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EXTREMAL PROBLEMS IN BMOA TYPE SPACES IN DOMAINS IN C^n

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Abstract. We provide some new sharp results on extremal problems in new BMOA type spaces in the unit ball and pseudoconvex domains. These generalize a known one dimensional result and in addition these results as far as we know are first sharp results of this type in higher dimension in BMOA type spaces.

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1. On a distance function in BMOA type spaces in bounded pseudoconvex domains.

In recent decades many papers appeared where various BMO spaces or BMOA type spaces were studied from various points of views in higher dimension in various domains in C^n. We refer for example to a series of papers of S. Krantz and coauthors (see [24], [23], [25] in particular) and also to [17], [10], [6], [9] in this direction.

Results we provide in this paper are as far as we know first sharp results on distance function in analytic BMOA-type spaces in higher dimension. Related results on other analytic function spaces in higher dimension were given earlier in [19]. We refer readers also to this paper for various other recent papers in this direction in analytic spaces in higher dimension in various types of domains.

The goal of first section is to provide full proof of known sharp theorem on an extremal problem related with the distance function in the unit disk case and then based on it for particular values of parameters we provide new results on distance function in more complicated new BMOA-type spaces in the unit ball and in bounded pseudoconvex domains with smooth boundary.

Let D be the unit disk in C, $dA(z)$ be the normalized Lebesgue measure on D so that $A(D) = 1$ and $d\zeta$ be the Lebesgue measure on the ∂D.

For $f \in H(D)$ and $f(z) = \sum_k a_k z^k$, define the fractional derivative of the function f as usual in the following manner

$$\mathcal{D}^\alpha f(z) = \sum_{k=0}^{\infty} (k+1)^\alpha a_k z^k, \alpha \in \mathbb{R}.$$

We will write $\mathcal{D}f(z)$ if $\alpha = 1$. Obviously, for all $\alpha \in \mathbb{R}$, $\mathcal{D}^\alpha f \in H(D)$ if $f \in H(D)$.

For $a \in D$, let $g(z,a) = log(\frac{1}{\varphi_a(z)})$ be the Green's function for D with pole at a, where $\varphi_a(z) = \frac{a-z}{1-\bar{a}z}$. For $0 < p < \infty$, $-2 < q < \infty$, $0 < s < \infty$, $-1 < q+s < \infty$, we say that $f \in F(p,q,s)$, if $f \in H(D)$ and

$$\|f\|^p_{F(p,q,s)} = \sup_{a \in D} \int_D |\mathcal{D}(f(z))|^p (1-|z|^2)^q g(z,a)^s dA(z) < \infty.$$

As we know(see ,for example, [2]), if $0 < p < \infty$, $-2 < q < \infty$, $0 < s < \infty$, $-1 < q+s < \infty$, $f \in F(p,q,s)$ if and only if

$$\sup_{a \in D} \int_D |\mathcal{D}(f(z))|^p (1-|z|^2)^q (1-|\varphi_a(z)|)^s dA(z) < \infty.$$

It is known (see ,for example, [2]) that $F(2,0,1) = BMOA$.

We recall that the weighted Bloch class $\mathcal{B}^\alpha(D)$, $\alpha > 0$, is the collection of the analytic functions on the D satisfying

$$\|f\|_{\mathcal{B}^\alpha} = \sup_{z \in D} |\mathcal{D}f(z)|(1-|z|^2)^\alpha < \infty.$$

Space $\mathcal{B}^\alpha(D)$ is a Banach space with the norm $\|f\|_{\mathcal{B}^\alpha}$. Note $\mathcal{B}^1(D) = \mathcal{B}(D)$ is a classical Bloch class (see [6] and the references there).

For $k > s$, $0 < p,q \le \infty$, the weighted analytic Besov space $\mathcal{B}^{q,p}_s(D)$ is the class of analytic functions satisfying (see [6])

$$\|f\|^q_{\mathcal{B}^{q,p}_s} = \int_0^1 \Big(\int_T |\mathcal{D}^k f(r\zeta)|^p |d\zeta| \Big)^{\frac{q}{p}} (1-r)^{(k-s)q-1} dr < \infty.$$

Quasinorm $\|f\|^q_{\mathcal{B}^{q,p}_s}$ does not depend on k. If $\min(p,q) \ge 1$, the class $\mathcal{B}^{q,p}_s(D)$ is a Banach space under the norm $\|f\|^q_{\mathcal{B}^{q,p}_s}$. If $\min(p,q) < 1$, then we have quasinormed class. For negative s and $p = q$ we have classical Bergman space A^p_{-sq-1}.Let also $A^p_0 = A^p$ and we define formally the HArdy space as A^p_{-1}(see, for example, [6]). The well-known so -called "duality" approach to extremal problems in theory of analytic functions leads to the following general formula

$$dist_Y(g,X) = \sup_{l\in X^\perp, \|l\|\leq 1} |l(g)| = \inf_{\varphi\in X} \|g-\varphi\|_Y,$$

where $g \in Y$, X is subspace of a normed space Y, $Y \in H(D)$ and $X^\perp$ is the orthogonal compliment of X in Y^*, the dual space of Y and l is linear functional on Y (see [5]).

