Acta Universitatis Apulensis No. 49/2017
ISSN: 1582-5329 pp. 79-92
http://www.uab.ro/auajournal/ doi: 10.17114/j.aua.2017.49.07

A CLASS OF COMPLEX-VALUED HARMONIC FUNCTIONS DEFINED BY EXTENDED MULTIPLIER DIZOK-SRIVASTAVA OPERATOR

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Abstract. In the present paper we define and investigate a family of complex-valued harmonic convex univalent functions defined by extended multiplier Dizok-Srivastava operator, we obtain the basic properties such as coefficient condition, distortion bounds, extreme points, inclusion result and integral operator.

2010 *Mathematics Subject Classification*: 30C45.

Keywords: Harmonic function, Multiplier, Dziok- Srivastava operator, extreme points.

1. Introduction

A continuous function $f = u + iv$ is a complex valued harmonic function in a simply connected complex domain $D \subset C$ if both u and v are real harmonic in D. It was shown by Clunie and Sheil-Small [5] that such harmonic function can be represented by $f = h + \overline{g}$, where h and g are analytic in D. Also, a necessary and sufficient condition for f to be locally univalent and sense preserving in D is that $\left|h^{'}(z)\right| > \left|g^{'}(z)\right|$, (see also, [7, 8] , [13] and [14]).

Denote by S_H the class of functions f that are harmonic univalent and sense-preserving in the open unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$ for which $f(0) = h(0) = f_z^{'}(0) - 1 = 0$. Then for $f = h + \overline{g} \in S_H$ we may express the analytic functions h and g as

$$h(z) = z + \sum_{k=2}^{\infty} a_k z^k \quad , \quad g(z) = \sum_{k=1}^{\infty} b_k z^k \quad |b_1| < 1. \tag{1.1}$$

Clunie and Shell-Small [5] investigated the class S_H as well as its geometric subclasses and obtained some coefficient bounds.

For complex parameters

$$\alpha_1, ..., \alpha_q \text{ and } \beta_1, ..., \beta_s \ \ (\beta_j \notin Z_0^- = \{0, -1, -2, ...\}; j = 1, 2, ..., s),$$

we now define the generalized hypergeometric function ${}_qF_s(\alpha_1,...,\alpha_q;\beta_1,...,\beta_s;z)$ by (see, for example, [15, p. 30])

$$ {}_qF_s(\alpha_1,...,\alpha_q;\beta_1,...,\beta_s;z)=\sum_{k=0}^{\infty}\frac{(\alpha_1)_k...(\alpha_q)_k}{(\beta_1)_k...(\beta_s)_k}.\frac{z^k}{k!} \tag{1.2}$$

$$(q\leq s+1;q,s\in\mathbb{N}_0=\mathbb{N}\cup\{0\},\mathbb{N}=\{1,2,...\};z\in U),$$

where $(\theta)_\nu$ is the Pochhammer symbol defined, in terms of the Gamma function Γ, by

$$(\theta)_\nu=\frac{\Gamma(\theta+\nu)}{\Gamma(\theta)}=\begin{cases}1 & (\nu=0;\theta\in\mathbb{C}\backslash\{0\}),\\ \theta(\theta-1)...(\theta+\nu-1) & (\nu\in N;\theta\in\mathbb{C}).\end{cases} \tag{1.3}$$

Corresponding to the function

$$h(\alpha_1,...,\alpha_q;\beta_1,...,\beta_s;z)=z\ {}_qF_s(\alpha_1,...,\alpha_q;\beta_1,...,\beta_s;z), \tag{1.4}$$

which is defined by the following Hadamard product (or convolution):

$$H(\alpha_1,...,\alpha_q;\beta_1,...,\beta_s;z)f(z)=h(\alpha_1,...,\alpha_q;\beta_1,...,\beta_s;z)*\phi(z). \tag{1.5}$$

We observe that

$$\begin{aligned}H(\alpha_1,...,\alpha_q;\beta_1,...,\beta_s)f(z) &= z+\sum_{k=2}^{\infty}\frac{(\alpha_1)_{k-1}...(\alpha_q)_{k-1}}{(\beta_1)_{k-1}...(\beta_s)_{k-1}(1)_{k-1}}a_kz^k,\\ &= z+\sum_{k=2}^{\infty}\Gamma_k(\alpha_1)a_kz^k, \end{aligned} \tag{1.6}$$

where

$$\Gamma_k(\alpha_1)=\frac{(\alpha_1)_{k-1}...(\alpha_q)_{k-1}}{(\beta_1)_{k-1}...(\beta_s)_{k-1}(1)_{k-1}}.$$

