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SECOND ORDER SEMILINEAR EVOLUTION EQUATIONS WITH INFINITE DELAY VIA MEASURE OF NONCOMPACTNESS

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Abstract. This paper is concerned with the existence of mild solutions for second order semi-linear functional with delay in Banach space . Our analysis is based on the technique of measure of noncompactness and Schauder's fixed point theorem. An example illustrating the abstract theory is given.

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1. Introduction

In this paper, we consider the existence of mild solutions of the following second order evolution equation

$$y'' - A(t)y = f(t, y_t),\ t \in J, \tag{1}$$

$$y_0 = \phi,\ y'(0) = \tilde{y}, \tag{2}$$

where $J = [0, T]$ and $\{A(t)\}_{0\leq t<+\infty}$ is a family of linear closed operators from E into E that generate an evolution system of operators $\{\mathcal{U}(t, s)\}_{(t,s)\in J\times J}$ for $0 \leq s \leq t < +\infty$, $f : J \times \mathcal{B} \to E$ be a Carathéodory function and $\mathcal{B}$ is an abstract phase space to be specified later, $\tilde{y} \in E$, $\phi \in \mathcal{B}$ and $(E, |\cdot|)$ a real Banach space.

For any continuous function y and any $t \geq 0$, we denote by y_t the element of $\mathcal{B}$ defined by $y_t(\theta) = y(t + \theta)$ for $\theta \in (-\infty, 0]$. Here $y_t(\cdot)$ represents the history of the state up to the present time t. We assume that the histories y_t belong to $\mathcal{B}$.

Evolution equations arise in many areas of applied mathematics [2, 28], this type of equations has received much attention in recent years [1]. There is many results concerning the second-order differential equations, see for exemple Fattorini [13], Travis and Webb [27], Henríquez and Vásquez [17], Travis and Webb [27], Baliki and Benchohra [7] and Henríquez *et al.* [18]. Among useful for the study of abstract second order equations is the existence of an evolution system $\mathcal{U}(t,s)$ for the homogenous equation

$$y''(t) = A(t)y(t), \quad \text{for} \quad t \geq 0.$$

For this purpose there are many techniques to show the existence of $\mathcal{U}(t,s)$ which has been developed by Kozak [21].

In recent years we see an increasing interest in infinite delay equations. The main reason is that equations of this type become more and more important for different applications. When the delay is infinite, the notion of the phase space $\mathcal{B}$ plays an important role in the study of both qualitative and quantitative theory. A usual choice is a seminormed space satisfying suitable axioms, which was introduced by Hale and Kato in [15], see also the books by Ahmed [2], Corduneanu and Lakshmikantham [10], Kappel and Schappacher [20]. For a detailed discussion and applications on this topic, we refer the reader to the book by Hale and Verduyn Lunel [16], Hino *et al.* [19], Wu [28] and Baghli and Benchohra [6] and the references therein.

In this paper we use the technique of measures of noncompactness. It is well known that this method provides an excellent tool for obtaining existence of solutions of nonlinear differential equation. This technique works fruitfully for both integral and differential equations. More details are found in Akhmerov *et al.* [3], Alv́ares [4], Banaś and Goebel [8], Guo *et al.* [14], Dudek and Olszowy [11], Olszowy and Wędrychowicz [24], Aissani and Benchohra [5], Henríquez *et al.* [18] and the references therein.

The paper is organized as follows. In Section two, we recall some definitions and facts about evolution system. In Section three, we give conditions guaranteeing the existence of mild solutions of the problem (1)-(2). In Section four we present an illustrative example.

2. Preliminaries

Let $C(J,E)$ the Banach space of all continuous functions y mapping J into E equipped with the norm

$$\|y\| = \sup\{|y(t)| : t \in J\}.$$

Let $\mathcal{C}$ be the space defined by

$$\mathcal{C} = \{y : (-\infty, T] \to E \text{ such that } y|_J \in C(J,E) \text{ and } y_0 \in \mathcal{B}\},$$

we denote by $y|_J$ the restriction of y to J.
A measurable function $u : J \to E$ is Bochner integrable if and only if $|u|$ is Lebesgue

integrable. For properties of the Bochner integral, see for instance, Yosida [29].
As usual, by $L^1(J,E)$ we denote the Banach space of measurable functions $u : J \to E$ which are Bochner integrable and normed by

$$\|u\|_{L^1} = \int_0^T |u(t)|dt.$$

In this paper, we will employ an axiomatic definition of the phase space $\mathcal{B}$ introduced by Hale and Kato in [15] and follow the terminology used in [19]. Thus, $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ will be a seminormed linear space of functions mapping $(-\infty, 0]$ into E, and satisfying the following axioms :

