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# A STUDY ON $\phi$-SYMMETRIC $\tau$-CURVATURE TENSOR IN K-CONTACT MANIFOLD

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Abstract. The aim of this paper is the study of curvature properties for globally $\phi$-$\tau$-symmetric and $\tau$-Ricci $\eta$-parallel K-contact manifolds.

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## 1. Introduction

The notion of local symmetry of a Riemannian manifold has been weakened by many authors in several ways to a different extent. In the context of contact geometry the notion of $\phi$-symmetry is introduced and studied by E. Boeckx, P. Buecken and L. Vanhecke [3] with several examples. As a weaker version of local symmetry, T. Takahashi [14] introduced the notion of locally $\phi$-symmetry on a Sasakian manifold.

In [9] M.M. Tripathi and et.al. introduced the $\tau$-curvature tensor which consists of known curvatures like conformal, concircular, projective, $M$-projective, $W_i$-curvature tensor($i = 0, \ldots, 9$) and $W_j^*$-curvature tensor($j = 0, 1$). Futher, in [10], [11] M.M. Tripathi and et.al. studied $\tau$-curvature tensor in K-contact, Sasakian and Semi-Riemannian manifolds. Later in [12] the authors studied some properties of $\tau$-curvature tensor and they obtained some interesting results.

Motivated by all these works in this paper we study the globally $\phi$-Symmetric $\tau$-curvature tensor in K-contact manifold.

The $\tau$-curvature tensor is given by ([10], [11])

$$\begin{aligned} \tau(X,Y)Z &= a_0R(X,Y)Z + a_1S(Y,Z)X + a_2S(X,Z)Y + a_3S(X,Y)Z \\ &+ a_4g(Y,Z)QX + a_5g(X,Z)QY + a_6g(X,Y)QZ \\ &+ a_7r[g(Y,Z)X - g(X,Z)Y], \end{aligned} \tag{1}$$

where $a_0, \ldots, a_7$ are some smooth functions on $M$. For different values of $a_0, \ldots, a_7$ the $\tau$-curvature tensor reduces to the curvature tensor $R$, Quasi-Conformal curvature tensor, Conformal curvature tensor, Conharmonic curvature tensor, Concircular curvature tensor, Pseudo-projective curvature tensor, Projective curvature tensor, $M$-projective curvature tensor, $W_i$-curvature tensors $(i = 0, \ldots, 9)$, $W_j^*$-curvature tensors $(j = 0, 1)$.

## 2. Preliminaries

A $(2n+1)$-dimensional manifold $M$ is said to be an almost contact metric structure $(\phi, \xi, \eta, g)$ if it carries a tensor field $\phi$ of type $(1,1)$, a vector field $\xi$, 1-form $\eta$ and a Riemannian metric $g$ on $M$ satisfy,

$$\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \eta \circ \phi = 0, \quad \phi\xi = 0, \tag{2}$$
$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad g(X, \xi) = \eta(X). \tag{3}$$

Thus a manifold $M$ equipped with this structure is called an almost contact metric manifold $(M, \phi, \xi, \eta, g)$.

If on $(M, \phi, \xi, \eta, g)$ the exterior derivative of 1-form $\eta$ satisfies,

$$d\eta(X, Y) = g(X, \phi Y), \tag{4}$$

then the manifold is said to a contact metric manifold.

If the contact metric structure is normal then it is called a Sasakian structure. Note that an almost contact metric manifold defines Sasakian structure if and only if,

$$(\nabla_X \phi)Y = g(X, Y)\xi - \eta(Y)X, \tag{5}$$

where $\nabla$ denotes the Riemannian connection on $M$. Contact metric manifold with structure tensor $(\phi, \xi, \eta, g)$ in which the Killing vector field $\xi$ satisfies the condition $\nabla_\xi \xi = 0$, then $M$ is called the $K$-contact manifold.