Various extremal problems in H^p Hardy classes in D based on duality approach we mentioned were discussed in [2] (Chapter 8). In particular for a function $K \in L^q(T)$ the following equality holds (see [2]), $1 \leq p < \infty$, $\frac{1}{p}+\frac{1}{q}=1$,

$$dist_{L^q}(K,H^q) = \inf_{g\in H^q, K\in L^q} \|K-g\|_{H^q} = \sup_{f\in H^p, \|f\|_{H^p}\leq 1} \frac{1}{2\pi}\Big|\int_{|\zeta|=1} f(\zeta)K(\zeta)d\zeta\Big|.$$

It is well known that if $p>1$ then the inf-dual extremal problem in the analytic H^p Hardy classes has a solution, it is unique if an extremal function exists (see for example [2] and references there). Note also that extremal problems for H^p spaces in multiply connected domains were studied before in [3], [7]. Various new results on extremal problems in A^p Bergman class and in its subspaces were obtained recently by many authors (see [4] and various references there). In this paper we will provide direct proofs for estimation of $dist_Y(f,X) = \inf_{g\in X}\|f-g\|_Y$, $X \subset Y$, $X, Y \subset H(D)$, $f \in Y$.

Let further $\Omega^k_{\alpha,\varepsilon} = \{z \in D : |D^k f(z)|(1-|z|^2)^\alpha \geq \varepsilon\}$, $\alpha \geq 0$, $\varepsilon > 0$. We will need the following definition.

Definition 1. *We say a positive Borel measure μ in D is s Carleson measure in the unit disk where $s<1$ or $s=1$ if*

$$\sup_{w\in D}\left[(1-|w|)^s \int_D \frac{d\mu(z)}{|1-\bar{w}z|^{2s}}\right] < \infty.$$

These measures studied actively in the unit disk and higher dimension recently.We refer for example to [10] for more information on these measures in higher dimension. For $s=1$ we have ordinary classical Carleson measures in the unit disk (see for example [6],[9] and various references there). Applying famous Fefferman duality theorem, P. Jones proved the following result.

Theorem 1. *([2]) Let $f \in \mathcal{B}$. Then the following are equivalent:*

(a) $d_1 = dist_{\mathcal{B}}(f, BMOA)$;

(b) $d_2 = \inf\{\varepsilon > 0 : \mathcal{X}_{\Omega^1_{1,\varepsilon}(f)}(z)\frac{dA(z)}{1-|z|^2}$ *is a Carleson measure*$\}$, *where $\mathcal{X}$ denotes the characteristic function of the mentioned set.*

Recently, R. Zhao (see [2]) and W. Xu (see [8]), repeating arguments of R. Zhao in the unit ball, obtained results on distances from Bloch functions to some Mobius invariant function spaces in one and higher dimensions in a relatively direct way. The goal of this paper is to develop further their ideas and present new sharp theorems in the higher dimension.

In this paper various sharp assertions for distance function will be given. We will indicate proofs of some assertions in details, short sketches of proofs in some other cases will be also provided.

Throughout the paper, we write C (sometimes with indexes) to denote positive constants which might be different at each occurrence (even in a chain of inequalities) ,but is independent of the functions or variables being discussed.

For the proof of one of the main results of this paper we will need the following estimate which can be found for example in [6].

Lemma 2. *(see [6]) Let $s > -1$, $r > 0$, $t > 0$ and $r + t - s > 2$. If $t < s + 2 < r$ then we have $\int_D \frac{(1-|z|^2)^s dA(z)}{|1-\bar{w}z|^r |1-\bar{a}z|^t} \leq \frac{C}{(1-|w|^2)^{r-s-2}|1-\bar{a}w|^t}$, $a, w \in D$.*

Note that $F(p, q, s) \subset \mathcal{B}^{\frac{q+2}{p}}$, $s \in (0, 1]$, (see [2]). Hence for $\alpha \geq \frac{q+2}{p}$, the problem of finding $dist_{\mathcal{B}^\alpha}(f, F(p, q, s))$ appears naturally.