(Λ_1) If $y : (-\infty, T) \to E, T > 0$, is continuous on $[0,T]$ and $y_0 \in \mathcal{B}$, then for any $t \in [0,T)$ the following conditions hold :
(i) $y_t \in \mathcal{B}$;
(ii) There exists a positive constant H such that $|y(t)| \leq H\|y_t\|_{\mathcal{B}}$;
(iii) There exist two functions $K(\cdot), M(\cdot) : J \to J$ independent of y with K continuous and M locally bounded such that :

$$\|y_t\|_{\mathcal{B}} \leq K(t)\sup\{\ |y(s)| : 0 \leq s \leq t\} + M(t)\|y_0\|_{\mathcal{B}}.$$

(Λ_2) For the function y in (Λ_1), y_t is a $\mathcal{B}-$valued continuous function for each $t \in J$.

(Λ_3) The space $\mathcal{B}$ is complete.

Remark 1. *In the sequel we set*

$$\gamma := \max\left\{\sup_{t\in J}\{K(t)\}, \sup_{t\in J}\{M(t)\}\right\}.$$

For other details we refer, for instance to the book by Hino *et al.* [19].

In what follows, let $\{A(t),\ t \geq 0\}$ be a family of closed linear operators on the Banach space E with domain $D(A(t))$ dense in E and independent of t.

In this work the existence of solution of problem (1)-(2) is related to the existence of an evolution operator $\mathcal{U}(t,s)$ for the following homogeneous problem

$$y''(t) = A(t)y(t) \qquad t \in J. \tag{3}$$

This concept of evolution operators has been developed by Kozak [21].

Definition 1. *A family $\mathcal{U}$ of bounded operators $\mathcal{U}(t,s) : E \to E$, $(t,s) \in \Delta := \{(t,s) \in J \times J : s \leq t\}$, is called an evolution operator of the equation (3) if de following conditions hold:*

(Π_1) *For any $x \in E$ the map $(t,s) \longmapsto \mathcal{U}(t,s)x$ is of continuously differentiable and*

- **(a)** *for any $t \in J$, $\mathcal{U}(t,t) = 0$.*
- **(b)** *for all $(t,s) \in \Delta$ and for any $x \in E$, $\frac{\partial}{\partial t}\mathcal{U}(t,s)x\big|_{t=s} = x$ and $\frac{\partial}{\partial s}\mathcal{U}(t,s)x\big|_{t=s} = -x$.*

(Π_2) *For all $(t,s) \in \Delta$, if $x \in D(A(t))$, then $\frac{\partial}{\partial s}\mathcal{U}(t,s)x \in D(A(t))$, the map $(t,s) \longmapsto \mathcal{U}(t,s)x$ is of class C^2 and*

- **(a)** $\frac{\partial^2}{\partial t^2}\mathcal{U}(t,s)x = A(t)\mathcal{U}(t,s)x$.
- **(b)** $\frac{\partial^2}{\partial s^2}\mathcal{U}(t,s)x = \mathcal{U}(t,s)A(s)x$
- **(c)** $\frac{\partial^2}{\partial s \partial t}\mathcal{U}(t,s)x\big|_{t=s} = 0$.

(Π_3) *For all $(t,s) \in \Delta$, then $\frac{\partial}{\partial s}\mathcal{U}(t,s)x \in D(A(t))$, there exist $\frac{\partial^3}{\partial t^2 \partial s}\mathcal{U}(t,s)x$, $\frac{\partial^3}{\partial s^2 \partial t}\mathcal{U}(t,s)x$ and*

- **(a)** $\frac{\partial^3}{\partial t^2 \partial s}\mathcal{U}(t,s)x = A(t)\frac{\partial}{\partial s}(t)\mathcal{U}(t,s)x$.
  Moreover, the map $(t,s) \longmapsto A(t)\frac{\partial}{\partial s}(t)\mathcal{U}(t,s)x$ is continuous.
- **(b)** $\frac{\partial^3}{\partial s^2 \partial t}\mathcal{U}(t,s)x = \frac{\partial}{\partial t}\mathcal{U}(t,s)A(s)x$.

