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SOME NEW GENERATING RELATIONS USING DECOMPOSITION TECHNIQUE

M. A. Khan, G. Yasmin and A. Muhyi

Abstract. The aim of this research paper is to develop a new form of the known decomposition technique employed by Manocha (1974), and to derive a new class of generating relations involving the Laguerre polynomials, Jacobi polynomials and Mittag-Leffler polynomials using this technique.

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1. Introduction

The classical Laguerre polynomials (LP) $L_n^{(\alpha)}(x)$ (of order α) [4] are defined as:

$$L_n^{(\alpha)}(x) = \sum_{k=0}^{n} \frac{(-1)^k (1+\alpha)_n \; x^k}{k!(n-k)!(1+\alpha)_k} \tag{1}$$

and specified by the generating function

$$\sum_{n=0}^{\infty} \frac{t^n}{(\alpha+1)_n} L_n^{(\alpha)}(x) = \exp(t)\; {}_0F_1[-;\alpha+1;-xt], \tag{2}$$

where α is non-negative integer and $|t| < 1$.

The LP $L_n^{(\alpha)}(x)$ are also defined in terms of the confluent hypergeometric function ${}_1F_1$ [4] (see also [6, 9]) as:

$$L_n^{(\alpha)}(x) = \frac{(\alpha+1)_n}{n!} {}_1F_1[-n;\alpha+1;x], \quad Re(\alpha) > -1 \tag{3}$$

The Jacobi polynomials (JP) $P_n^{(\alpha-n,\beta-n)}(x)$ are defined by [10]:

$$P_n^{(\alpha-n,\beta-n)}(x) = \sum_{k=0}^{n} \frac{(-1)^k(-\alpha-\beta)_n(-\alpha)_k}{k!(n-k)!(-\alpha-\beta)_k}\left(\frac{1-x}{2}\right)^{n-k} \tag{4}$$

and specified by the generating function

$$\sum_{n=0}^{\infty} P_n^{(\alpha-n,\beta-n)}(x)\frac{t^n}{(-\alpha-\beta)_n} = \exp\left(\frac{1}{2}(1-x)t\right){}_1F_1[-\alpha;-\alpha-\beta;-t]. \tag{5}$$

The JP $P_n^{(\alpha-n,\beta-n)}(x)$ are also defined in terms of the confluent hypergeometric function ${}_2F_1$ as:

$$P_n^{(\alpha-n,\beta-n)}(x) = \frac{(-\alpha-\beta)_n}{n!}\left(\frac{1-x}{2}\right)^n {}_2F_1\left[\begin{array}{cc} -n,-\alpha; & \\ & \dfrac{2}{1-x} \\ -\alpha-\beta; & \end{array}\right]. \tag{6}$$

The Mittag-Leffler polynomials (MLP) $g_n(x)$ [2] (see also [1, 12, 13]) are defined as:

$$g_n(x) = \sum_{k=0}^{n} \frac{(1-n)_k(1-x)_k\; x\; 2^k}{k!(2)_k} \tag{7}$$

and specified by the generating function

$$g_n(x) = 2x\;{}_2F_1[1-n,1-x;2;2]. \tag{8}$$

The MLP $g_n(x)$ are also defined in terms of the confluent hypergeometric function ${}_2F_1$ as:

$$\sum_{n=0}^{\infty} g_{n+1}(x)\frac{t^n}{n!} = 2x\exp(t){}_1F_1[1-x;2;-2t]. \tag{9}$$

Next, we recall that the generalized hypergeometric functions (GHF) are defined by [4, p.73 (2)] (see also [3, 5]) :

$${}_pF_q\left[\begin{array}{cc} \alpha_1,\alpha_2,..,\alpha_p; & \\ & z \\ \beta_1,\beta_2,...,\beta_q; & \end{array}\right] = 1+\sum_{n=1}^{\infty}\frac{\prod_{i=1}^{p}(\alpha_i)_n}{\prod_{j=1}^{q}(\beta_j)_n}\frac{z^n}{n!}, \quad \beta_j>0, j=0,1,...,q, \tag{10}$$

where $(\lambda)_n$ denotes the Pochhammer symbol (or the shifted factorial) defined by

$$(\lambda)_n = \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)}.$$

Also, it is shown that (see [4, p.22])

$$(\alpha)_{kn} = k^{nk}\left(\frac{\alpha}{k}\right)_n\left(\frac{\alpha+1}{k}\right)_n \cdots \left(\frac{\alpha+k-1}{k}\right)_n . \tag{11}$$

Decomposition technique is one of the most significant techniques used to derive the generating functions of the special polynomials which was first employed by Manocha by means of which the generating function is decomposed into two generating functions [7, 8, 10, 11]. This technique is based in part upon well known identities which are applied to derive certain generating relations for the polynomials.

We note the following known identities:

$$\sum_{n=0}^{\infty}\varphi(n) = \sum_{n=0}^{\infty}\varphi(2n) + \sum_{n=0}^{\infty}\varphi(2n+1), \tag{12}$$

$$\sum_{n=0}^{\infty}\varphi(n) = \sum_{n=0}^{\infty}\varphi(3n) + \sum_{n=0}^{\infty}\varphi(3n+1) + \sum_{n=0}^{\infty}\varphi(3n+2). \tag{13}$$

In the usual decomposition technique, the idea of separation of a power series into its even and odd parts, exhibited by the elementary identity (12) is at least as old as the series themselves. On account of the fact that, when (12) is applied to the generalized hypergeometric series (10), we get the result:

$${}_pF_q\left[\begin{array}{c} \alpha_1, \alpha_2, .., \alpha_p; \\ \beta_1, \beta_2, ..., \beta_q; \end{array} z\right] = {}_{2p}F_{2q+1}\left[\begin{array}{c} \frac{\alpha_1}{2}, \frac{\alpha_1+1}{2}, ..., \frac{\alpha_p}{2}, \frac{\alpha_p+1}{2}; \\ \frac{1}{2}, \frac{\beta_1}{2}, \frac{\beta_1+1}{2}, ..., \frac{\beta_q}{2}, \frac{\beta_q+1}{2}; \end{array} \frac{z^2}{4^{1-p+q}}\right]$$

$$+ z\frac{\alpha_1\alpha_2...\alpha_p}{\beta_1\beta_2...\beta_q}\ {}_{2p}F_{2q+1}\left[\begin{array}{c} \frac{\alpha_1+1}{2}, \frac{\alpha_1+2}{2}, ..., \frac{\alpha_p+1}{2}, \frac{\alpha_p+2}{2}; \\ \frac{3}{2}, \frac{\beta_1+1}{2}, \frac{\beta_1+2}{2}, ..., \frac{\beta_q+1}{2}, \frac{\beta_q+2}{2}; \end{array} \frac{z^2}{4^{1-p+q}}\right], \tag{14}$$

where, for convergence, $p \leqslant q$ and $|z| < \infty$, or $p = q+1$ and $|z| < 1$, it is being assumed that $\beta_j \neq 0, -1, -2, ..., \forall j \in \{1, 2, ..., q\}$.

Motivated by the work done in this direction [7, 8, 10, 11], in this paper, we first derive a formula for the generalized hypergeometric function then in view of this result and also by using the decomposition technique, we obtain certain new generating relations involving the Laguerre polynomials, the Jacobi polynomials and the Mittag-Leffler polynomials.

2. Main result

In order to establish a formula for the generalized hyprgeometric function which can be further utilized to obtain some other generating relations, we prove the following theorem.