If, for convenience, we write

$$H_{q,s}(\alpha_1)=H(\alpha_1,...,\alpha_q;\beta_1,...,\beta_s). \tag{1.7}$$

We defined the extended multiplier Dizok-Srivastava operator $D^{n,q,s}_{\lambda,\ell}$ as follows:

$$\begin{aligned}D^{0,q,s}_{\lambda,\ell}f(z) &= f(z)*H_{q,s}(\alpha_1),\\ D^{1,q,s}_{\lambda,\ell}f(z) &= \frac{\left(z^{\frac{\ell+1}{\lambda}-1}\left(f(z)*H_{q,s}(\alpha_1)\right)\right)'}{\frac{\ell+1}{\lambda}z^{\frac{\ell+1}{\lambda}-2}} \qquad (\lambda>0;\ell\geq0),\\ D^{2,q,s}_{\lambda,\ell}f(z) &= D_{\lambda,\ell}(D^{1,q,s}_{\lambda,\ell}f(z)),\end{aligned}$$

and (in general)

$$D_{\lambda,\ell}^{n,q,s}f(z) = D_{\lambda,\ell}(D_{\lambda,\ell}^{n-1,q,s}f(z)) \quad (n \in \mathbb{N}). \tag{1.8}$$

Then from (1.6) and (1.8), we see that

$$D_{\lambda,\ell}^{n,q,s}f(z) = z + \sum_{k=2}^{\infty} \Phi_{k,n}(\alpha_1, \lambda, \ell) a_k z^k \quad (n \in \mathbb{N}_0), \tag{1.9}$$

where

$$\Phi_{k,n}(\alpha_1, \lambda, \ell) = \left[\frac{\ell + 1 + \lambda(k-1)}{\ell + 1}\right]^n \Gamma_k(\alpha_1). \tag{1.10}$$

By specializing the parameters $q, s, \alpha_1, \beta_1, \ell$ and λ, we obtain the following operators studied by various authors:

(i) For $q = 2, s = 1$ and $\alpha = \ \alpha_2 = \beta_1 = 1$, we have $D_{\lambda,\ell}^{n,2,1}f(z) = I^n(\lambda, \ell)f(z)$ (see Catas [3]);

(ii) $D_{\lambda,0}^{n,2,1}f(z) = D_\lambda^n f(z)$ (see Al-Oboudi [1]) and $D_{1,0}^{n,2,1}f(z) = D^n f(z)$ (see Salagean [12]);

(iii)$D_{0,0}^{0,q,s}f(z) = H_{q,s}(\alpha_1)$ (see Dziok and Srivastava [6]).

We modified the extended multiplier Dizok-Srivastava operator of the harmonic function $f = h + \overline{g}$ given by (1.1) as

$$D_{\lambda,\ell}^{n,q,s}f(z) = D_{\lambda,\ell}^{n,q,s}h(z) + (-1)^n \overline{D_{\lambda,\ell}^{n,q,s}g(z)}, \tag{1.11}$$

where

$$D_{\lambda,\ell}^{n,q,s}h(z) = z + \sum_{k=2}^{\infty} \Phi_{k,n}(\alpha_1, \lambda, \ell) a_k z^k,$$

and

$$D_{\lambda,\ell}^{n,q,s}g(z) = \sum_{k=1}^{\infty} \Phi_{k,n}(\alpha_1, \lambda, \ell) b_k z^k.$$

Also let $S_{\overline{H}}$ denote the subclass of S_H consisting of functions $f = h + \overline{g}$ such that the functions h and g are of the form

$$h(z) = z - \sum_{k=2}^{\infty} |a_k| z^k \quad , \quad g(z) = (-1)^n \sum_{k=1}^{\infty} |b_k| z^k \quad |b_1| < 1. \tag{1.12}$$

We introduce here a new subclass $G_H^n((\alpha_1, \lambda, \ell), \gamma)$ of function of the form (1.1) using the extended multiplier Dizok-Srivastava operator of harmonic univalent functions .

Let $G_H^n((\alpha_1,\lambda,\ell),\gamma)$ denote the subfamily of convex harmonic functions $f\in S_H$ of the form $f=h+\overline{g}$ such that

$$\operatorname{Re}\left\{1+(1+e^{i\psi})\frac{z^2(D_{\lambda,\ell}^{n,q,s}h(z))''+\overline{(-1)^n2z(D_{\lambda,\ell}^{n,q,s}g(z))'+(-1)^nz^2(D_{\lambda,\ell}^{n,q,s}g(z))''}}{z(D_{\lambda,\ell}^{n,q,s}h(z))'-(-1)^n\overline{z(D_{\lambda,\ell}^{n,q,s}g(z))'}}\right\}\geq\gamma. \quad (1.13)$$

$$(0\leq\gamma<1;\psi\in\mathbb{R})$$

Finally, consider the subclass $\overline{G_H^n}((\alpha_1,\lambda,\ell),\gamma)$ of $G_H^n((\alpha_1,\lambda,\ell),\gamma)$ for h and g of the form (1.12) .

We note that:

(i) Putting $\lambda=\ell=n=0$ then $\overline{G_H^0}((\alpha_1,0,0),\gamma)=G_H([\alpha_1,\beta_1],\gamma)$ (see Chandrashekar et al. [4]);

(ii) Putting $q=2,s=1,\alpha_1=\alpha_2=\beta_1=1$, and $\lambda=\ell=n=0$ then $\overline{G_H^0}((1,0,0),\gamma)=HCV(k,\gamma)$ (see Kim et al. [19, $k=1$]).