Throughout this paper, we will use the following definition of the concept of measure of noncompactness [8].

Definition 2. *Let E be a Banach space and Ω_D the family of all nonempty and bounded subsets of E. The measure of noncompactness is a map*

$$\mu : \Omega_D \to [0; +\infty)$$

satisfying the following properties:

(i_1) $\mu(D) = 0$ if D is relatively compact,

(i_2) $\mu(\overline{D}) = \mu(D)$; $\overline{D}$ the hull of D,

(i_3) $\mu(C + D) \leq \mu(D) + \mu(D)$,

(i_4) $\mu(aD) = |a|\mu(D)$,

(i_5) $\mu(ConvD) = \mu(D)$; $ConvD$ the convex hull of D.

Denote by $\omega^T(y, \varepsilon)$ the modulus of continuity of y on the interval $[0, T]$ ie

$$\omega^T(y, \varepsilon) = \sup \{|y(t) - y(s)| ; t, s \in J], |t - s| \leq \varepsilon\}.$$

Moreover, let us put

$$\omega^T(D, \varepsilon) = \sup \left\{\omega^T(y, \varepsilon); y \in D\right\},$$

$$\omega_0^T(D) = \lim_{\varepsilon \to 0} \sup \omega^T(D, \varepsilon).$$

Lemma 1. *[8] If $\{D_n\}_{n=0}^{+\infty}$ is sequence of nonempty, bounded and closed subsets of E such that $D_{n+1} \subset D_n (n = 0, 1, 2...)$and if $\lim\limits_{n\to\infty} \mu(D_n) = 0$ for each $n \in \mathbb{N}$, then the intersection*

$$D_\infty = \cap_{n=0}^{+\infty} D_n$$

is nonempty and compact.

Lemma 2. *[9] If B is bounded subset of Banach space, then for each $\varepsilon > 0$ there is a sequence function $\{b_n\}_{n=0}^{\infty}$ such that*

$$\mu(B) \leq 2\mu(\{b_n\}_{n=0}^{\infty}) + \varepsilon.$$

We call $\tilde{B} \in L^1(J; E)$ uniformly integrable if there exists $\xi \in L^1(J; \mathbb{R}^+)$ such that $|x(s)| \leq \xi(s)$ for $x \in \tilde{B}$ and a.e. $s \in J$.

Lemma 3. *[22] If $\{B_n\}_{n=0}^{\infty} \subset L^1(J, E)$ is uniformly integrable, then $\mu(\{B_n\}_{n=0}^{\infty})$ is mesurable and*

$$\mu \left\{ \int_0^t B_n(s) ds \right\}_{n=0}^{\infty} \leq 2 \int_0^t \mu(\{B_n(s)\}_{n=0}^{\infty}) ds.$$

Theorem 4 (Schauder fixed point theorem)**.** *[12] Let X be a Banach space, $\mathcal{K}$ compact convex subset of X and $N : \mathcal{K} \to \mathcal{K}$ continuous map.Then N has at least one fixed point in $\mathcal{K}$.*

3. Main result

Definition 3. *A function $y \in \mathcal{C}$ is called a mild solution to the problem (1)-(2), if y is continuous and*

$$y(t) = \begin{cases} \phi(t), & \text{if } t \leq 0 \\ -\dfrac{\partial}{\partial s}\mathcal{U}(t, 0)\phi(0) + \mathcal{U}(t, 0)\tilde{y} + \displaystyle\int_0^t \mathcal{U}(t, s) f(s, y_s) ds, & \text{if } t \in J. \end{cases} \tag{4}$$

To prove our results we introduce the following conditions:

(H_1) There exists a constant $M \geq 1$ such that

$$\|\mathcal{U}(t,s)\|_{B(E)} \leq M \quad \text{for any } (t,s) \in \Delta.$$

(H_2) There exists a constant $\widetilde{M} \geq 0$ such that

$$\left\| \frac{\partial}{\partial s}\mathcal{U}(t,s) \right\|_{B(E)} \leq \widetilde{M}.$$

