ if*

$$(1-\alpha)\cos\theta\left[(k+1-\frac{k}{m})(1-e^{-m})+ke^{-m}\right]\leq\frac{1}{2},$$

is true.

Corollary 33. *Let $\alpha = 0$ in Theorem 31, then $[\mathcal{X}_m(f)](z)$ maps the class $\mathcal{L}(A, B, \theta)$ to the class $k-ST$ if*

$$(B-A)\cos\theta\left[(k+1-\frac{k}{m})(1-e^{-m})+ke^{-m}\right]\leq 1,$$

is true

Corollary 34. *Let $A = -\beta$, $B = \beta$, $\theta = 0$ and $\alpha = 0$ in Theorem 31, then $[\mathcal{X}_m(f)](z)$ maps the class $\mathcal{L}(-\beta,\beta,0;0) = \mathcal{D}(\beta)$ to the class $k-ST$ if*

$$\left[(k+1-\frac{k}{m})(1-e^{-m})+ke^{-m}\right]\leq\frac{1}{2\beta},$$

is true.

Remark 1. *Putting $\beta = 1$ and $k = 0$ in Corollary 34, we obtain the result obtained by Porwal and Kumar [14] Theorem 3.2, with $A = 1$, $B = -1$, $|\tau| = 1$, $\lambda = 0$ and $\alpha = 0$.*

Corollary 35. *Let $A = -\beta$, $B = \beta$ and $\theta = 0$ in Theorem 31, then $[\mathcal{X}_m(f)](z)$ maps the class $\mathcal{R}(\beta;\alpha)$ to the class $k-ST$ if*

$$(1-\alpha)\left[(k+1-\frac{k}{m})(1-e^{-m})+ke^{-m}\right]\leq\frac{1}{2\beta},$$

is true.

References

[1] M. K. Aouf, *On certain subclass of analytic p-valent functions of order alpha*, Rend. Mat., 7 (8) (1988), 89-104.

[2] T. R. Caplinger and W. M. Causey, *A class of univalent functions*, Proc. Am. Math. Soc., 39 (1973), 357-361.

[3] P. L. Duren, *Univalent functions*, Springer-Verlag, New York, 1983.

[4] O. P. Juneja and M. L. Mogra, *A class of univalent functions*, Bull. Sci. Math. 2^e Ser., 103 (4)(1979), 435-447.

[5] S. Kanas and H. M. Srivastava, *Linear operators associated with k-uniformly convex functions*, Integral Transform. Spec. Funct., 9 (2)(2000), 121-132.

[6] S. Kanas and A. Wisniowska, *Conic regions and k-starlike functions*, Rev. Roum. Math. Pures Appl., 54 (2000), 647-657.

[7] S. Kanas and A. Wisniowska, *Conic regions and k-uniform convexity*, Comput. Appl. Math., 105 (1999), 327-336.

[8] T. H. MacGregor, *The radius of convexity for starlike function of order α*, Proc. Am. Math. Soc., 14 (1963), 71-76.

[9] S. S. Miller and P. T. Mocanu, *Differntial Subordination: Theory and Applications*, Series on Monographs and Textbooks in Pure and Applied Mathematics, Vol. 225, Marcel Dekker Incorporated, NewYork and Basel, 2000.

[10] K. S. Padmanabhan, *On certain class of functions whose derivatives have a positive real part in the unit disc*, Ann. Polon. Math., 23 (1970), 73-81.

[11] B. Pinchuk, *On the starlike and convex functions of order α*, Duke Math. J., 35 (1968), 721-734.

[12] S. Porwal, *An application of a Poisson distribution series on certain analytic functions*, J. Complex Anal., Ar. ID 984135 (2014), 1-3.

[13] S. Porwal and K. K. Dixit, *An application of generalized Bessel functions on certain analytic functions*, Acta Universitatis Matthiae Belii, Series Mathematics, (2013), 51-57.

[14] S. Porwal and M. Kumar, *A unified study on starlike and convex functions associated with Poisson distribution series*, Afr. Mat., DOI:10.1007/s13370-016-0398-z, 2016.

[15] M. S. Robertson, *On the theory of univalent functions*, Ann. Math., 37 (1936), 374-408.

[16] A. K. Sharma, S. Porwal and K. K. Dixit, *Class mappings propertiws of convolutions involving certain univalent functions associated with hypergeometric functions*, Electronic J. Math. Anal. Appl., 1 (2) (2013), 326-333.

[17] S. L. Shukla and Dashrath, *On a class of univalent functions*, Soochow J. Math., 10 (1984), 117-126.

[18] N. Shukla and P. Shukla, *Mapping properties of analytic function defined by hypergeometric function*, II, Soochow J. Math., 25 (1) (1999), 29-36.

[19] H. M. Srivastava and S. Owa (Eds.), *Current topics in analytic function theory*, World Scientific Publishing Company, Singapore, New Jersey, London, Hong Kong, 1992.

[20] H. Tang and G. T. Deng, *Subordination and superordination preserving properties for a family of integral operators involving the noorintegral operator*, J. Egypt. Math. Soc., 22 (2014), 352-361.

R. M. El-Ashwah
Department of Mathematics, Faculty of science,
Damietta University,
Damietta, Egypt.
email: *r_elashwah@yahoo.com*

W. Y. Kota
Department of Mathematics, Faculty of science,
Damietta University,
Damietta, Egypt.
email: *wafaa_kota@yahoo.com*