In this paper, we obtain a sufficient coefficient condition for functions $f=h+\overline{g}$ to be in the class $G_H^n((\alpha_1,\lambda,\ell),\gamma)$ and show that this coefficient condition also is necessary for functions belonging to the class $\overline{G_H^n}((\alpha_1,\lambda,\ell),\gamma)$. Also distortion bound ,extreme points for functions in the class $\overline{G_H^n}((\alpha_1,\lambda,\ell),\gamma)$ and certain inclusion results and integral operator are obtained.

2. Coefficient condition for the class $G_H^n((\alpha_1,\lambda,\ell),\gamma)$

Unless otherwise mentioned, we assume throughou this paper that $0\leq\gamma<1$ and ψ is real and $\Phi_{k,n}(\alpha_1,\lambda,\ell)$ is given by (1.10). We begin with a sufficient condition for functions in the class $G_H^n((\alpha_1,\lambda,\ell),\gamma)$.

Theorem 1. *Let $f=h+\overline{g}$ be such that h and g are given by* (1.1)*. If*

$$\sum_{k=1}^{\infty}k\left(\tfrac{2k-1-\gamma}{1-\gamma}|a_k|+\tfrac{2k-1-\gamma}{1-\gamma}|b_k|\right)\Phi_{k,n}(\alpha_1,\lambda,\ell)\leq 2, \quad (2.1)$$

Then $f\in G_H^n((\alpha_1,\lambda,\ell),\gamma)$.

Proof. When the condition (2.1) holds for the coefficients of $f=h+\overline{g}$ it is shown that the inequality (1.13) is satisfied. Write the left side of inequality (1.13) as

$$\operatorname{Re}\left\{\frac{\begin{array}{c}z(D_{\lambda,\ell}^{n,q,s}h(z))+(1+e^{i\psi})z^2(D_{\lambda,\ell}^{n,q,s}h(z))''+\\(1+2e^{i\psi})(-1)^n\overline{z(D_{\lambda,\ell}^{n,q,s}g(z))'}+(1+e^{i\psi})(-1)^n\overline{z^2(D_{\lambda,\ell}^{n,q,s}g(z))''}\end{array}}{z(D_{\lambda,\ell}^{n,q,s}h(z))'-(-1)^n\overline{z(D_{\lambda,\ell}^{n,q,s}g(z))'}}\right\}=\operatorname{Re}\frac{A(z)}{B(z)}.$$

Since $\operatorname{Re}(w) \geq \gamma$ if and only if $|1-\gamma+w| > |1+\gamma-w|$, it suffices to show that

$$|A(z)+(1-\gamma)B(z)| - |A(z)-(1+\gamma)B(z)| \geq 0. \tag{2.2}$$

Substituting for $A(z)$ and $B(z)$ the appropriate in (2.2), we get

$$\begin{aligned} & |A(z)+(1-\gamma)B(z)| - |A(z)-(1+\gamma)B(z)| \\ \geq\ & (2-\gamma)\,|z| - \sum_{k=2}^{\infty} k(2k-\gamma)\Phi_{k,n}(\alpha_1,\lambda,\ell)|a_k|\,|z|^k \\ & - \sum_{k=1}^{\infty} k(2k+\gamma)\Phi_{k,n}(\alpha_1,\lambda,\ell)|b_k|\,|z|^k \\ & -\gamma\,|z| - \sum_{k=2}^{\infty} k(2k-2-\gamma)\Phi_{k,n}(\alpha_1,\lambda,\ell)|a_k|\,|z|^k \\ & - \sum_{k=1}^{\infty} k(2k+2+\gamma)\Phi_{k,n}(\alpha_1,\lambda,\ell)|b_k|\,|z|^k \\ \geq\ & (2-\gamma)\,|z| \left\{ \begin{array}{c} 1 - \sum_{k=2}^{\infty} k\frac{2k-1-\gamma}{1-\gamma}\Phi_{k,n}(\alpha_1,\lambda,\ell)|a_k| \\ - \sum_{k=1}^{\infty} k\frac{2k+1+\gamma}{1-\gamma}\Phi_{k,n}(\alpha_1,\lambda,\ell)|b_k| \end{array} \right\} \\ \geq\ & 0 \end{aligned}$$

by inequality (2.1), which implies that $f \in G_H((\alpha_1,\lambda,\ell),\gamma)$.

Now we obtain the necessary and sufficient condition for the function $f = h + \overline{g}$ *be such that* h *and* g *are given by* (1.12) *to be in* $\overline{G_H^n}$.

