Acta Universitatis Apulensis No. 48/2016
ISSN: 1582-5329 pp. 27-35
http://www.uab.ro/auajournal/ doi: 10.17114/j.aua.2016.48.03

A CONVOLUTION TYPE INEQUALITY FOR PSEUDO-INTEGRALS

B. Daraby

Abstract. Pseudo-analysis uses for the generalization of the classical analysis, where instead of the field of the numbers a semiring is defined on a real interval $[a,b] \subset [-\infty,\infty]$ with pseudo-addition $\oplus$ and with pseudo-multiplication $\odot$. Thus it would be an interesting topic to generalize an inequality from the from work of the classical analysis as special cases. In this paper we prove generalizations of the Bushell-Okrasinski's type inequality for pseudo-integrals.

2010 *Mathematics Subject Classification*: 03E72, 28E10, 26E50.

Keywords: inequality, Convolution type inequality, Pseudo-addition, Pseudo-multiplication; Pseudo-integral.

1. Introduction

Not long ago, H. Román-Flores et al. analyzed an interesting type of geometric inequalities for the Sugeno integrals with some applications to convex geometry in [12]. More precisely, a Prékopa-Leindler type inequality for fuzzy integrals was proven, and subsequently used for the characterization of some convexity properties of fuzzy measures.

In this paper, we prove Bushell-Okrasiaski inequality at two decreasing and increasing cases for two classes of pseudo-integrals. One of them, classes with pseudo-integrals where pseudo-operations are defined via a monotone and continuous generator function. The other one concerns the pseudo-integrals based on a semiring with an idempotent addition and a pseudo-multiplication generator.

The classical Bushell-Okrasinski [4] is a convolution type inequality. More precisely,

$$\int_0^x (x-t)^{s-1} g(t)^s dt \leq \Big(\int_0^x g(t) \Big)^s, \qquad 0 \leq x \leq b, \tag{1}$$

holds for a continuous and increasing function $g : [0,1] \to [0,\infty)$ and $s \geq 1$, $b \leq 1$. This inequality was used by Bushell and Okrasinski [4] in the study of solutions of

Volterra integral equations (also see [7]). Later on Walter and Weckesser [16] study some extensions of (1) and finally, after the change of variable $t = xs$, Malamud [6] analyze the B-O inequality (1) in the following new form:

$$s\int_0^1 (1-t)^{s-1} g(t)^s dt \leq \Big(\int_0^x g(t)\Big)^s.$$

H. Román-Flores et al [11] proved Bushell-Okrasinski type inequality for the Sugeno integrals at two cases in the following way:

Theorem 1. *(Fuzzy B-O inequality: decreasing case). Let $g : [0,1] \to [0,\infty)$ be a continuous and decreasing function. Then*

$$s ⨍_0^1 (1-t)^{s-1} g(t)^s dt \geq \Big(⨍_0^1 g(t)dt\Big)^s,$$

holds for all $s \geq 2$.

The following theorem establish an analogous result for the increasing case.

Theorem 2. *(Fuzzy B-O inequality: increasing case). Let $g : [0,1] \to [0,\infty)$ be a continuous and increasing function. Then*

$$s ⨍_0^1 t^{s-1} g(t)^s dt \geq \Big(⨍_0^1 g(t)dt\Big)^s,$$

holds for all $s \geq 2$.

2. Preliminaries

2.1. Pseudo-integrals

Let $[a,b]$ be a closed (in some cases can be considered semiclosed) subinterval of $[-\infty,\infty]$. The full order on $[a,b]$ will be denoted by $\preceq$. A binary operation $\oplus$ on $[a,b]$ is pseudo-addition if it is commutative, non-decreasing(with respect to $\preceq$), associative and with a zero (neutral) element denoted by $\mathbf{0}$. Let $[a,b]_+ = \{x | x \in [a,b], \mathbf{0} \preceq x\}$. A binary operation $\odot$ on $[a,b]$ is Pseudo-multiplication if it is commutative, positively non-decreasing, i.e., $x \preceq y$ implies $x \odot z \preceq y \odot z$ for all $z \in [a,b]_+$, associative and with a unit element $1 \in [a,b]$, i.e., for each $x \in [a,b]$, $1 \odot x = x$. We assume also $\mathbf{0} \odot x = \mathbf{0}$ and that $\odot$ is distributive over $\oplus$, i.e.,

$$x \odot (y \oplus z) = (x \odot y) \oplus (x \odot z).$$

The structure $([a, b], \oplus, \odot)$ is a semiring ([1, 2, 3, 5, 9, 15]). In this paper we will consider semirings with following continuous operations:
Case I. The pseudo-addition is idempotent operation and the pseudo-multiplication is not.