In the following theorem we show that in Zhao's theorems (see[2]) the well-known Moebius invariant Bloch classes can be replaced by same Bloch classes, but with general weights.

Theorem 3. *Let $1 \leq p < \infty$, $\alpha > 0$, $0 < s \leq 1$, $\alpha \geq \frac{q+2}{p}$, $q > \alpha(p-1) - s - 1$, $q > s - 2 + \alpha(p-1)$ and $f \in \mathcal{B}^\alpha$. Then the following are equivalent:*

(a) $d_1 = dist_{\mathcal{B}^\alpha}(f, F(p, q, s))$;

(b) $d_2 = \inf\{\varepsilon > 0 : \mathcal{X}_{\Omega^1_{\alpha,\varepsilon}(f)}(z) \frac{dA(z)}{(1-|z|^2)^{\alpha p - q - s}}$ is an s - Carleson measure$\}$.

Proof. First we show $d_1 \leq C d_2$. According to the Bergman representation formula (see [6]), we have

$$f(z) = C(\alpha) \int_D \mathcal{D}f(w)(1-|w|^2)^\alpha \mathcal{D}^{-1} \frac{1}{(1-\bar{w}z)^{\alpha+2}} dA(w) =$$

$$= C(\alpha) \int_{\Omega^1_{\alpha,\varepsilon}} \mathcal{D}f(w)(1-|w|^2)^\alpha \mathcal{D}^{-1} \frac{1}{(1-\bar{w}z)^{\alpha+2}} dA(w) +$$

$$+ C(\alpha) \int_{D \setminus \Omega^1_{\alpha,\varepsilon}} \mathcal{D}f(w)(1-|w|^2)^\alpha \mathcal{D}^{-1} \frac{1}{(1-\bar{w}z)^{\alpha+2}} dA(w) = f_1(z) + f_2(z);\ z \in D$$

where $C(\alpha)$ is the constant of the Bergman representation formula in the unit disk (see [9]). By

$$\mathcal{D}f_1(z) = C(\alpha)\int_{\Omega^1_{\alpha,\varepsilon}} \frac{\mathcal{D}f(w)(1-|w|^2)^{\alpha}}{(1-\bar{w}z)^{\alpha+2}}dA(w),$$

$$|\mathcal{D}f_1(z)| \leq C\int_{\Omega^1_{\alpha,\varepsilon}} \frac{|\mathcal{D}f(w)|(1-|w|^2)^{\alpha}}{|1-\bar{w}z|^{\alpha+2}}dA(w) \leq C\|f\|_{\mathcal{B}^{\alpha}}\frac{1}{(1-|z|)^{\alpha}},\ z\in D$$

Then $f_1 \in \mathcal{B}^{\alpha}$. By Lemma 2,

$$\int_D |\mathcal{D}f_1(z)|^p(1-|z|^2)^q(1-|\varphi_a(z)|^2)^s dA(z) \leq$$

$$\leq C_1\|f_1\|^{p-1}_{\mathcal{B}^{\alpha}}\int_D |\mathcal{D}f_1(z)|(1-|z|^2)^{q-(p-1)\alpha}(1-|\varphi_a(z)|^2)^s dA(z) \leq$$

$$\leq C_2\|f_1\|^{p-1}_{\mathcal{B}^{\alpha}}\int_D\int_{\Omega^1_{\alpha,\varepsilon}} \frac{|\mathcal{D}f(w)|(1-|w|^2)^{\alpha}}{|1-\bar{w}z|^{2+\alpha}}dA(w)(1-|z|^2)^{q-(p-1)\alpha}(1-|\varphi_a(z)|^2)^s dA(z) \leq$$

$$\leq C_3\|f_1\|^{p-1}_{\mathcal{B}^{\alpha}}\|f\|_{\mathcal{B}^{\alpha}}\int_{\Omega^1_{\alpha,\varepsilon}}(1-|a|^2)^s\int_D \frac{(1-|z|^2)^{q-(p-1)\alpha+s}dA(z)dA(w)}{|1-zw|^{2+\alpha}|1-\bar{a}z|^{2s}} \leq$$

$$\leq C\int_{\Omega^1_{\alpha,\varepsilon}} \frac{(1-|a|^2)^s}{|1-|w|^2|^{p\alpha-q-s}|1-\bar{a}w|^{2s}}dA(w).$$