Theorem 2. *Let* $f = h + \overline{g}$ *be such that* h *and* g *are given by* (1.12). *Then* $f \in \overline{G_H^n}((\alpha_1,\lambda,\ell),\gamma)$ *if and only if*

$$\sum_{k=1}^{\infty} k\left(\tfrac{2k-1-\gamma}{1-\gamma}|a_k| + \tfrac{2k-1-\gamma}{1-\gamma}|b_k|\right)\Phi_{k,n}(\alpha_1,\lambda,\ell) \leq 2, \tag{2.3}$$

Proof. Since $\overline{G_H^n}((\alpha_1,\lambda,\ell),\gamma) \subset G_H^n((\alpha_1,\lambda,\ell),\gamma)$, we only need to prove the necessary part of the theorem. Assume that $f \in \overline{G_H^n}((\alpha_1,\lambda,\ell),\gamma)$, then by virtue of (1.11) to (1.13), we obtain

$$\operatorname{Re}\left\{(1-\gamma)+(1+e^{i\psi})\frac{z^2(D_{\lambda,\ell}^{n,q,s}h(z))'' + \overline{2z(-1)^n(D_{\lambda,\ell}^{n,q,s}g(z))' + (-1)^n z^2(D_{\lambda,\ell}^{n,q,s}g(z))''}}{z(D_{\lambda,\ell}^{n,q,s}h(z))' - \overline{(-1)^n z(D_{\lambda,\ell}^{n,q,s}g(z))'}}\right\} \geq 0\ .$$

The above inequality is equivalent to

$$\operatorname{Re}\left\{\frac{z - \left(\begin{array}{c} \sum_{k=2}^{\infty} k\left[k(1+e^{i\psi}) - \gamma - e^{i\psi}\right]\Phi_{k,n}(\alpha_1,\lambda,\ell)|a_k|z^k \\ +(-1)^n\sum_{k=1}^{\infty} k\left[k(1+e^{i\psi}) + \gamma + e^{i\psi}\right]\Phi_{k,n}(\alpha_1,\lambda,\ell)|b_k|\overline{z}^k \end{array}\right)}{z - \sum_{k=2}^{\infty} k\Phi_{k,n}(\alpha_1,\lambda,\ell)|a_k|z^k + (-1)^n\sum_{k=1}^{\infty} k\Phi_{k,n}(\alpha_1,\lambda,\ell)|b_k|\overline{z}^k}\right\}$$

$$= \quad \mathrm{Re}\left\{ \frac{\begin{array}{c}(1-\gamma)-\sum_{k=2}^{\infty}k\left[k(1+e^{i\psi})-\gamma-e^{i\psi}\right]\Phi_{k,n}(\alpha_1,\lambda,\ell)|a_k|z^{k-1}\\ -(-1)^n\sum_{k=1}^{\infty}k\left[k(1+e^{i\psi})+\gamma+e^{i\psi}\right]\Phi_{k,n}(\alpha_1,\lambda,\ell)|b_k|\overline{z}^{k-1}\end{array}}{1-\sum_{k=2}^{\infty}k\Phi_{k,n}(\alpha_1,\lambda,\ell)|a_k|z^{k-1}+(-1)^n\sum_{k=1}^{\infty}k\Phi_{k,n}(\alpha_1,\lambda,\ell)|b_k|\overline{z}^{k-1}} \right\}$$

$$\geq \quad 0.$$

This condition must hold for all values of $z \in U$ and for real ψ, so that on taking $z = r < 1$ and $\psi = 0$, the above inequality reduces to

$$\frac{(1-\gamma)-\left[\begin{array}{c}\sum_{k=2}^{\infty}k(2k-1-\gamma)\ \Phi_{k,n}(\alpha_1,\lambda,\ell)|a_k|r^{k-1}\\ -\sum_{k=1}^{\infty}k(2k+1+\gamma)\ \Phi_{k,n}(\alpha_1,\lambda,\ell)|b_k|r^{k-1}\end{array}\right]}{1-\sum_{k=2}^{\infty}k\Phi_{k,n}(\alpha_1,\lambda,\ell)|a_k|r^{k-1}+\sum_{k=1}^{\infty}k\Phi_{k,n}(\alpha_1,\lambda,\ell)|b_k|r^{k-1}} \geq 0. \tag{2.4}$$

Letting $r \to 1^-$ through real values, we obtain the (2.3). This completes the proof of Theorem 2.

3. Distortion Bounds

The following theorem gives the distortion bounds for the functions $\overline{G_H^n}((\alpha_1,\lambda,\ell),\gamma)$, which yields a covering for this class .

Theorem 3. *Let* $f \in \overline{G_H^n}((\alpha_1,\lambda,\ell),\gamma)$. *Then for* $|b_1| < \frac{1-\gamma}{3+\gamma}$ we have

$$|f(z)| \leq (1+|b_1|)r + \frac{1}{2}\frac{1}{\Phi_{k,n}(\alpha_1,\lambda,\ell)}\left\{\frac{(1-\gamma)}{(3-\gamma)} - \frac{(3+\gamma)}{(3-\gamma)}|b_1|\right\}r^2 \qquad |z| = r < 1,$$

and

$$|f(z)| \geq (1-|b_1|)r - \frac{1}{2}\frac{1}{\Phi_{k,n}(\alpha_1,\lambda,\ell)}\left\{\frac{(1-\gamma)}{(3-\gamma)} - \frac{(3+\gamma)}{(3-\gamma)}|b_1|\right\}r^2 \qquad |z| = r < 1.$$

The results are sharp.