Theorem 1. *The following identity of the hypergeometric series holds true:*

$$
\begin{aligned}
&{}_pF_q\left[\begin{array}{c}\alpha_1,\alpha_2,..,\alpha_p;\\ \\ \beta_1,\beta_2,...,\beta_q;\end{array} z\right]\\
&= {}_{3p}F_{3q+2}\left[\begin{array}{c}\frac{\alpha_1}{3},\frac{\alpha_1+1}{3},\frac{\alpha_1+2}{3},\frac{\alpha_2}{3},\frac{\alpha_2+1}{3},\frac{\alpha_2+2}{3},...,\frac{\alpha_p}{3},\frac{\alpha_p+1}{3},\frac{\alpha_p+2}{3};\\ \frac{1}{3},\frac{2}{3},\frac{\beta_1}{3},\frac{\beta_1+1}{3},\frac{\beta_1+2}{3},\frac{\beta_2}{3},\frac{\beta_2+1}{3},\frac{\beta_2+2}{3},...,\frac{\beta_q}{3},\frac{\beta_q+1}{3},\frac{\beta_q+2}{3};\end{array} \frac{z^3}{27^{1-p+q}}\right]\\
&+z\frac{\alpha_1\alpha_2...\alpha_p}{\beta_1\beta_2...\beta_q}\,{}_{3p}F_{3q+2}\left[\begin{array}{c}\frac{\alpha_1+1}{3},\frac{\alpha_1+2}{3},\frac{\alpha_1+3}{3},\frac{\alpha_2+1}{3},\frac{\alpha_2+2}{3},\frac{\alpha_2+3}{3},...,\frac{\alpha_p+1}{3},\frac{\alpha_p+2}{3},\frac{\alpha_p+3}{3};\\ \frac{2}{3},\frac{4}{3},\frac{\beta_1+1}{3},\frac{\beta_1+2}{3},\frac{\beta_1+3}{3},\frac{\beta_2+1}{3},\frac{\beta_2+2}{3},\frac{\beta_2+3}{3},...,\frac{\beta_q+1}{3},\frac{\beta_q+2}{3},\frac{\beta_q+3}{3};\end{array} \frac{z^3}{27^{1-p+q}}\right]\\
&+\frac{z^2}{2}\frac{\alpha_1(\alpha_1+1)\alpha_2(\alpha_2+1)...\alpha_p(\alpha_p+1)}{\beta_1(\beta_1+1)\beta_2(\beta_2+1)...\beta_q(\beta_q+1)}\\
&{}_{3p}F_{3q+2}\left[\begin{array}{c}\frac{\alpha_1+2}{3},\frac{\alpha_1+3}{3},\frac{\alpha_1+4}{3},\frac{\alpha_2+2}{3},\frac{\alpha_2+3}{3},\frac{\alpha_2+4}{3},...,\frac{\alpha_p+2}{3},\frac{\alpha_p+3}{3},\frac{\alpha_p+4}{3};\\ \frac{4}{3},\frac{5}{3},\frac{\beta_1+2}{3},\frac{\beta_1+3}{3},\frac{\beta_1+4}{3},\frac{\beta_2+2}{3},\frac{\beta_2+3}{3},\frac{\beta_2+4}{3},...,\frac{\beta_q+2}{3},\frac{\beta_q+3}{3},\frac{\beta_q+4}{3};\end{array} \frac{z^3}{27^{1-p+q}}\right],
\end{aligned}
\tag{15}
$$

where, for convergence, $p \leqslant q$ *and* $|z| < \infty$, *or* $p = q + 1$ *and* $|z| < 1$, *it is being assumed that* $\beta_j \neq 0, -1, -2, ..., \forall j \in \{1, 2, ..., q\}$.

Proof. By making use of the elementary identity (13) in the series definition of the generalized hypergeometric function (10), we led to an equation of the form

$$
\begin{aligned}
{}_pF_q\left[\begin{array}{c}\alpha_1,\alpha_2,.......,\alpha_p;\\ \\ \beta_1,\beta_2,.......,\beta_q\end{array} z\right] &= \sum_{n=0}^{\infty}\frac{(\alpha_1)_{3n}(\alpha_2)_{3n}....(\alpha_p)_{3n}}{(\beta_1)_{3n}(\beta_2)_{3n}...(\beta_q)_{3n}}\frac{z^{3n}}{(3n)!}\\
&+ \sum_{n=0}^{\infty}\frac{(\alpha_1)_{3n+1}(\alpha_2)_{3n+1}....(\alpha_p)_{3n+1}}{(\beta_1)_{3n+1}(\beta_2)_{3n+1}...(\beta_q)_{3n+1}}\frac{z^{3n+1}}{(3n+1)!}\\
&+ \sum_{n=0}^{\infty}\frac{(\alpha_1)_{3n+2}(\alpha_2)_{3n+2}....(\alpha_p)_{3n+2}}{(\beta_1)_{3n+2}(\beta_2)_{3n+2}...(\beta_q)_{3n+2}}\frac{z^{3n+2}}{(3n+2)!}
\end{aligned}
\tag{16}
$$

Now, after some simplifications and then using the identity (11) in the resultant equation, we obtain

$$
{}_pF_q\left[\begin{array}{cc}\alpha_1,\alpha_2,.......,\alpha_p; & \\ & z\\ \beta_1,\beta_2,.......,\beta_q & \end{array}\right]
$$

$$
\begin{aligned}
=&\sum_{n=0}^{\infty}\frac{3^{3n}(\frac{\alpha_1}{3})_{3n}(\frac{\alpha_1+1}{3})_{3n}(\frac{\alpha_1+2}{3})_{3n}....3^{3n}(\frac{\alpha_p}{3})_{3n}(\frac{\alpha_p+1}{3})_{3n}(\frac{\alpha_p+2}{3})_{3n}}{3^{3n}(\frac{\beta_1}{3})_{3n}(\frac{\beta_1+1}{3})_{3n}(\frac{\beta_1+2}{3})_{3n}....3^{3n}(\frac{\beta_q}{3})_{3n}(\frac{\beta_q+1}{3})_{3n}(\frac{\beta_q+2}{3})_{3n}}\frac{z^{3n}}{n!3^{3n}(\frac{1}{3})_n(\frac{2}{3})_n}\\
+&\sum_{n=0}^{\infty}\frac{\alpha_1 3^{3n}(\frac{\alpha_1+1}{3})_{3n}(\frac{\alpha_1+2}{3})_{3n}(\frac{\alpha_1+3}{3})_{3n}....\alpha_p 3^{3n}(\frac{\alpha_p+1}{3})_{3n}(\frac{\alpha_p+2}{3})_{3n}(\frac{\alpha_p+3}{3})_{3n}}{\beta_1 3^{3n}(\frac{\beta_1+1}{3})_{3n}(\frac{\beta_1+2}{3})_{3n}(\frac{\beta_1+3}{3})_{3n}....\beta_q 3^{3n}(\frac{\beta_q+1}{3})_{3n}(\frac{\beta_q+2}{3})_{3n}(\frac{\beta_q+3}{3})_{3n}}\frac{z^{3n+1}}{3^{3n}(\frac{2}{3})_n(\frac{4}{3})_n n!}\\
+&\sum_{n=0}^{\infty}\frac{\alpha_1(\alpha_1+1)3^{3n}(\frac{\alpha_1+2}{3})_{3n}(\frac{\alpha_1+3}{3})_{3n}(\frac{\alpha_1+4}{3})_{3n}....\alpha_p(\alpha_p+1)3^{3n}(\frac{\alpha_p+2}{3})_{3n}(\frac{\alpha_p+3}{3})_{3n}(\frac{\alpha_p+4}{3})_{3n}}{\beta_1(\beta_1+1)3^{3n}(\frac{\beta_1+2}{3})_{3n}(\frac{\beta_1+3}{3})_{3n}(\frac{\beta_1+4}{3})_{3n}....\beta_q(\beta_q+1)3^{3n}(\frac{\beta_q+2}{3})_{3n}(\frac{\beta_q+3}{3})_{3n}(\frac{\beta_q+4}{3})_{3n}}\\
.&\frac{z^{3n+1}}{(2)3^{3n}(\frac{4}{3})_n(\frac{5}{3})_n n!}.
\end{aligned}
$$