(H_3) There exist a function $p \in L^1(J, \mathbb{R}_+)$ and a continuous nondecreasing function $\psi : [0, +\infty) \to (0, +\infty)$ such that:

$$|f(t,u)| \leq p(t)\psi(\|u\|_{\mathcal{B}}) \text{ for a.e. } t \in J \text{ and any } u \in \mathcal{B}.$$

(H_4) There exists a constant $R > 0$ such that

$$M\psi\Big(\gamma R + \gamma\|\phi\|_{\mathcal{B}}(\widetilde{M}+1) + \gamma M|\tilde{y}|\Big)\|p\|_{L^1} \leq R$$

.

(H_5) There exists a function $\sigma \in L^1(J, \mathbb{R}_+)$ such that for any nonempty bounded set $D \subset \mathcal{B}$ we have :

$$\mu(f(t,D)) \leq \sigma(t) \sup_{\theta \in (-\infty,0]} \mu(D(\theta)) \text{ for a.e } t \in J.$$

To establish our main theorem, we need the following lemma.

Lemma 5. *[24] Assume that the hypotheses $(H_1), (H_3)$ and (H_5) hold and a set $D \subset \mathcal{C}$ is bounded. Then*

$$\omega_0^T(F(D)) \leq 2M \int_0^T \mu(f(s, D_s)ds,$$

where

$$F(D) = \{Fy : y \in D\},$$

and

$$(Fy)(t) = \int_0^T \mathcal{U}(t,s)f(s,y_s)ds.$$

Theorem 6. *Assume that the hypotheses $(H_1)-(H_5)$ are fulfilled. Then the problem (1)-(2) admits at least one mild solution.*

Proof. It is clear that the fixed point of the operator $N : \mathcal{C} \to \mathcal{C}$ defined by $(Ny)(t) = \phi(t)$ if $t \leq 0$ and

$$(Ny)(t) = -\frac{\partial}{\partial s}\mathcal{U}(t,0)\phi(0) + \mathcal{U}(t,0)\tilde{y} + \int_0^t \mathcal{U}(t,s)f(s,y_s)ds, \text{ if } t \in J, \tag{5}$$

are mild solutions of problem (1)-(2).

For $\phi \in \mathcal{B}$, Let $x : (-\infty, T] \to E$ be the function defined by

$$x(t) = \begin{cases} -\frac{\partial}{\partial s}\mathcal{U}(t,0)\phi(0) + \mathcal{U}(t,0)\tilde{y}, & \text{if } t \in J \\ \\ \phi(t) & \text{if } t \in (-\infty, 0]. \end{cases}$$

Then $x_0 = \phi$. For any function $z \in \mathcal{C}$, we denote

$$y(t) = x(t) + z(t).$$

It is obvious that y satisfies (5) if and only if z satisfies $z_0 = 0$ and for all $t \in J$

$$z(t) = \int_0^t \mathcal{U}(t,s)f(t, x_s + z_s)ds. \tag{6}$$

In the sequel, we always denote $\mathcal{C}_0$ as a Banach space defined by

$$\mathcal{C}_0 = \{z \in \mathcal{C} : \; z_0 = 0\},$$

endowed with the the norm

$$\begin{aligned} \|z\|_{\mathcal{C}_0} &= \sup\{|z(t)|; t \in J\} + \|z_0\|_{\mathcal{B}} \\ &= \sup\{|z(t)|; t \in J\}. \end{aligned}$$

Now, we can consider the operator $L : \mathcal{C}_0 \to \mathcal{C}_0$ given by

$$(Lz)(t) = \int_0^t \mathcal{U}(t,s)f(s, z_s + x_s)ds, \text{ for } t \in J.$$

The problem (1)-(2) has a solution is equivalent to L has a fixed point. To prove this end, we start with the following estimation.