In a $(2n+1)$-dimensional K-contact manifold the following relations hold:

$$\nabla_X \xi = -\phi X, \tag{6}$$
$$g(R(\xi, X)Y, \xi) = g(X, Y) - \eta(X)\eta(Y), \tag{7}$$
$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y, \tag{8}$$
$$R(\xi, X)\xi = -X + \eta(X)\xi, \tag{9}$$
$$S(X, \xi) = 2n\eta(X), \tag{10}$$
$$S(\phi X, \phi Y) = S(X, Y) - 2n\eta(X)\eta(Y), \tag{11}$$

where $R$ and $S$ are the Riemannian curvature and the Ricci tensor of $M$, respectively.

**Definition 1.** *A K-contact manifold $M$ is said to be locally $\phi$-symmetric if*

$$\phi^2((\nabla_W R)(X,Y)Z) = 0, \tag{12}$$

*for all vector fields $X$, $Y$, $Z$ and $W$ which are orthogonal to $\xi$. The notion was introduced by T. Takahashi [14] for Sasakian manifolds.*

**Definition 2.** *A K-contact manifold $M$ is said to be globally $\phi$-symmetric if*

$$\phi^2((\nabla_W R)(X,Y)Z) = 0, \tag{13}$$

*for all arbitrary vector fields $X$, $Y$, $Z$ and $W$ on $M$.*

**Definition 3.** *A K-contact manifold $M$ is said to be globally $\phi$-$\tau$-symmetric if*

$$\phi^2((\nabla_W \tau)(X,Y)Z) = 0,$$

*for all arbitrary vector fields $X$, $Y$, $Z$, $W$ and $\tau$ is the curvature tensor.*

**Definition 4.** *The Ricci tensor of a K-contact manifold is said to be $\eta$-parallel if it satisfies*

$$(\nabla_X S)(\phi Y, \phi Z) = 0, \tag{14}$$

*for all vector fields $X, Y, Z$. This notion of Ricci $\eta$-parallelity was first introduced by M. Kon [7] in a Sasakian manifold*

## 3. Globally $\phi$-symmetric K-contact manifold

In this section, we define globally $\phi$-symmetric K-contact manifold. From (2) and (13), we have

$$-((\nabla_W R)(X,Y)Z) + \eta((\nabla_W R)(X,Y)Z)\xi = 0. \tag{15}$$

We know that

$$g((\nabla_W R)(X,Y)Z, \xi) = -g((\nabla_W R)(X,Y)\xi, Z). \tag{16}$$

From (16) and (15), we have

$$((\nabla_W R)(X,Y)Z) = -g((\nabla_W R)(X,Y)\xi, Z)\xi. \tag{17}$$

Differentiating (8) and with the help of (6), we obtain

$$(\nabla_W R)(X,Y)\xi = -g(\phi W, Y)X + g(X, \phi W)Y + R(X,Y)\phi W, \tag{18}$$

By using (18) in (17), we get

$$\begin{aligned}((\nabla_W R)(X,Y)Z) &= \{g(\phi W,Y)g(X,Z) - g(X,\phi W)g(Y,Z) \\ &- g(R(X,Y)\phi W, Z)\}\xi, \end{aligned} \tag{19}$$

Again, if (19) holds, then (16) and (18) implies that the manifold is globally $\phi$-symmetric.
Thus, we can state the following:

**Theorem 1.** *A K-contact manifold is globally $\phi$-symmetric if and only if the relation (19) holds for any vector fields $X$, $Y$, $Z$ and $W$ tangent to $M$.*

Next, putting $Z = \xi$ in (17) and by virtue of (16), we have

$$(\nabla_W R)(X,Y)\xi = 0, \tag{20}$$

for any vector fields $X, Y, Z, W$ tangent to $M$. From (20) and (19), we get

$$R(X,Y)\phi W = g(\phi W, Y)X - g(X, \phi W)Y. \tag{21}$$

From (21), we get

$$R(X,Y)W = g(W,Y)X - g(X,W)Y. \tag{22}$$

Thus, the manifold is of constant curvature. Hence, we state the following theorem:

**Theorem 2.** *A globally $\phi$-symmetric K-contact manifold is a space of constant curvature.*

## 4. Globally $\phi$-$\tau$-symmetric K-contact manifold

In this section, we define globally $\phi$-$\tau$-symmetric K-contact manifold by

$$\phi^2((\nabla_W \tau)(X,Y)Z) = 0, \tag{23}$$

for all arbitrary vector fields $X, Y, Z, W$ on $M$.
From (2) and (23), we have

$$-((\nabla_W \tau)(X,Y)Z) + \eta((\nabla_W \tau)(X,Y)Z)\xi = 0. \tag{24}$$

By taking an inner product with respect to $U$, we get

$$-g((\nabla_W \tau)(X,Y)Z, U) + \eta((\nabla_W \tau)(X,Y)Z)g(\xi, U) = 0, \tag{25}$$