Further, this result can be reduced as

$$
{}_pF_q\left[\begin{array}{cc}\alpha_1,\alpha_2,...,\alpha_p; & \\ & z\\ \beta_1,\beta_2,..,\beta_q & \end{array}\right]
$$

$$
\begin{aligned}
=&\ {}_{3p}F_{3q+2}\left[\begin{array}{cc}\frac{\alpha_1}{3},\frac{\alpha_1+1}{3},\frac{\alpha_1+2}{3},\frac{\alpha_2}{3},\frac{\alpha_2+1}{3},\frac{\alpha_2+2}{3},...,\frac{\alpha_p}{3},\frac{\alpha_p+1}{3},\frac{\alpha_p+2}{3}; & \frac{z^3}{27^{1-p+q}}\\ \frac{1}{3},\frac{2}{3},\frac{\beta_1}{3},\frac{\beta_1+1}{3},\frac{\beta_1+2}{3},\frac{\beta_2}{3},\frac{\beta_2+1}{3},\frac{\beta_2+2}{3},...,\frac{\beta_q}{3},\frac{\beta_q+1}{3},\frac{\beta_q+2}{3}; & \end{array}\right]\\
+&\ z\frac{\alpha_1\alpha_2...\alpha_p}{\beta_1\beta_2...\beta_q}\,{}_{3p}F_{3q+2}\left[\begin{array}{cc}\frac{\alpha_1+1}{3},\frac{\alpha_1+2}{3},\frac{\alpha_1+3}{3},\frac{\alpha_2+1}{3},\frac{\alpha_2+2}{3},\frac{\alpha_2+3}{3},...,\frac{\alpha_p+1}{3},\frac{\alpha_p+2}{3},\frac{\alpha_p+3}{3}; & \frac{z^3}{27^{1-p+q}}\\ \frac{2}{3},\frac{4}{3},\frac{\beta_1+1}{3},\frac{\beta_1+2}{3},\frac{\beta_1+3}{3},\frac{\beta_2+1}{3},\frac{\beta_2+2}{3},\frac{\beta_2+3}{3},...,\frac{\beta_q+1}{3},\frac{\beta_q+2}{3},\frac{\beta_q+3}{3}; & \end{array}\right]\\
+&\ \frac{z^2}{2}\frac{\alpha_1(\alpha_1+1)\alpha_2(\alpha_2+1)...\alpha_p(\alpha_p+1)}{\beta_1(\beta_1+1)\beta_2(\beta_2+1)...\beta_q(\beta_q+1)}\\
&\ {}_{3p}F_{3q+2}\left[\begin{array}{cc}\frac{\alpha_1+2}{3},\frac{\alpha_1+3}{3},\frac{\alpha_1+4}{3},\frac{\alpha_2+2}{3},\frac{\alpha_2+3}{3},\frac{\alpha_2+4}{3},...,\frac{\alpha_p+2}{3},\frac{\alpha_p+3}{3},\frac{\alpha_p+4}{3}; & \frac{z^3}{27^{1-p+q}}\\ \frac{4}{3},\frac{5}{3},\frac{\beta_1+2}{3},\frac{\beta_1+3}{3},\frac{\beta_1+4}{3},\frac{\beta_2+2}{3},\frac{\beta_2+3}{3},\frac{\beta_2+4}{3},...,\frac{\beta_q+2}{3},\frac{\beta_q+3}{3},\frac{\beta_q+4}{3}; & \end{array}\right],
\end{aligned}
$$

which yields the assertion (15) of Theorem 1 .

In the next section, we derive certain new generating relations involving the Laguerre polynomials, the Jacobi polynomials and the Mittag-Leffler polynomials using the results obtained in this section.

3. Applications

Lemma 2. *For $t \in \mathbb{R}$ the following formula holds true:*

$$
\exp(jt) = A(t) + jB(t) + j^2C(t), \tag{17}
$$

where $j^3 = -1$ *and*

$$A(t) = \sum_{n=0}^{\infty} \frac{(-1)^n t^{3n}}{(3n)!}, B(t) = \sum_{n=0}^{\infty} \frac{(-1)^n t^{3n+1}}{(3n+1)!}, C(t) = \sum_{n=0}^{\infty} \frac{(-1)^n t^{3n+2}}{(3n+2)!}.$$

Theorem 3. *The following generating relations for the Jacobi polynomials holds true:*

$$\begin{aligned}
&\sum_{n=0}^{\infty} P_{3n}^{(\alpha-3n,\beta-3n)}(x) \frac{t^n}{(-\alpha-\beta)_{3n}} \\
&= A(\frac{1}{2}(1-x)jt^{\frac{1}{3}})\,{}_3F_5\left[\begin{array}{c} \frac{-\alpha}{3}, \frac{-\alpha+1}{3}, \frac{-\alpha+2}{3}; \\ \frac{1}{3}, \frac{2}{3}, \frac{-\alpha-\beta}{3}, \frac{-\alpha-\beta+1}{3}, \frac{-\alpha-\beta+2}{3}; \end{array} \frac{-t}{27}\right] \\
&+ \frac{\alpha j t^{\frac{1}{3}}}{\alpha+\beta} C(\frac{1}{2}(1-x)jt^{\frac{1}{3}})\,{}_3F_5\left[\begin{array}{c} \frac{-\alpha+1}{3}, \frac{-\alpha+2}{3}, \frac{-\alpha+3}{3}; \\ \frac{2}{3}, \frac{4}{3}, \frac{-\alpha-\beta+1}{3}, \frac{-\alpha-\beta+2}{3}, \frac{-\alpha-\beta+3}{3}; \end{array} \frac{-t}{27}\right] \\
&+ \frac{\alpha(\alpha-1)j^2 t^{\frac{2}{3}} B(\frac{1}{2}(1-x)jt^{\frac{1}{3}})}{2(\alpha+\beta)(\alpha+\beta-1)}\,{}_3F_5\left[\begin{array}{c} \frac{-\alpha+2}{3}, \frac{-\alpha+3}{3}, \frac{-\alpha+4}{3}; \\ \frac{4}{3}, \frac{5}{3}, \frac{-\alpha-\beta+2}{3}, \frac{-\alpha-\beta+3}{3}, \frac{-\alpha-\beta+4}{3}; \end{array} \frac{-t}{27}\right], \qquad (18)
\end{aligned}$$

$$\begin{aligned}
&\sum_{n=0}^{\infty} P_{3n+1}^{(\alpha-3n-1,\beta-3n-1)}(x) \frac{t^n}{(-\alpha-\beta)_{3n}} \\
&= \frac{-(\alpha-\beta)B(\frac{1}{2}(1-x)jt^{\frac{1}{3}})}{jt^{\frac{1}{3}}}\,{}_3F_5\left[\begin{array}{c} \frac{-\alpha}{3}, \frac{-\alpha+1}{3}, \frac{-\alpha+2}{3}; \\ \frac{1}{3}, \frac{2}{3}, \frac{-\alpha-\beta}{3}, \frac{-\alpha-\beta+1}{3}, \frac{-\alpha-\beta+2}{3}; \end{array} \frac{-t}{27}\right] \\
&+ \alpha A(\frac{1}{2}(1-x)jt^{\frac{1}{3}})\,{}_3F_5\left[\begin{array}{c} \frac{-\alpha+1}{3}, \frac{-\alpha+2}{3}, \frac{-\alpha+3}{3}; \\ \frac{2}{3}, \frac{4}{3}, \frac{-\alpha-\beta+1}{3}, \frac{-\alpha-\beta+2}{3}, \frac{-\alpha-\beta+3}{3}; \end{array} \frac{-t}{27}\right] \\
&+ \frac{\alpha(\alpha-1)jt^{\frac{1}{3}} A(\frac{1}{2}(1-x)jt^{\frac{1}{3}})}{2(\beta+\alpha-1)}\,{}_3F_5\left[\begin{array}{c} \frac{-\alpha+2}{3}, \frac{-\alpha+3}{3}, \frac{-\alpha+4}{3}; \\ \frac{4}{3}, \frac{5}{3}, \frac{-\alpha-\beta+2}{3}, \frac{-\alpha-\beta+3}{3}, \frac{-\alpha-\beta+4}{3}; \end{array} \frac{-t}{27}\right] \qquad (19)
\end{aligned}$$

and

$$\sum_{n=0}^{\infty} P_{3n+2}^{(\alpha-3n-2,\beta-3n-2)}(x)\frac{t^n}{(-\alpha-\beta)_{3n}}$$