Proof. Let $f \in \overline{G_H^n}((\alpha_1,\lambda,\ell),\gamma)$. Taking the absolute value of f, we have

$$
\begin{aligned}
|f(z)| &\leq (1+|b_1|)r + \sum_{k=2}^{\infty}(|a_k|+|b_k|)r^k \\
&\leq (1+|b_1|)r + \sum_{k=2}^{\infty}(|a_k|+|b_k|)r^2 \\
&\leq (1+|b_1|)r + \frac{1-\gamma}{(3-\gamma)\,\Phi_{2,n}(\alpha_1,\lambda,\ell)} \sum_{k=2}^{\infty}\left(\frac{3-\gamma}{1-\gamma}\,|a_k| + \frac{3-\gamma}{1-\gamma}\,|b_k|\right)\Phi_{k,n}(\alpha_1,\lambda,\ell)r^2 \\
&\leq (1+|b_1|)r + \frac{(1-\gamma)}{(3-\gamma)\,\Phi_{2,n}(\alpha_1,\lambda,\ell)} \sum_{k=2}^{\infty} k\left(\frac{2k-1-\gamma}{1-\gamma}\,|a_k|\right. \\
&\quad \left. + \frac{2k+1+\gamma}{1-\gamma}\,|b_k|\right)\Phi_{k,n}(\alpha_1,\lambda,\ell)r^2 \\
&\leq (1+|b_1|)r + \frac{(1-\gamma)}{(3-\gamma)\,\Phi_{2,n}(\alpha_1,\lambda,\ell)}\frac{1}{2}\left(1-\frac{3+\gamma}{1-\gamma}\,|b_1|\right)r^2 \\
&\leq (1+|b_1|)r + \frac{1}{2\Phi_{2,n}(\alpha_1,\lambda,\ell)}\left(\frac{(1-\gamma)}{(3-\gamma)} - \frac{(3+\gamma)}{(3-\gamma)}\,|b_1|\right)r^2
\end{aligned}
$$

and

$$
\begin{aligned}
|f(z)| &\geq (1-|b_1|)r - \sum_{k=2}^{\infty}(|a_k|+|b_k|)r^k \\
&\geq (1-|b_1|)r - \sum_{k=2}^{\infty}(|a_k|+|b_k|)r^2 \\
&\geq (1-|b_1|)r - \frac{1-\gamma}{(3-\gamma)\,\Phi_{2,n}(\alpha_1,\lambda,\ell)} \sum_{k=2}^{\infty}\left(\frac{3-\gamma}{1-\gamma}\,|a_k| + \frac{3-\gamma}{1-\gamma}\,|b_k|\right)\Phi_{k,n}(\alpha_1,\lambda,\ell)r^2 \\
&\geq (1+|b_1|)r - \frac{(1-\gamma)}{(3-\gamma)\,\Phi_{2,n}(\alpha_1,\lambda,\ell)} \sum_{k=2}^{\infty} k\left(\frac{2k-1-\gamma}{1-\gamma}\,|a_k|\right. \\
&\quad \left. + \frac{2k+1+\gamma}{1-\gamma}\,|b_k|\right)\Phi_{k,n}(\alpha_1,\lambda,\ell)r^2 \\
&\geq (1-|b_1|)r - \frac{(1-\gamma)}{(3-\gamma)\,\Phi_{2,n}(\alpha_1,\lambda,\ell)}\frac{1}{2}\left(1-\frac{3+\gamma}{1-\gamma}\,|b_1|\right)r^2 \\
&\geq (1+|b_1|)r - \frac{1}{2\Phi_{2,n}(\alpha_1,\lambda,\ell)}\left(\frac{(1-\gamma)}{(3-\gamma)} - \frac{(3+\gamma)}{(3-\gamma)}\,|b_1|\right)r^2
\end{aligned}
$$

Remark 1.

Putting $q = 2, s = 1, \alpha_1 = \alpha_2 = \beta_1 = 1$, and $\lambda = \ell = n = 0$ we improve the results obtained by Kim et al. [9,with $k = 1$].