Let $\{e_i : i = 1, 2, \ldots, 2n+1\}$ be an orthonormal basis of the tangent space at any point of the manifold. Putting $X = U = e_i$ in (25) and taking summation over $i$, we get

$$-g((\nabla_W \tau)(e_i, Y)Z, e_i) + \eta((\nabla_W \tau)(e_i, Y)Z)g(\xi, e_i) = 0, \tag{26}$$

with the help of (1) and on simplification, we obtain

$$\begin{aligned}
- &\quad [a_0 + (2n+1)a_1 + a_2 + a_3](\nabla_W S)(Y, Z) - [a_4 + 2na_7](\nabla_W r)g(Y, Z) \\
- &\quad a_5 g((\nabla_W Q)Y, Z) - a_6 g((\nabla_W Q)Z, Y) + a_0 \eta((\nabla_W R)(\xi, Y)Z) + a_1(\nabla_W S)(Y, Z) \\
+ &\quad a_2(\nabla_W S)(\xi, Z)\eta(Y) + a_3(\nabla_W S)(Y, \xi)\eta(Z) + a_4 g(Y, Z)\eta((\nabla_W Q)\xi) \\
+ &\quad a_5 \eta(Z)\eta((\nabla_W Q)Y) + a_6 \eta(Y)\eta((\nabla_W Q)Z) + a_7(\nabla_W r)[g(Y, Z) - \eta(Y)\eta(Z)] = 0.
\end{aligned} \tag{27}$$

Putting $Z = \xi$ in (27) and on simplification, we get

$$(\nabla_W S)(Y, \xi) = \frac{[a_4 + 2na_7](\nabla_W r)}{[-a_0 - 2na_1 - a_2 - a_6]}\eta(Y), \tag{28}$$

if $Y = \xi$ in (28), we get

$$\frac{[a_4 + 2na_7](\nabla_W r)}{[-a_0 - 2na_1 - a_2 - a_6]} = 0. \tag{29}$$

The above equation (29) implies that $\frac{[a_4+2na_7]}{[-a_0-2na_1-a_2-a_6]} \neq 0$,

$$(\nabla_W r) = 0 \Longrightarrow r \;\; is \;\; constant. \tag{30}$$

From (30) and (28), we have

$$(\nabla_W S)(Y, \xi) = 0, \tag{31}$$

we know that

$$(\nabla_W S)(Y, \xi) = \nabla_W S(Y, \xi) - S(\nabla_W Y, \xi) - S(Y, \nabla_W \xi). \tag{32}$$

By using (6) and (10) in (32), we get

$$(\nabla_W S)(Y, \xi) = S(Y, \phi W) - 2ng(Y, \phi W). \tag{33}$$

From (33) and (31), we have

$$S(Y, \phi W) = 2ng(Y, \phi W). \tag{34}$$

Replacing $W = \phi W$ in (34), we have

$$S(Y, W) = 2ng(Y, W). \tag{35}$$

Hence we can state the following:

**Theorem 3.** *A globally $\phi$-$\tau$-symmetric K-contact manifold is an Einstein manifold.*

## 5. $\tau$-Ricci $\eta$-parallel K-contact manifold

In this section, we examine the notion of $\tau$-Ricci $\eta$-parallelity for a K-contact manifold. At first, we give the definition of $\tau$-Ricci $\eta$-parallelity:

**Definition 5.** *The $\tau$-Ricci tensor of a K-contact manifold is said to be $\eta$-parallel if it satisfies*

$$(\nabla_X S_\tau)(\phi Y, \phi Z) = 0. \tag{36}$$

*for all vector fields $X, Y, Z$.*

From, (1) we have

$$S_\tau(Y,Z) = [a_0 + (2n+1)a_1 + a_2 + a_3 + a_5 + a_6]S(Y,Z) + [a_4 + 2na_7]g(Y,Z). \tag{37}$$