- (a) $x \oplus y = \sup(x, y)$, $\odot$ is arbitrary not idempotent pseudo-multiplication on the interval $[a, b]$. We have $\mathbf{0} = a$ and the idempotent operation sup induces a full order in the following way: $x \preceq y$ if and only if $\sup(x, y) = y$.
- (b) $x \oplus y = \inf(x, y)$, $\odot$ is arbitrary not idempotent pseudo-multiplication on the interval $[a, b]$. We have $\mathbf{0} = b$ and the the idempotent operation inf induces a full order in the following way: $x \preceq y$ if and only if $\inf(x, y) = y$.

Case II. The pseudo-operations are defined by a monotone and continuous function $g : [a, b] \to [0, \infty]$ (additive generator of $\oplus$), i.e., pseudo-operations are given with

$$x \oplus y = g^{-1}(g(x) + g(y)) \qquad and \qquad x \odot y = g^{-1}(g(x).g(y)).$$

If the zero element for the pseudo-addition is a, we will consider increasing generators.Then $g(a) = 0$ and $g(b) = \infty$. If the zero element for the pseudo-addition is b, we will consider decreasing generators. Then $g(b) = 0$ and $g(a) = \infty$.
If the generator g is increasing (respectively decreasing), the operation $\oplus$ induce the usual order (respectively opposite to the usual order) on the interval $[a, b]$ in the following way: $x \preceq y$ if and only if $g(x) \preceq g(y)$.
Case III. Both operation are idempotent. We have

- (a) $x \oplus y = \sup(x, y)$, $x \odot y = \inf(x, y)$, on the interval $[a, b]$. We have $\mathbf{0} = a$ and $\mathbf{1} = b$. The idempotent operation *sup* induces a usual order ($x \prec y$ if and only if $\sup(x, y) = y$).
- (b) $x \oplus y = \inf(x, y)$, $x \odot y = \sup(x, y)$, on the interval $[a, b]$. We have $\mathbf{0} = b$ and $\mathbf{1} = a$. The idempotent operation inf induces an order opposite to the usual order($x \preceq y$ if and only if $\inf(x, y) = y$).

2.2. Explicit forms of special Pseudo-integrals

We shall consider the semiring $([a, b], \oplus, \odot)$ for three (with completely different behaviour) cases, namely I(a), II, and III(a). Observe that the cases I(b) and III(b) are linked to the cases I(a) and III(a) by duality. First case is when pseudo-operations are generated by a monotone and continuous function $g : [a, b] \to [0, \infty]$, case then the pseudo-integral for a measurable function $f : X \to [a, b]$ is given by,

$$\int_X^{\oplus} f \odot dm = g^{-1}\left(\int_X (gof) d(gom)\right), \tag{2}$$

Where the integral applied on the right side is the standard Lebesgue integral. In spacial case, when $X = [c, d]$, $\mathcal{A} = \mathcal{B}(X)$ and $m = g^{-1}o\lambda$, λ the standard Lebesgue measure on $[c, d]$, then we use notation

$$\int_{[c,d]}^{\oplus} f(x)dx = \int_{X}^{\oplus} f \odot dm.$$

By(2),

$$\int_{[c,d]}^{\oplus} f(x)dx = g^{-1}\big(\int_{c}^{d} g(f(x))dx\big),$$

i.e., we have recovered the g-integral, (see[8, 9]).
Second case is when the semiring is of the form $([a, b], \sup, \odot)$, case I(a) and III(a). We will consider complete sup-measure m only and $\mathcal{A} = 2^x$, i.e., for any system $(A_i)_i \in I$ of measurable sets,

$$m(\underset{i\in I}{\cup} A_i) = \sup_{i\in I} m(A_i)$$

Recall that if X is countable (especially, if X is finite) then any σ-sup-measure m is complete and, moreover, $m(A) = \sup_{x\in A} \psi(X)$, where $\psi : X \to [a, b]$ is a density function given by $\psi(x) = m(\{x\})$. Then the pseudo-integral for a function $f : X \to [a, b]$ is given by

$$\int_{X}^{\oplus} f \odot dm = \sup_{x\in X}(f(x) \odot \psi(x)),$$

where function ψ defines sup-measure m.