For any $z \in \mathcal{C}_0$ and $t \in J$, we have

$$\begin{aligned}
\|z_t + x_t\|_{\mathcal{B}} &\leq \|z_t\|_{\mathcal{B}} + \|x_t\|_{\mathcal{B}} \\
&\leq K(t)|z(t)| + K(t)\|\frac{\partial}{\partial s}\mathcal{U}(t,0)\|_{B(E)}\|\phi\|_{\mathcal{B}} \\
&+ K(t)\|\mathcal{U}(t,0)\|_{B(E)}|\tilde{y}| + M(t)\|\phi\|_{\mathcal{B}} \\
&\leq \gamma\|z\|_{\mathcal{C}_0} + \gamma\widetilde{M}\|\phi\|_{\mathcal{B}} + \gamma M|\tilde{y}| + \gamma\|\phi\|_{\mathcal{B}} \\
&\leq \gamma\|z\|_{\mathcal{C}_0} + \gamma\|\phi\|_{\mathcal{B}}(\widetilde{M}+1) + \gamma M|\tilde{y}|. \qquad (7)
\end{aligned}$$

Now, we will show that the operator L satisfied the conditions of Schauder's fixed point theorem.

We define

$$B_R = \{z \in \mathcal{C}_0 : \|z\|_{\mathcal{C}_0} \leq R\}.$$

The set B_R is nonempty convex and closed. Let $z \in B_R$

$$\begin{aligned}
|L(z)(t)| &\leq \int_0^t \|\mathcal{U}(t,s)\|_{B(E)}|f(s, x_s + z_s)|\, ds \\
&\leq M\int_0^t p(s)\psi(\|z_s + x_s\|_{\mathcal{B}})ds. \\
&\leq M\psi\Big(\gamma\|z\|_{\mathcal{C}_0} + \gamma\|\phi\|_{\mathcal{B}}(\widetilde{M}+1) + \gamma M|\tilde{y}|\Big)\int_0^t p(s)ds \\
&\leq M\psi\big(\gamma R + \gamma\|\phi\|_{\mathcal{B}}(\widetilde{M}+1) + \gamma M|\tilde{y}|\big)\|p\|_{L^1} \leq R.
\end{aligned}$$

Thus the operator L maps B_R into itself.

Step 1. L is continuous.
Let $(z^n)_{n\in\mathbb{N}}$ be a sequence in B_R such that $z^n \to z$ in B_R, then for any $t \in J$ we obtain

$$\begin{aligned}
|(Lz^n)(t) - (Lz)(t)| &\leq \int_0^t \|\mathcal{U}(t,s)\|_{B(E)}|f(t, x_s + z_s^n) - f(t, x_s + z_s)|\, ds \\
&\leq M\int_0^t |f(s, z_s^n + x_s) - f(s, z_s + x_s)|\, ds.
\end{aligned}$$

Hence, from the Lebesgue dominated convergence theorem we obtain

$$\|Lz_k - Lz\|_{\mathcal{C}_0} \to 0 \quad \text{as} \quad n \to +\infty.$$

So L is continuous.

Consider the measure of noncompacteness μ^* defined on the family of bounded subsets of the space $\mathcal{C}_0$ by

$$\mu^*(D) = e^{-\tau\tilde{\sigma}(t)}\omega_0^T(D) + \sup_{t\in J} e^{-\tau\tilde{\sigma}(t)}\mu(D(t)),$$

where

$$\tilde{\sigma}(t) = M\int_0^t \sigma(s)ds, \qquad \tau \geq 6.$$

Step 2. $D_\infty = \cap_{n=0}^{+\infty} D_n$ is compact.

In the sequel, we consider the sequence of sets $\{D_n\}_{n=0}^{+\infty}$ defined by induction as follows :

$$D_0 = B_R,\ D_{n+1} = Conv(N(D_n)) \text{ for } n = 0, 1, 2, \cdots \text{and } D_\infty = \cap_{n=0}^{+\infty} D_n$$

this sequence is nondecreasing, i.e. $D_n \subset D_{n+1}$ for each n.