By $\mathcal{X}_{\Omega^1_{\alpha,\varepsilon}}\frac{dA(z)}{(1-|z|^2)^{\alpha p-q-s}}$ is an s-Carleson measure, $f_1 \in F(p,q,s)$. Also we have

$$|\mathcal{D}f_2(z)| \leq C\int_{D\setminus\Omega^1_{\alpha,\varepsilon}} \frac{|\mathcal{D}f(w)|(1-|w|^2)^{\alpha}}{|1-\bar{w}z|^{\alpha+2}}dA(w) \leq C\varepsilon\int_D \frac{dA(w)}{|1-\bar{w}z|^{2+\alpha}} \leq \frac{C\varepsilon}{(1-|z|)^{\alpha}}.$$

So, $dist_{\mathcal{B}^{\alpha}}(f, F(p,q,s)) \leq \|f-f_1\|_{\mathcal{B}^{\alpha}} = \|f_2\|_{\mathcal{B}^{\alpha}} < \varepsilon$.

It remains to show that $d_1 > d_2$. If $d_1 < d_2$ then we can find two numbers ε, ε_1 such that $\varepsilon > \varepsilon_1 > 0$ and a function $f_{\varepsilon_1} \in F(p;q;s)$, $\|f-f_{\varepsilon_1}\|_{\mathcal{B}^{\alpha}} \leq \varepsilon_1$ and $\frac{\mathcal{X}_{\Omega^1_{\alpha,\varepsilon}}(z)}{(1-|z|^2)^{\alpha p-q-s}}$ is not a s-Carleson measure. Since $(|Df(z)|-|Df_{\varepsilon_1}(z)|)(1-|z|^2)^{\alpha} \leq \varepsilon_1$ we can easily obtain

$$(\varepsilon-\varepsilon_1)\mathcal{X}_{\Omega^1_{\alpha,\varepsilon}}(z)dA(z) \leq C|\mathcal{D}f_{\varepsilon_1}(z)|(1-|z|^2)^{\alpha},$$

where $\mathcal{X}_{\Omega^1_{\alpha,\varepsilon}}$ is defined above. Hence from this equation and the fact that $f_{\varepsilon_1} \in F(p;q;s)$ and hence we arrive at contradiction. The theorem is proved.

The carefull analysis and inspection of the proof of this result shows that some similar results in this direction should be valid in more complicated domains such as the unit ball or the unit polydisk.

We extend the previous theorem to the unit ball case and then even to bounded strongly pseudoconvex domain with smooth boundary for particular values of parameters. Let B be the unit ball in $\mathbb{C}^n$, $n \in \mathbb{N}$. Let $H(B)$ be the space of all analytic functions in B. Let also dv be usual Lebeques measure on B.

Let

$$\mathcal{B}^\alpha(B) = \{f \in H(B) : |\mathcal{D}f(z)|(1-|z|)^\alpha < +\infty\}, \alpha \geq 0.$$

Let

$$F(p,q)(B) = \left\{ g \in H(B) : \sup_{a\in B} \int_B \frac{|\mathcal{D}\}|^p(1-|w|)^{q+1}dv(w)}{|1-\bar{a}w|^{n+1}}(1-|a|) < \infty \right\},$$

$1 \leq p < \infty$, $q \in (0,\infty)$, n is a natural number and where $\mathcal{D}^{\tilde{k}}$ is a usual fractional derivative of analytic f function in B, $(\mathcal{D}^{\tilde{k}}f)(z) = \sum\limits_{|k|\geq 0}(|k|+1)^{\tilde{k}}a_k z^k$; $f(z) = \sum\limits_{|k|\geq 0} a_k z^k$. These are Banach spaces.We refer for them [17] and [9]. There is an embedding between these two spaces (see [17]) and dist problem can be hence posed again. Let $\alpha \geq \frac{q+2}{p}$ then $F(p,q) \subset B_\alpha$; $1 \leq p < \infty$, $q \in (0,\infty)$. The following definition can be seen in [6] and in [9].

Definition 2. *We say μ a positive Borel measure in B is a Carleson measure in the unit ball if*

$$\sup_{w\in B}\left[(1-|w|)\int_B \frac{d\mu(z)}{|1-\bar{w}z|^{n+1}}\right] < \infty.$$

Note for $n=1$ we have usual Carleson measures in the unit disk.