Replacing $Y = \phi Y$ and $Z = \phi Z$, then we have

$$\begin{aligned} S_\tau(\phi Y, \phi Z) &= [a_0 + (2n+1)a_1 + a_2 + a_3 + a_5 + a_6]S(\phi Y, \phi Z) \\ &+ [a_4 + 2na_7]rg(\phi Y, \phi Z). \end{aligned} \tag{38}$$

Differentiating (38) with respect to $X$, we get

$$\begin{aligned} (\nabla_X S_\tau)(\phi Y, \phi Z) &= [a_0 + (2n+1)a_1 + a_2 + a_3 + a_5 + a_6](\nabla_X S)(\phi Y, \phi Z) \\ &+ [a_4 + 2na_7](\nabla_X r)g(\phi Y, \phi Z). \end{aligned} \tag{39}$$

Again, differentiating (11) and by virtue of (5), we obtain

$$\begin{aligned} (\nabla_X S)(\phi Y, \phi Z) &= (\nabla_X S)(Y,Z) + 2n[g(\phi X, Y)\eta(Z) + g(\phi X, Z)\eta(Y)] \\ &+ \eta(Y)S(X, \phi Z) + \eta(Z)S(\phi Y, X) \end{aligned} \tag{40}$$

By using (40) in (39), we have

$$\begin{aligned} (\nabla_X S_\tau)(\phi Y, \phi Z) &= [a_0 + (2n+1)a_1 + a_2 + a_3 + a_5 + a_6]\{(\nabla_X S)(Y,Z) + 2n[g(\phi X, Y)\eta(Z) \\ &+ g(\phi X, Z)\eta(Y)] + \eta(Y)S(X, \phi Z) + \eta(Z)S(\phi Y, X)\} \\ &+ [a_4 + 2na_7](\nabla_X r)g(\phi Y, \phi Z). \end{aligned} \tag{41}$$

If $(\nabla_X S_\tau)(\phi Y, \phi Z) = 0$, we get

$$\begin{aligned} (\nabla_X S)(Y,Z) &= -\frac{[a_4 + 2na_7](\nabla_X r)}{[a_0 + (2n+1)a_1 + a_2 + a_3 + a_5 + a_6]} g(\phi Y, \phi Z) \\ &- 2n[g(\phi X, Y)\eta(Z) + g(\phi X, Z)\eta(Y)] \\ &- \eta(Y)S(X, \phi Z) - \eta(Z)S(\phi Y, X). \end{aligned} \tag{42}$$

Hence we can state the following:

**Theorem 4.** *A K-contact manifold is $\tau$-Ricci $\eta$-parallel if and only if the equation (42) holds with* $[a_0 + (2n+1)a_1 + a_2 + a_3 + a_5 + a_6] \neq 0$.

Now, let $\{e_i : i = 1, 2, \dots, (2n+1)\}$, be an orthonormal basis of the tangent space at any point. Taking $Y = Z = e_i$ in (42) and then taking summation over $i$, we get

$$\begin{aligned}(\nabla_X S)(e_i, e_i) &= -\frac{[a_4 + 2na_7](\nabla_X r)}{[a_0 + (2n+1)a_1 + a_2 + a_3 + a_5 + a_6]} g(\phi e_i, \phi e_i) \\ &- 2n[g(\phi X, e_i)\eta(e_i) + g(\phi X, e_i)\eta(e_i)] \\ &- \eta(e_i)S(X, \phi e_i) - \eta(e_i)S(\phi e_i, X). \qquad (43)\end{aligned}$$

On simplification of (43), we get

$$\left[1 + \frac{(2n+1)[a_4 + 2na_7]}{[a_0 + (2n+1)a_1 + a_2 + a_3 + a_5 + a_6]}\right](\nabla_X r) = 0. \qquad (44)$$

So we have $(\nabla_X r) = 0$, which implies $r$ is constant, where $r$ is the scalar curvature of the manifold $M$. Hence we state the following theorem:

**Theorem 5.** *If a K-contact manifold is $\tau$-Ricci $\eta$-parallel, then the scalar curvature is constant.*

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