We know from Lemma 2 that for each $\varepsilon > 0$ there is a sequence function $\{W_k\}_{k=0}^{\infty} \subset (LD_n)(t))$ such that

$$\mu((LD_n)(t)) \leq 2\mu(\{W_k\}_{k=0}^{\infty}) + \varepsilon.$$

This implies that there is a sequence $\{Q^k\}_{k=0}^{\infty} \subset W_k$ such that

$$W_k = (LQ^k)(s) \text{ for } k = 0, 1, 2, \ldots$$

Using the properties of μ, Lemma 2, Lemma 3 and assumptions (H_4), (H_5), we get

$$
\begin{aligned}
\mu(D_{n+1}(t)) &= \mu(Conv(LD_n)(t)) \\
&= \mu((LD_n)(t)) \\
&\leq 2\mu(\{W_k\}_{k=0}^{\infty}) + \varepsilon \\
&\leq 2\mu(\{(LQ^k)(s)\}_{k=0}^{\infty}) + \varepsilon \\
&\leq 4\mu\left(\left\{\int_0^t \mathcal{U}(t,s)f(s,Q_s^k)ds\right\}_{k=0}^{\infty}\right) + \varepsilon \\
&\leq 4M\int_0^t \sigma(s) \sup_{\theta\in(-\infty,0]} \mu(\{Q^k(s+\theta))\}_{k=0}^{\infty} + \{x_s\})ds + \varepsilon. \\
&\leq 4M\int_0^t \sigma(s) \sup_{\theta\in(-\infty,0]} \mu(D_n(s+\theta) + \{x_s\}))ds + \varepsilon. \\
&\leq 4M\int_0^t \sigma(s) \sup_{\theta\in(-\infty,0]} \mu(D_n(s+\theta)))ds + \varepsilon. \\
&\leq 4M\int_0^t \sigma(s) \sup_{u\in(-\infty,s]} \mu(D_n(u))ds + \varepsilon. \\
&\leq 4M\int_0^t \sigma(s) \sup_{u\in[0,s]} \mu(D_n(u)))ds + \varepsilon \\
&\leq 2M\int_0^t \sigma(s)\mu(D_n(s))ds + \varepsilon \\
&\leq 4\int_0^t \sigma(s)e^{\tau\tilde{\sigma}(s)}e^{-\tau\tilde{\sigma}(s)}\mu(D_n(s))ds + \varepsilon \\
&\leq 4\int_0^t \sigma(s)e^{\tau\tilde{\sigma}(s)} \sup_{s\in[0,t]} e^{-\tau\tilde{\sigma}(s)}\mu(D_n(s))ds + \varepsilon \\
&\leq \frac{4}{\tau}e^{\tau\tilde{\sigma}(t)} \sup_{t\in J} e^{-\tau\tilde{\sigma}(t)}\mu(D_n(t)) + \varepsilon.
\end{aligned}
$$

Since ε is arbitrary, we get

$$
\sup_{t\in J} e^{-\tau\tilde{\sigma}(t)}\mu(D_{n+1}(t) \leq \frac{4}{\tau}\left(\sup_{t\in J} e^{-\tau\tilde{\sigma}(t)}\mu(D_n(t)\right).
$$

Then

$$
\sup_{t\in J} e^{-\tau\tilde{\sigma}(t)}\mu(D_{n+1}(t)) \leq \frac{4}{\tau}\left(e^{-\tau\tilde{\sigma}(t)}\omega_0^T(D_n) + \sup_{t\in J]} e^{-\tau\tilde{\sigma}(t)}\mu(D_n(t)\right). \tag{8}
$$

Now, applying Lemma 5 and using assumptions $(H_4), (H_5)$ and (8), we derive

$$\begin{aligned} \omega_0^T(D_{n+1}) &= \omega_0^T(Conv(LD_n)) \\ &= \omega_0^T(LD_n) \\ &\leq 2M\int_0^T \mu(f(s,(D_n)_s + \{x_s\})ds \\ &\leq 2M\int_0^t \sigma(s) \sup_{\theta\in(-\infty,0]} \mu((D_n)_s) + \{x_s\})ds \\ &\leq \frac{2}{\tau}e^{\tau\tilde{\sigma}(T)}(\sup_{t\in J} e^{-\tau\tilde{\sigma}(t)}\mu(D_n(t)). \end{aligned}$$

Then

$$e^{-\tau\tilde{\sigma}(t)}\omega_0^T(D_{n+1}) \leq \frac{4}{\tau}\left(e^{-\tau\tilde{\sigma}(t)}\omega_0^T(D_n) + \sup_{t\in J} e^{-\tau\tilde{\sigma}(t)}\mu(D_n(t)\right). \tag{9}$$