Theorem 4. *Let $\Omega^2_{\alpha,\varepsilon} = \{z \in B : |\mathcal{D}f(z)|(1-|z|)^\alpha > \varepsilon\}$. Let $1 \leq p < \infty$, $\alpha \geq \frac{q+2}{p}$; $q_1 < q < q_0$, $q_0 = q_0(\alpha,p)$; $q_1 = q_1(\alpha,p)$. Then let $f \in \mathcal{B}^\alpha$. Then*

$$dist_{B^\alpha}(f, F(p,q)) \asymp \inf\left\{\varepsilon > 0 : \left[\lambda_{\Omega^2_{\alpha,\varepsilon}}(z)\right] \frac{dv(z)}{(1-|z|^2)^{\alpha p-q-1}} \text{ is a Carleson measure } \right\}.$$

Proof. Note in the unit ball we have that

$$\sup_{z\in B}|(\mathcal{D}^k f)(z)|(1-|z|)^{k-s} \leq c_1 \sup_{a\in B}\int_B \frac{|(\mathcal{D}^k f)(z)|^p(1-|z|)^{p(k-s)-1}d\nu(z)}{|1-z\bar{a}|^{n+1}}(1-|a|),$$

$1 \leq p < \infty$,$k > s$,$s \in R$(see [6]and see [9]).

For $D^k f = Dg$,where g is an analytic function in the unit ball and for $(k-s) = \frac{q+2}{p}$, where $q$is positive we have the same embedding as in the unit disk case that is $F(p,q,1) \subset B^{\frac{q+2}{p}}$.(see the previous theorem)

The rest is the repetition of arguments above (the unit disk case) based on the following three lemmas.(their complete analogues were used in the unit disk proof which we provided above)

Lemma 5. *([9]) We have*

$$\int_B \frac{(1-|z|)^s dv(z)}{|1-\bar{w}z|^r |1-\bar{a}z|^t} \le \frac{c(1-|w|)^{-r+s+n+1}}{|1-\bar{a}w|^t};$$

where $r, t > 0,\ s > -1$; $r+t-s > n+1$; $a, w \in B,\ t < s+n+1 < r$.

(the analogue of this lemma is valid also in Euclidean spaceR^n ,see [28])

Lemma 6. *([9]) For each f function analytic in the unit ball we have the following Bergman integral representation.*

$$f(z) = c(\alpha) \int_B \frac{f(w)(1-|w|)^\alpha dv(w)}{(1-\bar{w}z)^{\alpha+n+1}},$$

for all $\alpha > -1$ and all z in the unit ball.

It is easy to observe that we use milder version of this lemma during the proof in the unit ball ,to be more precise we need this lemma for all analytic f functions taken from analytic B^α space in the unit ball. Further precisely this fact is valid also in bounded strongly pseudoconvex domains with smooth boundary (see for this, for example, [10] or [16] and references there) The following is the well-known Forelli-Rudin estimate. It is also needed for our proof. In [10] or [16] we can see this lemma also in bounded pseudoconvex domains with smooth boundary.

Lemma 7. *([9]) Let $\alpha > -1$ then we have that*

$$\int_B \frac{(1-|z|^2)^\alpha dv(z)}{|1-z\bar{w}|^{\beta+n+1}} \le c(1-|w|)^{\alpha-\beta},\ \beta > \alpha,\ w \in B$$

We use the important observarion is that these three lemmas are valid also for bounded pseudoconvex domains with smooth boundary. We refer the reader to [10] and [16] and [18] for these results. Even further, we note the same sharp result on distances even with the same proof is valid in bounded pseudoconvex domains with smooth boundary D in C^n . We need some some standard definitions for these domains. Let D be the bounded strictly pseudoconvex domain in $\mathbb{C}^n$ with smooth boundary. Let further $H(D)$ be the space of all analytic functions on D.

Now shortly we provide also a chain of introductiory material,facts,properties and estimates on the geometry of strongly pseudconvex domains which we will use partially. Practically all of them on bounded pseudoconvex domains are taken from recent interesting papers of Abate and coauthors [14], [18]. In particular, we following these papers provide several results on the boundary behavior of Kobayashi balls, and we formulate a vital submean property for nonnegative plurisubharmonic functions in Kobayashi balls and Forelli -Rudin type estimate which are crucial for our proof in this setting. We will also provide properties of r-lattices (see [14])which are needed for an embedding to pose our distance problem. Let $D \subset\subset \mathbb{C}^n$ be a bounded strongly pseudoconvex domain in $\mathbb{C}^n$. We shall use the following notations:

- $\delta : D \to \mathbb{R}^+$ will denote the Euclidean distance from the boundary, that is $\delta(z) = d(z, \partial D)$;
- given two non-negative functions $f, g : D \to \mathbb{R}^+$ we shall write $f \preceq g$ to say that there is $C > 0$ such that $f(z) \leq Cg(z)$ for all $z \in D$. The constant C is independent of $z \in D$, but it might depend on other parameters (r, θ, etc.);
- ν (or sometimes dv) will be the Lebesque measure;
- $H(D)$ will denote the space of holomorphic functions on D, endowed with the topology of uniform convergence on compact subsets;
- $K : D \times D \to \mathbb{C}$ will be the Bergman kernel of D; The K_t is a kernel of type t, (see [16]). Note $K = K_{n+1}$ (see [14], [16]);
- given $r \in (0, 1)$ and $z_0 \in D$, we shall denote by $B_D(z_0, r)$ the Kobayashi ball of center z_0 and radius $\frac{1}{2}\log\frac{1+r}{1-r}$.