$$\begin{aligned}
&= \frac{(\alpha-\beta)_2 C(\frac{1}{2}(1-x)jt^{\frac{1}{3}})}{j^2t^{\frac{2}{3}}}\,{}_3F_5\left[\begin{array}{c}\frac{-\alpha}{3},\frac{-\alpha+1}{3},\frac{-\alpha+2}{3};\\ \frac{1}{3},\frac{2}{3},\frac{-\alpha-\beta}{3},\frac{-\alpha-\beta+1}{3},\frac{-\alpha-\beta+2}{3};\end{array}\frac{-t}{27}\right]\\
&+ \frac{\alpha(1-\alpha-\beta)B(\frac{1}{2}(1-x)jt^{\frac{1}{3}})}{jt^{\frac{1}{3}}}\,{}_3F_5\left[\begin{array}{c}\frac{-\alpha+1}{3},\frac{-\alpha+2}{3},\frac{-\alpha+3}{3};\\ \frac{2}{3},\frac{4}{3},\frac{-\alpha-\beta+1}{3},\frac{-\alpha-\beta+2}{3},\frac{-\alpha-\beta+3}{3};\end{array}\frac{-t}{27}\right]\\
&+ \frac{\alpha(\alpha-1)A(\frac{1}{2}(1-x)jt^{\frac{1}{3}})}{2}\,{}_3F_5\left[\begin{array}{c}\frac{-\alpha+2}{3},\frac{-\alpha+3}{3},\frac{-\alpha+4}{3};\\ \frac{4}{3},\frac{5}{3},\frac{-\alpha-\beta+2}{3},\frac{-\alpha-\beta+3}{3},\frac{-\alpha-\beta+4}{3};\end{array}\frac{-t}{27}\right]. \qquad (20)
\end{aligned}$$

Proof. On using identity (13) and the result (15) in the left and right hand sides of the equation (5) respectively, we get

$$\sum_{n=0}^{\infty} P_{3n}^{(\alpha-3n,\beta-3n)}(x)\frac{t^{3n}}{(-\alpha-\beta)_{3n}} + \sum_{n=0}^{\infty} P_{3n+1}^{(\alpha-3n-1,\beta-3n-1)}\frac{t^{3n+1}}{(-\alpha-\beta)_{3n+1}} + \sum_{n=0}^{\infty} P_{3n+2}^{(\alpha-3n-2,\beta-3n-2)}\frac{t^{3n+2}}{(-\alpha-\beta)_{3n+2}}$$

$$\begin{aligned}
&= \exp\left(\frac{1}{2}(1-x)t\right){}_3F_5\left[\begin{array}{c}\frac{-\alpha}{3},\frac{-\alpha+1}{3},\frac{-\alpha+2}{3};\\ \frac{1}{3},\frac{2}{3},\frac{-\alpha-\beta}{3},\frac{-\alpha-\beta+1}{3},\frac{-\alpha-\beta+2}{3};\end{array}\frac{-t^3}{27}\right]\\
&- \frac{\alpha t}{\alpha+\beta}\exp\left(\frac{1}{2}(1-x)t\right){}_3F_5\left[\begin{array}{c}\frac{-\alpha+1}{3},\frac{-\alpha+2}{3},\frac{-\alpha+3}{3};\\ \frac{2}{3},\frac{4}{3},\frac{-\alpha-\beta+1}{3},\frac{-\alpha-\beta+2}{3},\frac{-\alpha-\beta+3}{3};\end{array}\frac{-t^3}{27}\right]\\
&+ \frac{\alpha(\alpha-1)t^2\exp\left(\frac{1}{2}(1-x)t\right)}{2(\alpha+\beta)(\alpha+\beta-1)}\,{}_3F_5\left[\begin{array}{c}\frac{-\alpha+2}{3},\frac{-\alpha+3}{3},\frac{-\alpha+4}{3};\\ \frac{4}{3},\frac{5}{3},\frac{-\alpha-\beta+2}{3},\frac{-\alpha-\beta+3}{3},\frac{-\alpha-\beta+4}{3};\end{array}\frac{-t^3}{27}\right]. \qquad (21)
\end{aligned}$$

Further, replacing t by jt in equation (21), we obtain

$$\sum_{n=0}^{\infty} P_{3n}^{(\alpha-3n,\beta-3n)}(x)\frac{(-t^3)^n}{(-\alpha-\beta)_{3n}} + jt\sum_{n=0}^{\infty} P_{3n+1}^{(\alpha-3n-1,\beta-3n-1)}\frac{(-t^3)^n}{(-\alpha-\beta)_{3n+1}} + j^2t^2\sum_{n=0}^{\infty} P_{3n+2}^{(\alpha-3n-2,\beta-3n-2)}\frac{(-t^3)^n}{(-\alpha-\beta)_{3n+2}}$$

$$= \exp\Big(\frac{1}{2}(1-x)jt\Big)\,{}_3F_5\left[\begin{array}{c}\frac{-\alpha}{3},\frac{-\alpha+1}{3},\frac{-\alpha+2}{3};\\ \frac{1}{3},\frac{2}{3},\frac{-\alpha-\beta}{3},\frac{-\alpha-\beta+1}{3},\frac{-\alpha-\beta+2}{3};\end{array}\ \frac{t^3}{27}\right]$$
$$- \frac{\alpha jt}{\alpha+\beta}\exp\Big(\frac{1}{2}(1-x)jt\Big)\,{}_3F_5\left[\begin{array}{c}\frac{-\alpha+1}{3},\frac{-\alpha+2}{3},\frac{-\alpha+3}{3};\\ \frac{2}{3},\frac{4}{3},\frac{-\alpha-\beta+1}{3},\frac{-\alpha-\beta+2}{3},\frac{-\alpha-\beta+3}{3};\end{array}\ \frac{t^3}{27}\right]$$
$$+ \frac{\alpha(\alpha-1)j^2t^2\exp\big(\frac{1}{2}(1-x)jt\big)}{2(\alpha+\beta)(\alpha+\beta-1)}\,{}_3F_5\left[\begin{array}{c}\frac{-\alpha+2}{3},\frac{-\alpha+3}{3},\frac{-\alpha+4}{3};\\ \frac{4}{3},\frac{5}{3},\frac{-\alpha-\beta+2}{3},\frac{-\alpha-\beta+3}{3},\frac{-\alpha-\beta+4}{3};\end{array}\ \frac{t^3}{27}\right], \quad (22)$$

which on using the identity (17) in the r.h.s of the above equation, gives

$$\sum_{n=0}^{\infty} P_{3n}^{(\alpha-3n,\beta-3n)}(x)\frac{(-t^3)^n}{(-\alpha-\beta)_{3n}} + jt\sum_{n=0}^{\infty} P_{3n+1}^{(\alpha-3n-1,\beta-3n-1)}\frac{(-t^3)^n}{(-\alpha-\beta)_{3n+1}} + j^2t^2\sum_{n=0}^{\infty} P_{3n+2}^{(\alpha-3n-2,\beta-3n-2)}\frac{(-t^3)^n}{(-\alpha-\beta)_{3n+2}}$$

$$= [A(\frac{1}{2}(1-x)t) + jB(\frac{1}{2}(1-x)t) + j^2C(\frac{1}{2}(1-x)t)]\,{}_3F_5\left[\begin{array}{c}\frac{-\alpha}{3},\frac{-\alpha+1}{3},\frac{-\alpha+2}{3};\\ \frac{1}{3},\frac{2}{3},\frac{-\alpha-\beta}{3},\frac{-\alpha-\beta+1}{3},\frac{-\alpha-\beta+2}{3};\end{array}\ \frac{t^3}{27}\right]$$
$$- \frac{\alpha jt[A(\frac{1}{2}(1-x)t) + jB(\frac{1}{2}(1-x)t) + j^2C(\frac{1}{2}(1-x)t)]}{\alpha+\beta}\,{}_3F_5\left[\begin{array}{c}\frac{-\alpha+1}{3},\frac{-\alpha+2}{3},\frac{-\alpha+3}{3};\\ \frac{2}{3},\frac{4}{3},\frac{-\alpha-\beta+1}{3},\frac{-\alpha-\beta+2}{3},\frac{-\alpha-\beta+3}{3};\end{array}\ \frac{t^3}{27}\right]$$
$$+ \frac{\alpha(\alpha-1)j^2t^2[A(\frac{1}{2}(1-x)t) + jB(\frac{1}{2}(1-x)t) + j^2C(\frac{1}{2}(1-x)t)]}{2(\alpha+\beta)(\alpha+\beta-1)}\cdot$$
$${}_3F_5\left[\begin{array}{c}\frac{-\alpha+2}{3},\frac{-\alpha+3}{3},\frac{-\alpha+4}{3};\\ \frac{4}{3},\frac{5}{3},\frac{-\alpha-\beta+2}{3},\frac{-\alpha-\beta+3}{3},\frac{-\alpha-\beta+4}{3};\end{array}\ \frac{t^3}{27}\right]. \quad (23)$$