Corollary 1. *Let* $f \in \overline{G_H^n}((\alpha_1, \lambda, \ell), \gamma)$, *then for* $|b_1| < \frac{6\Phi_{2,n}(\alpha_1,\lambda,\ell)-1-(2\Phi_{2,n}(\alpha_1,\lambda,\ell)-1)\gamma}{3(2\Phi_{2,n}(\alpha_1,\lambda,\ell)-1)-(2\Phi_{2,n}(\alpha_1,\lambda,\ell)+1)\gamma}$ *the set*

$$\left\{ w : |w| < \frac{6\Phi_{2,n}(\alpha_1, \lambda, \ell) - 1 - (2\Phi_{2,n}(\alpha_1, \lambda, \ell) - 1)\gamma}{2\,(3-\gamma)\,\Phi_{2,n}(\alpha_1, \lambda, \ell)} - \frac{3(2\Phi_{2,n}(\alpha_1, \lambda, \ell) - 1) - (2\Phi_{2,n}(\alpha_1, \lambda, \ell) + 1)\gamma}{2\,(3-\gamma)\,\Phi_{2,n}(\alpha_1, \lambda, \ell)} |b_1| \right\}$$

is included in $f(U)$.

4. Extreme Points and Inclusion Results

We determine the extreme points of closed convex hulls of the class $\overline{G_H^n}((\alpha_1, \lambda, \ell), \gamma)$, denoted by $clco\overline{G_H^n}((\alpha_1, \lambda, \ell), \gamma)$.

Theorem 4. *Let* $f = h + \overline{g}$ *be such that* h *and* g are *given by (1.12). Then* $f \in clco\overline{G_H^n}((\alpha_1, \lambda, \ell), \gamma)$ *if and only if* f *can be expressed as*

$$f(z) = \sum_{k=1}^{\infty} \left[X_k h_k(z) + Y_k g_k(z) \right], \tag{4.1}$$

where

$$\begin{aligned} h_1(z) &= z, \\ h_k(z) &= z - \frac{(1-\gamma)}{k(2k-1-\gamma)\ \Phi_{k,n}(\alpha_1, \lambda, \ell)} z^k \quad (k \geq 2), \end{aligned}$$

$$g_k(z) = z + (-1)^n \frac{(1-\gamma)}{k(2k+1+\gamma)\ \Phi_{k,n}(\alpha_1, \lambda, \ell)} \overline{z}^k \ \ (k \geq 2),$$

$$X_k \geq 0, Y_k \geq 0, \quad \sum_{k=1}^{\infty} [X_k + Y_k] = 1.$$

In particular, the extreme points of the class $\overline{G_H^n}((\alpha_1, \lambda, \ell), \gamma)$ are $\{h_k\}$ and $\{g_k\}$ respectively.

Proof. First, we note that for f as in the theorem above, we may write

$$\begin{aligned} f(z) &= \sum_{k=1}^{\infty}\left[X_k h_k(z)+Y_k g_k(z)\right] \\ &= \sum_{k=1}^{\infty}\left[X_k+Y_k\right] z-\sum_{k=2}^{\infty}\frac{(1-\gamma)}{k(2k-1-\gamma)\ \Phi_{k,n}(\alpha_1,\lambda,\ell)}X_k z^k \\ &\quad +(-1)^n\sum_{k=1}^{\infty}\frac{(1-\gamma)}{k(2k+1+\gamma)\ \Phi_{k,n}(\alpha_1,\lambda,\ell)}Y_k\overline{z}^k \\ &= z-\sum_{k=2}^{\infty}A_k z^k+(-1)^n\sum_{k=1}^{\infty}B_k\overline{z}^k \end{aligned}$$

where

$$A_k=\frac{(1-\gamma)}{k(2k-1-\gamma)\ \Phi_{k,n}(\alpha_1,\lambda,\ell)}X_k, \quad \textit{and} \quad B_k=\frac{(1-\gamma)}{k(2k+1+\gamma)\ \Phi_{k,n}(\alpha_1,\lambda,\ell)}Y_k.$$

Therefore

$$\begin{aligned} &\sum_{k=2}^{\infty}\frac{k(2k-1-\gamma)\ \Phi_{k,n}(\alpha_1,\lambda,\ell)}{(1-\gamma)}A_k+\sum_{k=1}^{\infty}\frac{k(2k+1+\gamma)\ \Phi_{k,n}(\alpha_1,\lambda,\ell)}{(1-\gamma)}B_k \\ &= \sum_{k=2}^{\infty}X_k+\sum_{k=1}^{\infty}Y_k \\ &= 1-X_1\leq 1, \end{aligned}$$

and hence $f(z)\in clco\overline{G_H^n}((\alpha_1,\lambda,\ell),\gamma)$.
Conversely, suppose that $f(z)\in clco\overline{G_H^n}((\alpha_1,\lambda,\ell),\gamma)$. Setting

$$X_k=\frac{k(2k-1-\gamma)\ \Phi_{k,n}(\alpha_1,\lambda,\ell)}{(1-\gamma)}A_k\ \ (k\geq 2)$$

and

$$Y_k=\frac{k(2k+1+\gamma)\ \Phi_{k,n}(\alpha_1,\lambda,\ell)}{(1-\gamma)}B_k\ \ (k\geq 1),$$