See, e.g., [18], [33], [34] for definitions, basic properties and applications to geometric function theory of the Kobayashi distance; [30], [31] for definitions and basic properties of the Bergman kernel.

We add some basic facts on pseudoconvex domains now and much details on Kobayashi distance and Kobayashi balls, it is the main tool for our proof (see also [14] and references there). Given $z_0 \in D$ and $0 < r < 1$, let $B_D(z_0, r)$ denote the ball of center z_0 and radius $\frac{1}{2}\log\frac{1+r}{1-r}$ for the Kobayashi distance k_D of D (that is, of radius r with respect to the pseudohyperbolic distance $\rho = tanh(k_D)$; see [14]). The basic fact for Kobayashi balls $B(z, r)$ is the following. It is possible to prove [14], [18] for D strongly pseudoconvex that a finite positive measure μ is a Carleson measure of $A^p(D)$ for all positive p if and only if for some (and hence all) $0 < r < 1$ there is a constant $C_r > 0$ such that

$$\mu(B_D(z_0, r)) \leq C_r \nu(B_D(z_0, r))$$

for all $z_0 \in D$. (The proof of this equivalence in [18] relied on Cima and Mercer's characterization [32]). Here ν (or below sometimes dv) is a standard Lebeques measure on D.

We now recall first definition and main properties of the Kobayashi distance which can be seen in various books and papers; we refer, for example, to [18], [33] and [34] for details. Let k_Δ denote the Poincare distance on the unit disk $\Delta \subset \mathbb{C}^n$. If X is a complex manifold, the Lempert function $\delta_X{:}X \times X \to \mathbb{R}^+$ of X is defined by

$$\delta_X(z, w) = \inf\{k_\Delta(\zeta, \eta)|\ \textit{there exists a holomorphic}\ \phi :\ \Delta\ \to X$$
$$\textit{with}\ \phi(\zeta) = z\ \textit{and}\ \phi(\eta) = w\}$$

for all $z, w \in X$. The Kobayashi pseudodistance $k_X : X \times X \to \mathbb{R}^+$ of X is the smallest pseudodistance on X bounded below by δX. We say that X is (Kobayashi) hyperbolic if k_X is a true distance and in that case it is known that the metric topology induced by k_X coincides with the manifold topology of X (see, e.g., Proposition 2.3.10 in [18]). For instance, all bounded domains are hyperbolic (see, e.g., Theorem 2.3.14 in [18]). The following properties are well known in literature. The Kobayashi (pseudo)distance is contracted by holomorphic maps: if $f : X \to Y$ is a holomorphic map then

$$\forall z, w \in X \quad k_Y(f(z), f(w)) \leq k_X(z, w).$$

Next the Kobayashi distance is invariant under biholomorphisms, and decreases under inclusions: if $D_1 \subset D_2 \subset\subset \mathbb{C}^n$ are two bounded domains we have $k_{D_2}(z, w) \leq k_{D_1}(z, w)$ for all $z, w \in D_1$. Further the Kobayashi distance of the unit disk coincides with the Poincare distance. Also, the Kobayashi distance of the unit ball $B \subset \mathbb{C}^n$ coincides with the well known in many applications so-called Bergman distance (see, e.g., Corollary 2.3.6 in [18].

If X is a hyperbolic manifold, $z_0 \in X$ and $r \in (0; 1)$ we shall denote by $B_X(z_0, r)$ the Kobayashi ball of center z_0 and radius $\frac{1}{2} \log \frac{1+r}{1-r}$:

$$B_X(z_0, r) = \{z \in X|\ \ tanh\ k_X(z_0, z) < r\}.$$

We can see that $\rho_X = tanh\ k_X$ is still a distance on X, because tanh is a strictly convex function on $\mathbb{R}^+$. In particular, ρ_B is the pseudohyperboic distance of B.