Finally, equating the coefficients of like powers of j on both sides of equation (23) (assuming t, α and β to be real numbers) and then replacing t by $jt^{\frac{1}{3}}$ in the resultant equation, we obtain

$$\sum_{n=0}^{\infty}\frac{P_{3n}^{(\alpha-3n,\beta-3n)}(x)\ t^n}{(-\alpha-\beta)_{3n}} = A(\frac{1}{2}(1-x)jt^{\frac{1}{3}})\,{}_3F_5\left[\begin{array}{c}\frac{-\alpha}{3},\frac{-\alpha+1}{3},\frac{-\alpha+2}{3};\\ \frac{1}{3},\frac{2}{3},\frac{-\alpha-\beta}{3},\frac{-\alpha-\beta+1}{3},\frac{-\alpha-\beta+2}{3};\end{array}\ \frac{-t}{27}\right]$$

$$+ \frac{\alpha jt^{\frac{1}{3}}}{\alpha+\beta}C(\frac{1}{2}(1-x)jt^{\frac{1}{3}})\,{}_3F_5\left[\begin{array}{c}\frac{-\alpha+1}{3},\frac{-\alpha+2}{3},\frac{-\alpha+3}{3};\\ \frac{2}{3},\frac{4}{3},\frac{-\alpha-\beta+1}{3},\frac{-\alpha-\beta+2}{3},\frac{-\alpha-\beta+3}{3};\end{array}\ \frac{-t}{27}\right]$$
$$+ \frac{\alpha(\alpha-1)j^2t^{\frac{2}{3}}B(\frac{1}{2}(1-x)jt^{\frac{1}{3}})}{2(\alpha+\beta)(\alpha+\beta-1)}\,{}_3F_5\left[\begin{array}{c}\frac{-\alpha+2}{3},\frac{-\alpha+3}{3},\frac{-\alpha+4}{3};\\ \frac{4}{3},\frac{5}{3},\frac{-\alpha-\beta+2}{3},\frac{-\alpha-\beta+3}{3},\frac{-\alpha-\beta+4}{3};\end{array}\ \frac{-t}{27}\right],$$

$$\sum_{n=0}^{\infty} \frac{P_{3n+1}^{(\alpha-3n-1,\beta-3n-1)}(x)\ t^n}{(-\alpha-\beta)_{3n}} = \frac{-(\alpha-\beta)B(\frac{1}{2}(1-x)jt^{\frac{1}{3}})}{jt^{\frac{1}{3}}} {}_3F_5\left[\begin{array}{c} \frac{-\alpha}{3}, \frac{-\alpha+1}{3}, \frac{-\alpha+2}{3}; \\ \frac{1}{3}, \frac{2}{3}, \frac{-\alpha-\beta}{3}, \frac{-\alpha-\beta+1}{3}, \frac{-\alpha-\beta+2}{3}; \end{array} \frac{-t}{27}\right]$$

$$+ \alpha A(\frac{1}{2}(1-x)jt^{\frac{1}{3}}) {}_3F_5\left[\begin{array}{c} \frac{-\alpha+1}{3}, \frac{-\alpha+2}{3}, \frac{-\alpha+3}{3}; \\ \frac{2}{3}, \frac{4}{3}, \frac{-\alpha-\beta+1}{3}, \frac{-\alpha-\beta+2}{3}, \frac{-\alpha-\beta+3}{3}; \end{array} \frac{-t}{27}\right]$$

$$+ \frac{\alpha(\alpha-1)jt^{\frac{1}{3}} A(\frac{1}{2}(1-x)jt^{\frac{1}{3}})}{2(\beta+\alpha-1)} {}_3F_5\left[\begin{array}{c} \frac{-\alpha+2}{3}, \frac{-\alpha+3}{3}, \frac{-\alpha+4}{3}; \\ \frac{4}{3}, \frac{5}{3}, \frac{-\alpha-\beta+2}{3}, \frac{-\alpha-\beta+3}{3}, \frac{-\alpha-\beta+4}{3}; \end{array} \frac{-t}{27}\right]$$

and

$$\sum_{n=0}^{\infty} \frac{P_{3n+2}^{(\alpha-3n-2,\beta-3n-2)}(x)\ t^n}{(-\alpha-\beta)_{3n}} = \frac{(\alpha-\beta)_2 C(\frac{1}{2}(1-x)jt^{\frac{1}{3}})}{j^2 t^{\frac{2}{3}}} {}_3F_5\left[\begin{array}{c} \frac{-\alpha}{3}, \frac{-\alpha+1}{3}, \frac{-\alpha+2}{3}; \\ \frac{1}{3}, \frac{2}{3}, \frac{-\alpha-\beta}{3}, \frac{-\alpha-\beta+1}{3}, \frac{-\alpha-\beta+2}{3}; \end{array} \frac{-t}{27}\right]$$

$$+ \frac{\alpha(1-\alpha-\beta)B(\frac{1}{2}(1-x)jt^{\frac{1}{3}})}{jt^{\frac{1}{3}}} {}_3F_5\left[\begin{array}{c} \frac{-\alpha+1}{3}, \frac{-\alpha+2}{3}, \frac{-\alpha+3}{3}; \\ \frac{2}{3}, \frac{4}{3}, \frac{-\alpha-\beta+1}{3}, \frac{-\alpha-\beta+2}{3}, \frac{-\alpha-\beta+3}{3}; \end{array} \frac{-t}{27}\right]$$

$$+ \frac{\alpha(\alpha-1)A(\frac{1}{2}(1-x)jt^{\frac{1}{3}})}{2} {}_3F_5\left[\begin{array}{c} \frac{-\alpha+2}{3}, \frac{-\alpha+3}{3}, \frac{-\alpha+4}{3}; \\ \frac{4}{3}, \frac{5}{3}, \frac{-\alpha-\beta+2}{3}, \frac{-\alpha-\beta+3}{3}, \frac{-\alpha-\beta+4}{3}; \end{array} \frac{-t}{27}\right]$$

which yield the assertions (18), (19) and (20) respectively of Theorem 3 .