where $\sum_{k=1}^{\infty}[X_k + Y_k] = 1$. Then

$$\begin{aligned} f(z) &= z - \sum_{k=2}^{\infty} A_k z^k + (-1)^n \sum_{k=1}^{\infty} B_k \overline{z}^k, \qquad A_k,\ B_k \geq 0 \\ &= z - \sum_{k=2}^{\infty} \frac{(1-\gamma)}{k(2k-1-\gamma)\ \Phi_{k,n}(\alpha_1,\lambda,\ell)} X_k z^k \\ &\quad + \sum_{k=1}^{\infty} \frac{(1-\gamma)}{k(2k+1+\gamma)\ \Phi_{k,n}(\alpha_1,\lambda,\ell)} Y_k \overline{z}^k \\ &= z + \sum_{k=2}^{\infty}(h_k(z)-z)X_k + \sum_{k=1}^{\infty}(g_k(z)-z)Y_k \\ &= \sum_{k=2}^{\infty}(X_k h_k(z) + Y_k g_k(z)) \end{aligned}$$

as required. This complete the proof.

Now we show that $\overline{G_H^n}((\alpha_1,\lambda,\ell),\gamma)$ is closed under convex combinations of its members.

Theorem 5. *The family $\overline{G_H^n}((\alpha_1,\lambda,\ell),\gamma)$ is closed under convex combinations.*

Proof. For $i = 1, 2, 3, ...,$ suppose that $f_i \in \overline{G_H^n}((\alpha_1,\lambda,\ell),\gamma)$, where

$$f_i(z) = z - \sum_{k=2}^{\infty} a_{i,k} z^k + (-1)^n \sum_{k=2}^{\infty} b_{i,k} \overline{z}^k.$$

Then, by inequality (2.3)

$$\begin{aligned} &\sum_{k=2}^{\infty} \frac{k(2k-1-\gamma)\ \Phi_{k,n}(\alpha_1,\lambda,\ell)}{(1-\gamma)} a_{i,k} + \\ &\sum_{k=1}^{\infty} \frac{k(2k+1+\gamma)\ \Phi_{k,n}(\alpha_1,\lambda,\ell)}{(1-\gamma)} b_{i,k} \\ \leq\ & 1. \end{aligned} \tag{4.2}$$

For $\sum\limits_{i=1}^{\infty} t_i = 1; 0 \leq t_i \leq 1$, the convex linear combination of f_i may be written as

$$\sum_{i=1}^{\infty} t_i f_i(z) = z - \sum_{k=2}^{\infty}\left(\sum_{i=1}^{\infty} t_i a_{i,k}\right) z^k - (-1)^n \sum_{k=1}^{\infty}\left(\sum_{i=1}^{\infty} t_i b_{i.k}\right) \overline{z}^k.$$

Using the inequality (4.2), we obtain

$$\begin{aligned} & \sum_{k=2}^{\infty} \frac{k(2k-1-\gamma)\ \Phi_{k,n}(\alpha_1,\lambda,\ell)}{(1-\gamma)} \left(\sum_{i=1}^{\infty} t_i a_{i,k}\right) \\ & \quad + \sum_{k=1}^{\infty} \frac{k(2k+1+\gamma)\ \Phi_{k,n}(\alpha_1,\lambda,\ell)}{(1-\gamma)} \left(\sum_{i=1}^{\infty} t_i b_{i.k}\right) \\ = & \sum_{i=1}^{\infty} t_i \left(\sum_{k=2}^{\infty} \frac{k(2k-1-\gamma)\ \Phi_{k,n}(\alpha_1,\lambda,\ell)}{(1-\gamma)} a_{i,k} \right. \\ & \quad \left. + \sum_{k=1}^{\infty} \frac{k(2k+1+\gamma)\ \Phi_{k,n}(\alpha_1,\lambda,\ell)}{(1-\gamma)} b_{i,k}\right) \\ \leq & \sum_{i=1}^{\infty} t_i = 1, \end{aligned}$$

and therefore $\sum_{i=1}^{\infty} t_i f_i(z) \in \overline{G_H^n}((\alpha_1,\lambda,\ell),\gamma)$.

Theorem 6. *For* $0 \leq \delta \leq \gamma < 1$, let $f(z) \in \overline{G_H^n}((\alpha_1,\lambda,\ell),\gamma)$ and $F(z) \in \overline{G_H^n}((\alpha_1,\lambda,\ell),\delta)$. Then

$$f(z) * F(z) \in \varsigma_H^n((\alpha_1,\lambda,\ell),\gamma) \subset \varsigma_H^n((\alpha_1,\lambda,\ell),\delta).$$

Proof. Let $f(z) = z - \sum_{k=2}^{\infty} a_k z^k - \sum_{k=2}^{\infty} \overline{b}_k \overline{z}^k \in \overline{G_H^n}((\alpha_1,\lambda,\ell),\gamma)$ and

$$F(z) = z - \sum_{k=2}^{\infty} A_k z^k + (-1)^n \sum_{k=1}^{\infty} \overline{B}_k \overline{z}^k \in \overline{G_H^n}((\alpha_1,\lambda,\ell),\delta). \tag{4.3}$$