The Kobayashi distance of bounded strongly pseudoconvex domains with smooth boundary has several important properties. First of all, it is complete (see, e.g., Corollary 2.3.53 in [18]), and hence closed Kobayashi balls are compact. It is vital that we can describe the boundary behavior of the Kobayashi distance: if $D \subset\subset \mathbb{C}^n$ is a strongly pseudoconvex bounded domain and $z_0 \in D$, there exist $c_0, C_0 > 0$ such that

$$\forall z \in D \qquad c_0 - \frac{1}{2} \log d(z, \partial D) \leq k_D(z_0, z) \leq C_0 - \frac{1}{2} \log d(z, \partial D),$$

where $d(\cdot, \partial D)$ denotes the Euclidean distance from the boundary of D (see Theorems 2.3.51 and 2.3.52 in [18]).

Lemma 8 (see [14], [18]). *Let $D \subset\subset \mathbb{C}^n$ be a strongly pseudoconvex bounded domain. Then there exist $c_1 > 0$ and, for each $r \in (0;1)$, a $C_{1,r} > 0$ depending on r such that*

$$c_1 r^{2n} d(z_0, \partial D)^{n+1} \le \nu(B_D(z_0, r)) \le C_{1,r} d(z_0, \partial D)^{n+1}$$

for every $z_0 \in D$ and $r \in (0,1)$.

Let us now recall a number of results proved in [18]. The first two give information about the shape of Kobayashi balls:

Lemma 9 (Lemma 2.1 in [18]). *Let $D \subset\subset \mathbb{C}^n$ be a bounded strongly pseudoconvex domain, and $r \in (0,1)$. Then*

$$\nu(B_D(\cdot, r)) \approx \delta^{n+1},$$

(where the constant depends on r).

Lemma 10 (Lemma 2.2 in [18]). *Let $D \subset\subset \mathbb{C}^n$ be a bounded strongly pseudoconvex domain. Then there is $C > 0$ such that*

$$\frac{C}{1-r}\delta(z_0) \ge \delta(z) \ge \frac{1-r}{C}\delta(z_0)$$

for all $r \in (0,1), z_0 \in D$ and $z \in B_D(z_0, r)$.

Definition 3. *Let $D \subset\subset \mathbb{C}^n$ be a bounded domain, and $r > 0$. An r-lattice in D is a sequence $\{a_k\} \subset D$ such that $D = \bigcup_k B_D(a_k, r)$ and there exists $m > 0$ such that any point in D belongs to at most m balls of the form $B_D(a_k, R)$, where $R = \frac{1}{2}(1+r)$.*

The existence of r-lattices in bounded strongly pseudoconvex domains is ensured by the following

Lemma 11 (Lemma 2.5 in [18]). *Let $D \subset\subset \mathbb{C}^n$ be a bounded strongly pseudoconvex domain. Then for every $r \in (0,1)$ there exists an r-lattice in D, that is there exist $m \in N$ and a sequence $\{a_k\} \subset D$ of points such that $D = \bigcup_{k=0}^{\infty} B_D(a_k, r)$ and no point of D belongs to more than m of the balls $B_D(a_k, R)$, where $R = \frac{1}{2}(1+r)$,.*

We will call r-lattice sometimes the family $B_D(a_k, r)$. Dealing with K unweighted Bergman kernel we always assume (we assume here the kernel is positive otherwise we simply add modulus) $K(z, a_k) \asymp K(a_k, a_k)$ for any $z \in B_D(a_k, r)$, $r \in (0;1)$

(see [14], [18]) for proof. This leads to the same properity for all weighted Bergman kernels K_t, so that $t = m(n+1)$, where $m \in N$. This follows directly from definition of weighted Bergman kernel via Henkin-Ramirez function (see [16] and references there for this well-known defintion). We assume here and sometimes below that the Bergman kernel is positive otherwise we simply add modulus.

We mention a submean estimate for nonnegative plurisubharmonic functions on Kobayashi balls:

Lemma 12 (Corollary 2.8 in [18]). *Let $D \subset\subset \mathbb{C}^n$ be a bounded strongly pseudoconvex domain. Given $r \in (0,1)$, set $R = \frac{1}{2}(1+r) \in (0,1)$. Then there exists a $C_r > 0$ depending on r such that*

$$\forall z_0 \in D, \quad \forall z \in B_D(z_0, r) \quad \chi(z) \leq \frac{C_r}{\nu(B_D(z_0,r))} \int\limits_{B_D(z_0,R)} \chi d\nu$$

for every nonnegative plurisubharmonic function $\chi : D \to \mathbb{R}^+$.

We will use this lemma for proof for $\chi = |f(z)|^q, f \in H(D), q \in (0;\infty)$.