Theorem 4. *The following generating relations of Laguerre polynomial holds true:*

$$\sum_{n=0}^{\infty} L_{3n}^{(\alpha)}(x) \frac{t^n}{(\alpha+1)_{3n}} = A(jt^{\frac{1}{3}}) {}_0F_5\left[\ -; \frac{1}{3}, \frac{2}{3}, \frac{\alpha+1}{3}, \frac{\alpha+2}{3}, \frac{\alpha+3}{3}; \ -\frac{x^3t}{729}\right]$$

$$+ \frac{jxt^{\frac{1}{3}}}{\alpha+1} C(jt^{\frac{1}{3}}) {}_0F_5\left[\ -; \frac{2}{3}, \frac{4}{3}, \frac{\alpha+2}{3}, \frac{\alpha+3}{3}, \frac{\alpha+4}{3}; \ -\frac{x^3t}{729}\right]$$

$$+ \frac{j^2x^2t^{\frac{2}{3}}}{2(\alpha+1)_2} B(jt^{\frac{1}{3}}) {}_0F_5\left[\ -; \frac{4}{3}, \frac{5}{3}, \frac{\alpha+1}{3}, \frac{\alpha+4}{3}, \frac{\alpha+5}{3}; \ -\frac{x^3t}{729}\right], \quad (24)$$

$$\sum_{n=0}^{\infty} L_{3n+1}^{(\alpha)}(x) \frac{t^n}{(\alpha+1)_{3n}} = \frac{\alpha+1}{jt^{\frac{1}{3}}} B(jt^{\frac{1}{3}}) {}_0F_5\left[\ -; \frac{1}{3}, \frac{2}{3}, \frac{\alpha+1}{3}, \frac{\alpha+2}{3}, \frac{\alpha+3}{3}; \ -\frac{x^3t}{729}\right]$$

$$-\,xA(jt^{\frac{1}{3}})\,{}_0F_5\left[\quad -;\frac{2}{3},\frac{4}{3},\frac{\alpha+2}{3},\frac{\alpha+3}{3},\frac{\alpha+4}{3};\quad -\frac{x^3t}{729}\right]$$

$$-\,\frac{jxt^{\frac{1}{3}}}{2(\alpha+2)}C(jt^{\frac{1}{3}})\,{}_0F_5\left[\quad -;\frac{4}{3},\frac{5}{3},\frac{\alpha+1}{3},\frac{\alpha+4}{3},\frac{\alpha+5}{3};\quad -\frac{x^3t}{729}\right] \quad (25)$$

and

$$\sum_{n=0}^{\infty} L^{(\alpha)}_{3n+2}(x)\frac{t^n}{(\alpha+1)_{3n}} = \frac{(\alpha+1)_2}{j^2t^{\frac{2}{3}}}C(jt^{\frac{1}{3}})\,{}_0F_5\left[\quad -;\frac{1}{3},\frac{2}{3},\frac{\alpha+1}{3},\frac{\alpha+2}{3},\frac{\alpha+3}{3};\ -\frac{x^3t}{729}\right]$$

$$-\,\frac{x(\alpha+2)}{jt^{\frac{1}{3}}}B(jt^{\frac{1}{3}})\,{}_0F_5\left[\quad -;\frac{2}{3},\frac{4}{3},\frac{\alpha+2}{3},\frac{\alpha+3}{3},\frac{\alpha+4}{3};\quad -\frac{x^3t}{729}\right]$$

$$+\,\frac{x^2}{2}A(jt^{\frac{1}{3}})\,{}_0F_5\left[\quad -;\frac{4}{3},\frac{5}{3},\frac{\alpha+1}{3},\frac{\alpha+4}{3},\frac{\alpha+5}{3};\quad -\frac{x^3t}{729}\right] \quad (26)$$

Proof. On using the identity (13) and the result (15) in the left and right hand sides of the equation (2) respectively, we get

$$\sum_{n=0}^{\infty} L^{(\alpha)}_{3n}(x)\frac{t^{3n}}{(\alpha+1)_{3n}} + \sum_{n=0}^{\infty} L^{(\alpha)}_{3n+1}(x)\frac{t^{3n+1}}{(\alpha+1)_{3n+1}} + \sum_{n=0}^{\infty} L^{(\alpha)}_{3n+2}(x)\frac{t^{3n+2}}{(\alpha+1)_{3n+2}}$$

$$\begin{aligned}
&= \quad e^t\,{}_0F_5\left[\quad -;\frac{1}{3},\frac{2}{3},\frac{\alpha+1}{3},\frac{\alpha+2}{3},\frac{\alpha+3}{3};\quad -\frac{x^3t^3}{729}\right]\\
&- \quad \frac{xt}{\alpha+1}e^t\,{}_0F_5\left[\quad -;\frac{2}{3},\frac{4}{3},\frac{\alpha+2}{3},\frac{\alpha+3}{3},\frac{\alpha+4}{3};\quad -\frac{x^3t^3}{729}\right]\\
&+ \quad \frac{x^2t^2}{2(\alpha+1)_2}e^t\,{}_0F_5\left[\quad -;\frac{4}{3},\frac{5}{3},\frac{\alpha+1}{3},\frac{\alpha+4}{3},\frac{\alpha+5}{3};\quad -\frac{x^3t^3}{729}\right]
\end{aligned}$$

Now, replacing t by jt in the above equation and then using the identity (17) in the resultant equation, we get

$$\sum_{n=0}^{\infty} L^{(\alpha)}_{3n}(x)\frac{(-t^3)^n}{(\alpha+1)_{3n}} + jt\sum_{n=0}^{\infty} L^{(\alpha)}_{3n+1}(x)\frac{(-t^3)^n}{(\alpha+1)_{3n+1}} + j^2t^2\sum_{n=0}^{\infty} L^{(\alpha)}_{3n+2}(x)\frac{(-t^3)^n}{(\alpha+1)_{3n+2}}$$

$$\begin{aligned}
&= [A(t)+jB(t)+j^2C(t)]{}_0F_5\left[\begin{matrix} -; \frac{1}{3}, \frac{2}{3}, \frac{\alpha+1}{3}, \frac{\alpha+2}{3}, \frac{\alpha+3}{3}; & \frac{x^3t^3}{729}\end{matrix}\right] \\
&- \frac{xjt}{\alpha+1}[A(t)+jB(t)+j^2C(t)]{}_0F_5\left[\begin{matrix} -; \frac{2}{3}, \frac{4}{3}, \frac{\alpha+2}{3}, \frac{\alpha+3}{3}, \frac{\alpha+4}{3}; & \frac{x^3t^3}{729}\end{matrix}\right] \\
&+ \frac{x^2j^2t^2}{2(\alpha+1)_2}[A(t)+jB(t)+j^2C(t)]{}_0F_5\left[\begin{matrix} -; \frac{4}{3}, \frac{5}{3}, \frac{\alpha+1}{3}, \frac{\alpha+4}{3}, \frac{\alpha+5}{3}; & -\frac{x^3t^3}{729}\end{matrix}\right]
\end{aligned}$$

Further, equating the coefficients of the like powers of j on both sides of the above equation and then replacing t by $jt^{\frac{1}{3}}$ in the resultant equation, we obtain

$$\begin{aligned}
\sum_{n=0}^{\infty} L_{3n}^{(\alpha)}(x)\frac{t^n}{(\alpha+1)_{3n}} &= A(jt^{\frac{1}{3}})\ {}_0F_5\left[\begin{matrix} -; \frac{1}{3}, \frac{2}{3}, \frac{\alpha+1}{3}, \frac{\alpha+2}{3}, \frac{\alpha+3}{3}; & -\frac{x^3t}{729}\end{matrix}\right] \\
&+ \frac{jxt^{\frac{1}{3}}}{\alpha+1}C(jt^{\frac{1}{3}})\ {}_0F_5\left[\begin{matrix} -; \frac{2}{3}, \frac{4}{3}, \frac{\alpha+2}{3}, \frac{\alpha+3}{3}, \frac{\alpha+4}{3}; & -\frac{x^3t}{729}\end{matrix}\right] \\
&+ \frac{j^2x^2t^{\frac{2}{3}}}{2(\alpha+1)_2}B(jt^{\frac{1}{3}})\ {}_0F_5\left[\begin{matrix} -; \frac{4}{3}, \frac{5}{3}, \frac{\alpha+1}{3}, \frac{\alpha+4}{3}, \frac{\alpha+5}{3}; & -\frac{x^3t}{729}\end{matrix}\right],
\end{aligned}$$