Then

$$f(z) * F(z) = z + \sum_{k=2}^{\infty} a_k A_k z^k + (-1)^n \sum_{k=1}^{\infty} \overline{b}_k \overline{B}_k \overline{z}^k.$$

We note that $|A_k| \leq 1$ and $|B_k| \leq 1$. Now we have

$$\begin{aligned} & \sum_{k=2}^{\infty} \frac{k(2k-1-\delta)\Phi_{k,n}(\alpha_1,\lambda,\ell)}{(1-\delta)} |a_k|\,|A_k| + \sum_{k=1}^{\infty} \frac{k(2k+1+\delta)\Phi_{k,n}(\alpha_1,\lambda,\ell)}{(1-\delta)} |b_k|\,|B_k| \\ \leq & \sum_{k=2}^{\infty} \frac{k(2k-1-\delta)\Phi_{k,n}(\alpha_1,\lambda,\ell)}{(1-\delta)} |a_k| + \sum_{k=1}^{\infty} \frac{k(2k+1+\delta)\Phi_{k,n}(\alpha_1,\lambda,\ell)}{(1-\delta)} |b_k| \\ \leq & \sum_{k=2}^{\infty} \frac{k(2k-1-\gamma)\ \Phi_{k,n}(\alpha_1,\lambda,\ell)}{(1-\gamma)} |a_k| + \sum_{k=1}^{\infty} \frac{k(2k+1+\gamma)\ \Phi_{k,n}(\alpha_1,\lambda,\ell)}{(1-\gamma)} |b_k| \leq 1, \end{aligned}$$

using Theorem 2 since $f \in \overline{G_H^n}((\alpha_1,\lambda,\ell),\gamma)$ and $0 \leq \delta \leq \gamma < 1$. This proves that $f(z) * F(z) \in \overline{G_H^n}((\alpha_1,\lambda,\ell),\delta)$.

5. Integral Operator

Now, we examine a closure property of the class $\overline{G_H^n}((\alpha_1,\lambda,\ell),\gamma)$ under the generalized Bernardi-Libera- Livingston integral operator $L_c(f)$ which is defined by (see [2], [10] and [11])

$$L_c(f) = \frac{c+1}{z^c}\int_0^z t^{c-1} f(t)dt \qquad (c > -1).$$

Theorem 7. *Let* $f \in \overline{G_H^n}((\alpha_1,\lambda,\ell),\gamma)$. *Then* $L_c(f(z)) \in \overline{G_H^n}((\alpha_1,\lambda,\ell),\gamma)$

Proof. From the representation of $L_c(f(z))$, it follows that

$$\begin{aligned} L_c(f) &= \frac{c+1}{z^c}\int_0^z t^{c-1}[h(t)+\overline{g(t)}]dt. \\ &= \frac{c+1}{z^c}\left(\int_0^z t^{c-1}\left(t-\sum_{k=2}^{\infty} a_k t^n\right)dt - (-1)^n \overline{\int_0^z t^{c-1}\left(\sum_{k=1}^{\infty} a_k t^n\right)dt}\right) \\ &= z - \sum_{k=2}^{\infty} A_k z^k + (-1)^n \sum_{k=1}^{\infty} B_k \overline{z}^k \end{aligned}$$

where

$$A_k = \frac{c+1}{c+n} a_k; \quad B_k = \frac{c+1}{c+n} b_k$$

Therefore,

$$\begin{aligned}
& \sum_{k=1}^{\infty} k\left(\tfrac{2k-1-\gamma}{1-\gamma}(\frac{c+1}{c+n}|a_k|)+\tfrac{2k-1-\gamma}{1-\gamma}(\frac{c+1}{c+n}|b_k|)\right)\Phi_{k,n}(\alpha_1,\lambda,\ell) \\
\leq \; & \sum_{k=1}^{\infty} k\left(\tfrac{2k-1-\gamma}{1-\gamma}|a_k|+\tfrac{2k-1-\gamma}{1-\gamma}|b_k|\right)\Phi_{k,n}(\alpha_1,\lambda,\ell) \\
\leq \; & 2(1-\gamma).
\end{aligned}$$

Since $f \in \overline{G_H^n}((\alpha_1,\lambda,\ell),\gamma)$, therefore by Theorem 2, $L_c(f(z)) \in \overline{G_H^n}((\alpha_1,\lambda,\ell),\gamma)$.

Remark 2. (i) By specializing the parameters $q, s, \alpha_1, \beta_1, n, \ell$ and λ, we can obtain new results for the subclass of analytic univalent functions mantionent in the introduction,

(ii) Putting $\lambda = \ell = n = 0$ in our results we obtain the results obtained by Chandrashekar et al. [4],

(iii)Putting $q = 2, s = 1, \alpha_1 = \alpha_2 = \beta_1 = 1$, and $\lambda = \ell = n = 0$ in our results we obtain the results obtained by Kim et al. [9, with $k = 1$].

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