From (8) and (9),

$$\begin{aligned} &e^{-\tau\tilde{\sigma}(t)}\omega_0^T(D_{n+1}) + \sup_{t\in J} e^{-\tau\tilde{\sigma}(t)}\mu(D_{n+1}(t)) \\ &\leq \frac{6}{\tau}\left(e^{-\tau\tilde{\sigma}(t)}\omega_0^T(D_n) + \sup_{t\in J} e^{-\tau\tilde{\sigma}(t)}\mu(D_n(t)\right). \end{aligned}$$

Hence we get

$$\mu^*(D_{n+1}) \leq \frac{6}{\tau}\mu^*(D_n).$$

By method of mathematical induction, we can prove

$$\mu^*(D_{n+1}) \leq \left(\frac{6}{\tau}\right)^{n+1}\mu^*(D_0).$$

Hence, in view of the assumption (H_6), we get

$$\lim_{n\to+\infty} \mu^*(D_n) = 0.$$

Taking into account Lemma 1 we infer that $D_\infty = \cap_{n=0}^{+\infty} D_n$ is nonempty, convex and compact. Thus by Schauder's fixed point theorem the operator $N : D_\infty \to D_\infty$ has at least one fixed point which is a mild solution of problem (1)-(2).

4. An example

Consider the second order Cauchy problem

$$\begin{cases} \dfrac{\partial^2}{\partial t^2}y(t,\tau) = & \dfrac{\partial^2}{\partial \tau^2}y(t,\tau) + a(t)\dfrac{\partial}{\partial t}y(t,\tau) \\ & + \displaystyle\int_{-\infty}^{t} b(t-s)\arctan(y(s,\tau))ds \quad t \in J := [0,T],\ \tau \in [0,2\pi], \\ y(t,0) = & y(t,2\pi) = 0 \qquad t \in J, \\ y(\theta,\tau) = & \phi(\theta,\tau), \quad \dfrac{\partial}{\partial t}y(0,\tau) = \psi(\tau) \qquad \theta \in (-\infty,0], \tau \in [0,2\pi] \end{cases} \tag{10}$$

where we assume that $a,\ b, : J \to \mathbb{R}$ are continuous functions, $\phi(\theta,\cdot) \in \mathcal{B}$.

Let $X = L^2(\mathbb{R},\mathbb{C})$ the space of 2π-periodic 2-integrable functions from $\mathbb{R}$ into $\mathbb{C}$, and $H^2(\mathbb{R},\mathbb{C})$ denotes the Sobolev space 2π-periodic functions $x : \mathbb{R} \to \mathbb{C}$ such that $x'' \in L^2(\mathbb{R},\mathbb{C})$.

We consider the operator $A_1y(\tau) = y''(\tau)$ with domain $D(A_1) = H^2(\mathbb{R},\mathbb{C})$, infinitesimal generator of strongly continuous cosine function $C(t)$ on X. Moreover, we take $A_2(t)y(s) = a(t)y'(s)$ defined on $H^1(\mathbb{R},\mathbb{C})$, and consider the closed linear operator $A(t) = A_1 + A_2(t)$ which generates an evolution operator $\mathcal{U}$ defined by

$$\mathcal{U}(t,s) = \sum_{n\in\mathbb{Z}} z_n(t,s)\langle x, w_n\rangle w_n,$$

where z_n is a solution to the following scalar initial value problem

$$\begin{cases} z''(t) = -n^2 z(t) + ina(t)z(t) \\ z(s) = 0, \qquad z'(s) = z_1. \end{cases} \tag{11}$$

Define the operator $f : J \times \mathcal{B} \to X$ by

$$f(t,\varphi)(\tau) = \int_{-\infty}^{t} b(t-s)\varphi(s)(\tau)ds, \quad \tau \in [0,2\pi],$$

$$w(t)(\tau) = y(t,\tau),\ t \geq 0,\ \tau \in [0,2\pi],$$

$$\phi(s)(\tau) = \arctan(y(s,\tau)),\ -\infty < s \leq 0,\ \tau \in [0,2\pi],$$

and

$$\frac{d}{dt}w(0)(\tau) = \frac{\partial}{\partial t}y(0,\tau),\ \tau \in [0,2\pi].$$

Consequently, (10) can be written in the abstract form (1)-(2) with A and f defined above. Now, the existence of a mild solutions can be deduced from an application of Theorem 6.

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