$$\begin{aligned}
\sum_{n=0}^{\infty} L_{3n+1}^{(\alpha)}(x)\frac{t^n}{(\alpha+1)_{3n}} &= \frac{\alpha+1}{jt^{\frac{1}{3}}}B(jt^{\frac{1}{3}})\ {}_0F_5\left[\begin{matrix} -; \frac{1}{3}, \frac{2}{3}, \frac{\alpha+1}{3}, \frac{\alpha+2}{3}, \frac{\alpha+3}{3}; & -\frac{x^3t}{729}\end{matrix}\right] \\
&- xA(jt^{\frac{1}{3}})\ {}_0F_5\left[\begin{matrix} -; \frac{2}{3}, \frac{4}{3}, \frac{\alpha+2}{3}, \frac{\alpha+3}{3}, \frac{\alpha+4}{3}; & -\frac{x^3t}{729}\end{matrix}\right] \\
&- \frac{jxt^{\frac{1}{3}}}{2(\alpha+2)}C(jt^{\frac{1}{3}})\ {}_0F_5\left[\begin{matrix} -; \frac{4}{3}, \frac{5}{3}, \frac{\alpha+1}{3}, \frac{\alpha+4}{3}, \frac{\alpha+5}{3}; & -\frac{x^3t}{729}\end{matrix}\right]
\end{aligned}$$

and

$$\begin{aligned}
\sum_{n=0}^{\infty} L_{3n+2}^{(\alpha)}(x)\frac{t^n}{(\alpha+1)_{3n}} &= \frac{(\alpha+1)_2}{j^2t^{\frac{2}{3}}}C(jt^{\frac{1}{3}})\ {}_0F_5\left[\begin{matrix} -; \frac{1}{3}, \frac{2}{3}, \frac{\alpha+1}{3}, \frac{\alpha+2}{3}, \frac{\alpha+3}{3}; & -\frac{x^3t}{729}\end{matrix}\right] \\
&- \frac{x(\alpha+2)}{jt^{\frac{1}{3}}}B(jt^{\frac{1}{3}})\ {}_0F_5\left[\begin{matrix} -; \frac{2}{3}, \frac{4}{3}, \frac{\alpha+2}{3}, \frac{\alpha+3}{3}, \frac{\alpha+4}{3}; & -\frac{x^3t}{729}\end{matrix}\right] \\
&+ \frac{x^2}{2}A(jt^{\frac{1}{3}})\ {}_0F_5\left[\begin{matrix} -; \frac{4}{3}, \frac{5}{3}, \frac{\alpha+1}{3}, \frac{\alpha+4}{3}, \frac{\alpha+5}{3}; & -\frac{x^3t}{729}\end{matrix}\right]
\end{aligned}$$

which yield the assertions (24), (25) and (26) respectively of Theorem 4 .

Theorem 5. *The following generating relations of Mittag - Leffler plynomial holds true:*

$$
\begin{aligned}
\sum_{n=0}^{\infty} \frac{g_{3n+1}(x)\ t^n}{(3n)!} &= 2x\ A(jt^{\frac{1}{3}})\ {}_3F_5\left[\begin{array}{l} \frac{1-x}{3}, \frac{2-x}{3}, \frac{3-x}{3}; \\ \frac{1}{3}, \frac{2}{3}, \frac{2}{3}, \frac{3}{3}, \frac{4}{3}; \end{array} \frac{-8t}{27}\right] \\
&+ 2x(1-x)jt^{\frac{1}{3}}C(jt^{\frac{1}{3}})\ {}_3F_5\left[\begin{array}{l} \frac{2-x}{3}, \frac{3-x}{3}, \frac{4-x}{3}; \\ \frac{2}{3}, \frac{4}{3}, \frac{3}{3}, \frac{4}{3}, \frac{5}{3}; \end{array} \frac{-8t}{27}\right] \\
&- \frac{2}{3}x(1-x)(2-x)j^2t^{\frac{2}{3}}B(jt^{\frac{1}{3}})\,{}_3F_5\left[\begin{array}{l} \frac{3-x}{3}, \frac{4-x}{3}, \frac{5-x}{3}; \\ \frac{4}{3}, \frac{5}{3}, \frac{4}{3}, \frac{5}{3}, \frac{6}{3}; \end{array} \frac{-8t}{27}\right]
\end{aligned} \tag{27}
$$

$$
\begin{aligned}
\sum_{n=0}^{\infty} \frac{g_{3n+2}(x)\ t^n}{(3n+1)!} &= \frac{2x}{jt^{\frac{1}{3}}}B(jt^{\frac{1}{3}})\,{}_3F_5\left[\begin{array}{l} \frac{1-x}{3}, \frac{2-x}{3}, \frac{3-x}{3}; \\ \frac{1}{3}, \frac{2}{3}, \frac{2}{3}, \frac{3}{3}, \frac{4}{3}; \end{array} \frac{-8t}{27}\right] \\
&- 2x(1-x)A(jt^{\frac{1}{3}})\,{}_3F_5\left[\begin{array}{l} \frac{2-x}{3}, \frac{3-x}{3}, \frac{4-x}{3}; \\ \frac{2}{3}, \frac{4}{3}, \frac{3}{3}, \frac{4}{3}, \frac{5}{3}; \end{array} \frac{-8t}{27}\right] \\
&- \frac{2x(1-x)(2-x)jt^{\frac{1}{3}}C(jt^{\frac{1}{3}})}{3}\,{}_3F_5\left[\begin{array}{l} \frac{3-x}{3}, \frac{4-x}{3}, \frac{5-x}{3}; \\ \frac{4}{3}, \frac{5}{3}, \frac{4}{3}, \frac{5}{3}, \frac{6}{3}; \end{array} \frac{-8t}{27}\right]
\end{aligned} \tag{28}
$$

and

$$
\begin{aligned}
\sum_{n=0}^{\infty} \frac{g_{3n+3}(x)\ t^n}{(3n+2)!} &= \frac{2x}{j^2t^{\frac{2}{3}}}C(jt^{\frac{1}{3}})\,{}_3F_5\left[\begin{array}{l} \frac{1-x}{3}, \frac{2-x}{3}, \frac{3-x}{3}; \\ \frac{1}{3}, \frac{2}{3}, \frac{2}{3}, \frac{3}{3}, \frac{4}{3}; \end{array} \frac{-8t}{27}\right] \\
&- \frac{2x(1-x)}{jt^{\frac{1}{3}}}B(jt^{\frac{1}{3}})\,{}_3F_5\left[\begin{array}{l} \frac{2-x}{3}, \frac{3-x}{3}, \frac{4-x}{3}; \\ \frac{2}{3}, \frac{4}{3}, \frac{3}{3}, \frac{4}{3}, \frac{5}{3}; \end{array} \frac{-8t}{27}\right] \\
&+ \frac{2}{3}x(1-x)(2-x)A(jt^{\frac{1}{3}})\,{}_3F_5\left[\begin{array}{l} \frac{3-x}{3}, \frac{4-x}{3}, \frac{5-x}{3}; \\ \frac{4}{3}, \frac{5}{3}, \frac{4}{3}, \frac{5}{3}, \frac{6}{3}; \end{array} \frac{-8t}{27}\right]
\end{aligned} \tag{29}
$$

Proof. On using equation (13) and the result (15) in the left and right hand sides of the equation (9) respectively, we get

$$\sum_{n=0}^{\infty} g_{3n+1}(x)\frac{t^{3n}}{(3n)!} + \sum_{n=0}^{\infty} g_{3n+2}(x)\frac{t^{3n+1}}{(3n+1)!} + \sum_{n=0}^{\infty} g_{3n+3}(x)\frac{t^{3n+2}}{(3n+2)!}$$

$$\begin{aligned} = & \ 2xe^t {}_3F_5\left[\begin{array}{c} \frac{-\alpha}{3}, \frac{-\alpha+1}{3}, \frac{-\alpha+2}{3}; \\ \frac{1}{3}, \frac{2}{3}, \frac{-\alpha-\beta}{3}, \frac{-\alpha-\beta+1}{3}, \frac{-\alpha-\beta+2}{3}; \end{array} \frac{-t^3}{27}\right] \\ - & \ 2x(1-x)te^t {}_3F_5\left[\begin{array}{c} \frac{-\alpha+1}{3}, \frac{-\alpha+2}{3}, \frac{-\alpha+3}{3}; \\ \frac{2}{3}, \frac{4}{3}, \frac{-\alpha-\beta+1}{3}, \frac{-\alpha-\beta+2}{3}, \frac{-\alpha-\beta+3}{3}; \end{array} \frac{-t^3}{27}\right] \\ + & \ \frac{2x(1-x)(2-x)t^2e^t}{3} {}_3F_5\left[\begin{array}{c} \frac{-\alpha+2}{3}, \frac{-\alpha+3}{3}, \frac{-\alpha+4}{3}; \\ \frac{4}{3}, \frac{5}{3}, \frac{-\alpha-\beta+2}{3}, \frac{-\alpha-\beta+3}{3}, \frac{-\alpha-\beta+4}{3}; \end{array} \frac{-t^3}{27}\right] \end{aligned} \quad (30)$$