We now collect a few facts on the (possibly weighted) L^p-norms of the Bergman kernel and the normalized Bergman kernel. The first result is classical (see, e.g., [14], [18]):

Lemma 13 (see [14]). *Let $D \subset\subset \mathbb{C}^n$ be a bounded strongly pseudoconvex domain. Then*

$$\|K(\cdot, z_0)\|_2 = \sqrt{K(z_0,z_0)} \approx \delta^{-\frac{n+1}{2}}(z_0) \quad \textit{and} \quad \|k_{z_0}\|_2 \equiv 1$$

for all $z_0 \in D$.

The next result contains the weighted L^p-estimates we shall need in context of pseudoconves domains. (so- called Forelli-Rudin type estimtes):

Theorem 14. *(see [14]) Let $D \subset\subset \mathbb{C}^n$ be a bounded strongly pseudoconvex domain, and let $z_0 \in D$ and $1 \leq p < \infty$. Then*

$$\int\limits_D |K(\zeta, z_0)|^p \delta^\beta(\zeta) d\nu(\zeta) \preceq \begin{cases} \delta^{\beta-(n+1)(p-1)}(z_0), & for \ -1 < \beta < (n+1)(p-1); \\ |\log \delta(z_0)|, & for \ \beta = (n+1)(p-1); \\ 1, & for \ \beta > (n+1)(p-1). \end{cases}$$

A complete analogue of this theorem is valid also for all (unweighted) K_t Bergman kernels of t type,where $t > 0$ (see for example [16]).

Let

$$\mathcal{B}^\alpha(D) = \left\{ f \in H(D) : \sup_{z \in D} |\bigtriangledown f(z)| \delta(z)^\alpha < +\infty \right\}, \alpha \geq 0.$$

, where the differential operator is the gradient of f (see [16])

Let

$$F(p,q)(D) = \left\{ f \in H(D) : \sup_{a \in D} \int_D | \bigtriangledown f(w)|^p \delta(w)^{q+1} |K_{n+1}(a,w)| dv(w) \delta(a) < \infty \right\},$$

$1 \leq p < \infty$

These are Banach spaces.

There is an embedding between these two spaces in bounded strongly pseudoconvex domains with smooth boundary(see [16],[17]) and hence dist problem can be posed again. Let $\alpha \geq \frac{q+2}{p}$ then we have $F(p,q) \subset B_\alpha(D)$; $1 \leq p < \infty$, $q \in (0,\infty)$. The short proof of this fact follows immediately from properties of r-lattices we provided in pseudoconvex domains and the estimate from below of the Bergman kernel on the Kobayashi ball (see remark before lemma 1.9) and [18] and the plurisubharmonicity of modulus of analytic function (see lemma 1.9) and this proof repeats the well-known classical short proof of the unit disk case (see, for example, [12] and [13] for such a proof).

Definition 4. *We say that positive Borel measure μ in D is a Carleson measure in a bounded strictly pseudoconvex domain with smooth boundary if*

$$\sup_{w \in D} \left[\delta(w) \int_\Omega |K_{n+1}(w,z)| d\mu(z) \right] < \infty.$$

This definition is a direct generalization of the Carleson measure in the unit ball.

Theorem 15. . *Let $1 < p < \infty$,and let $D_{\alpha,\varepsilon} = \{z \in D : | \bigtriangledown f(z)| \delta(z)^\alpha > \varepsilon\}$. Let also $\alpha \geq \frac{q+2}{p}$; $q_1 < q < q_0$, $q_0 = q_0(\alpha,p)$; $q_1 = q_1(\alpha,p)$. Then let $f \in B_\alpha(D)$. Then*

$$dist_{B_\alpha}(f, F(p,q)) \asymp$$

$$\inf \left\{ \varepsilon > 0 : \left[\lambda_{D_{\alpha,\varepsilon}}(z) \right] K_{p\alpha-q-1}(z,z)] dv(z) \textit{ is a Carleson measure } \right\}.$$

The Proof of this theorem is simply repetition of arguments we provided above in the unit disk then in the unit ball cases and remarks we made there and lemmas we provided in this setting of general pseudoconvex domains. This result is the first sharp result on distances in bounded strongly pseudoconvex domains with smooth boundary in BMOA type spaces. Note that same type sharp distance theorems, but under additonal conditions on Bergman kernel can be provided even in other domains in higher dimension, for example in tubular domains, in weakly pseudoconvex domains, in Siegel domains, in bounded symmetric domains , where complete analogues of our lemmas used in proofs above can be seen. For example in tubular domains over symmetric cones such lemmas can be seen in [27] and also in [26]. Note also following the proof it can be easily seen that mentioned additional condition on Kernel is the lemma which was provided by us after theorem 1.These results will be discussed in our other papers.

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