Replacing t by jt in the equation (30) and then using the identity (17) in the resultant equation, we obtain

$$\sum_{n=0}^{\infty} g_{3n+1}(x)\frac{(-t^3)^n}{(3n)!} + jt\sum_{n=0}^{\infty} g_{3n+2}(x)\frac{(-t^3)^n}{(3n+1)!} + j^2t^2\sum_{n=0}^{\infty} g_{3n+3}(x)\frac{(-t^3)^n}{(3n+2)!}$$

$$\begin{aligned} = & \ 2x\ [A(t)+jB(t)+j^2C(t)]\ {}_3F_5\left[\begin{array}{c} \frac{-\alpha}{3}, \frac{-\alpha+1}{3}, \frac{-\alpha+2}{3}; \\ \frac{1}{3}, \frac{2}{3}, \frac{-\alpha-\beta}{3}, \frac{-\alpha-\beta+1}{3}, \frac{-\alpha-\beta+2}{3}; \end{array} \frac{-t^3}{27}\right] \\ - & \ 2x\ (1-x)jt[A(t)+jB(t)+j^2C(t)]\ {}_3F_5\left[\begin{array}{c} \frac{-\alpha+1}{3}, \frac{-\alpha+2}{3}, \frac{-\alpha+3}{3}; \\ \frac{2}{3}, \frac{4}{3}, \frac{-\alpha-\beta+1}{3}, \frac{-\alpha-\beta+2}{3}, \frac{-\alpha-\beta+3}{3}; \end{array} \frac{-t^3}{27}\right] \\ + & \ \frac{2x(1-x)(2-x)j^2t^2e^t}{3}[A(t)+jB(t)+j^2C(t)]\ {}_3F_5\left[\begin{array}{c} \frac{-\alpha+2}{3}, \frac{-\alpha+3}{3}, \frac{-\alpha+4}{3}; \\ \frac{4}{3}, \frac{5}{3}, \frac{-\alpha-\beta+2}{3}, \frac{-\alpha-\beta+3}{3}, \frac{-\alpha-\beta+4}{3}; \end{array} \frac{-t^3}{27}\right] \end{aligned}$$

Finally, equating the coefficients of like powers of j on both sides of the above equation and then replacing t by $jt^{\frac{1}{3}}$ in the resultant equation, we obtain

$$\sum_{n=0}^{\infty} g_{3n+1}(x)\frac{t^n}{(3n)!} = 2x\ A(jt^{\frac{1}{3}})\ {}_3F_5\left[\begin{array}{lr} \frac{1-x}{3}, \frac{2-x}{3}, \frac{3-x}{3}; & \frac{-8t}{27} \\ \frac{1}{3}, \frac{2}{3}, \frac{2}{3}, \frac{3}{3}, \frac{4}{3}; & \end{array}\right]$$

$$+\ 2x\ (1-x)jt^{\frac{1}{3}}C(jt^{\frac{1}{3}})\ {}_3F_5\left[\begin{array}{lr} \frac{2-x}{3}, \frac{3-x}{3}, \frac{4-x}{3}; & \frac{-8t}{27} \\ \frac{2}{3}, \frac{4}{3}, \frac{3}{3}, \frac{4}{3}, \frac{5}{3}; & \end{array}\right]$$

$$-\ \frac{2}{3}x(1-x)(2-x)j^2t^{\frac{2}{3}}B(jt^{\frac{1}{3}})\ {}_3F_5\left[\begin{array}{lr} \frac{3-x}{3}, \frac{4-x}{3}, \frac{5-x}{3}; & \frac{-8t}{27} \\ \frac{4}{3}, \frac{5}{3}, \frac{4}{3}, \frac{5}{3}, \frac{6}{3}; & \end{array}\right],$$

$$\sum_{n=0}^{\infty} g_{3n+2}(x)\frac{t^n}{(3n+1)!} = \frac{2x}{jt^{\frac{1}{3}}}B(jt^{\frac{1}{3}})\ {}_3F_5\left[\begin{array}{lr} \frac{1-x}{3}, \frac{2-x}{3}, \frac{3-x}{3}; & \frac{-8t}{27} \\ \frac{1}{3}, \frac{2}{3}, \frac{2}{3}, \frac{3}{3}, \frac{4}{3}; & \end{array}\right]$$

$$-\ 2x\ (1-x)A(jt^{\frac{1}{3}})\ {}_3F_5\left[\begin{array}{lr} \frac{2-x}{3}, \frac{3-x}{3}, \frac{4-x}{3}; & \frac{-8t}{27} \\ \frac{2}{3}, \frac{4}{3}, \frac{3}{3}, \frac{4}{3}, \frac{5}{3}; & \end{array}\right]$$

$$-\ \frac{2}{3}x(1-x)(2-x)jt^{\frac{1}{3}}C(jt^{\frac{1}{3}})\ {}_3F_5\left[\begin{array}{lr} \frac{3-x}{3}, \frac{4-x}{3}, \frac{5-x}{3}; & \frac{-8t}{27} \\ \frac{4}{3}, \frac{5}{3}, \frac{4}{3}, \frac{5}{3}, \frac{6}{3}; & \end{array}\right]$$

and

$$\sum_{n=0}^{\infty} g_{3n+3}(x)\frac{t^n}{(3n+2)!} = \frac{2x}{j^2t^{\frac{2}{3}}}C(jt^{\frac{1}{3}})\ {}_3F_5\left[\begin{array}{lr} \frac{1-x}{3}, \frac{2-x}{3}, \frac{3-x}{3}; & \frac{-8t}{27} \\ \frac{1}{3}, \frac{2}{3}, \frac{2}{3}, \frac{3}{3}, \frac{4}{3}; & \end{array}\right]$$

$$-\ \frac{2x(1-x)}{jt^{\frac{1}{3}}}B(jt^{\frac{1}{3}})\ {}_3F_5\left[\begin{array}{lr} \frac{2-x}{3}, \frac{3-x}{3}, \frac{4-x}{3}; & \frac{-8t}{27} \\ \frac{2}{3}, \frac{4}{3}, \frac{3}{3}, \frac{4}{3}, \frac{5}{3}; & \end{array}\right]$$

$$+\ \frac{2}{3}x(1-x)(2-x)A(jt^{\frac{1}{3}})\ {}_3F_5\left[\begin{array}{lr} \frac{3-x}{3}, \frac{4-x}{3}, \frac{5-x}{3}; & \frac{-8t}{27} \\ \frac{4}{3}, \frac{5}{3}, \frac{4}{3}, \frac{5}{3}, \frac{6}{3}; & \end{array}\right]$$

which yield the assertions (27), (28) and (29) respectively of Theorem 5.

In this paper, it has been shown that decomposition technique can be utilize to obtain certain new generating relations of some special functions with decomposition of a generating function into three generating relations. This process can be extended to establish more general form of this technique.Exploring the possibility of using the method we outline here to derive generating relations for other special functions is a further research problems.

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Mumtaz Ahmad Khan
Department of Applied Mathematics, Faculty of Engineering and Technology
Aligarh Muslim University,
Aligarh-202002, India
email: *mumtaz_ahmad_khan2008@yahoo.com*

Ghazala Yasmin
Department of Applied Mathematics, Faculty of Engineering and Technology
Aligarh Muslim University,
Aligarh-202002, India
email: *ghazala30@gmail.com*

Abdulghani Muhyi
Department of Applied Mathematics, Faculty of Engineering and Technology
Aligarh Muslim University,
Aligarh-202002, India
email: *abdalganihameed@gmail.com*