Theorem 3. *Let m be a* sup-*measure on* $([0, \infty], \mathfrak{B}([0, \infty]))$*, where* $\mathfrak{B}([0, \infty])$ *is the Borel* σ*-algebra on* $[0, \infty]$, $m(A) = esssup_{\mu}(\psi(x)|x \in A)$*, where* $\psi : [0, \infty] \to [0, \infty]$ *is a continuous density. Then for any pseudo-addition* $\oplus$ *with a generator* g *there exists a family* $\{m_\lambda\}$ *of* $\oplus_\lambda$*-measure on* $([0, \infty[, \mathfrak{B})$*, where* $\oplus_\lambda$ *is generated by* g^λ *(the function* g *of the power* λ*),* $\lambda \in]0, \infty[$*, such that* $\lim_{\lambda\to\infty} m_\lambda = m$.

For any continuous function $f : [0, \infty] \to [0, \infty]$ the integral $\int^{\oplus} f \odot dm$ can be obtained as a limit of g-integrals, [3, 10].

Theorem 4. *Let* $([0, \infty], \sup, \odot)$ *be a semiring with* $\odot$ *generated by some increasing generator* g*, i.e., we have* $x \odot y = g^{-1}(g(x)g(y))$ *for every* $x, y \in [a, b]$*. Let* m *be the same as in Theorem 3. Then there exists a family* $\{m_\lambda\}$ *of* $\oplus_\lambda$*-measure, where* $\oplus_\lambda$ *is generated by* g^λ, $\lambda \in]0, \infty[$*, such that for every continuous function* $f : [0, \infty] \to [0, \infty]$

$$\int^{\sup} f \odot dm = \lim_{\lambda\to\infty} \int^{\oplus_\lambda} f \odot dm_\lambda = \lim_{\lambda\to\infty} (g^\lambda)^{-1}\big(\int g^\lambda(f(x))dx\big).$$

Now we recall generalization of the Jensen inequality for pseudo-integral that proved by E. Pap et al. on [**?**].

Theorem 5. *Let $\Phi : [a,b] \to [a,b]$ be a convex and nondecreasing function. If a generator $g : [a,b] \to [a,b]$ of the pseudo-addition $\oplus$ and the pseudo-multiplication $\odot$ is a convex and increasing function, then for any measurable function $f : [0,1] \to [a,b]$ we have*

$$\Phi\big(\int_{[0,1]}^{\oplus} f(x)dx\big) \leq \int_{[0,1]}^{\oplus} \Phi(f(x))dx.$$

Theorem 6. *Let $\Phi : [a,b] \to [a,b]$ be a convex and nondecreasing function, and the pseudo-multiplication $\odot$ is represented by a convex and increasing generator g. Let m be the same as in Theorem 3. Then for any continuous function $f : [0,1] \to [a,b]$ we have*

$$\Phi\big(\int_{[0,1]}^{\sup} f \odot dm\big) \leq \int_{[0,1]}^{\sup} \Phi(f) \odot dm.$$

The following Theorem shows that the Chebyshev's inequality for pseudo-integrals that is proved in [1].

Theorem 7. *Let $u, v : [0,1] \to [a,b]$ be two measurable functions and let a generator $g : [a,b] \to [0,\infty)$ of the pseudo-addition $\oplus$ and the pseudo-multiplication $\odot$ be an increasing function. If u and v are comonotone functions, then the inequality*

$$\int_{[0,1]}^{\oplus} (u \odot v)dx \geq \big(\int_{[0,1]}^{\oplus} udx\big) \odot \big(\int_{[0,1]}^{\oplus} vdx\big),$$

hold and the reserve inequality holds whenever u and v are countermonotone functions.

3. Main results

In this section, we prove two Bushell-Okrasiaski inequalities for pseudo-integrals.

Theorem 8. *(Pseudo Bushell-Okrasiaski inequality: decreasing case) Let $f : [0,1] \to]a,b[$ be a continuous and decreasing function. If a generator $g :]a,b[\to]a,b[$ of the pseudo-addition $\oplus$ and the pseudo-multiplication $\odot$ is a convex and increasing function, then*

$$\int_{[0,1]}^{\oplus} (1-t)^{s-1} \odot f^s(t)dt \geq \frac{1}{s} \odot \big(\int_{[0,1]}^{\oplus} f(t)dt\big)^s,$$

holds for all $s \geq 2$.

Proof. By the definition of pseudo-integral and pseudo-operations we have

$$\begin{aligned}\int_{[0,1]}^{\oplus}(1-t)^{s-1}\odot f^s(t)dt &= g^{-1}\big(\int_0^1 g\big[(1-t)^{s-1}\odot f^s(t)\big]dt\big)\\ &= g^{-1}\big(\int_0^1 g\big[g^{-1}(g((1-t)^{s-1})g(f^s(t))\big]dt\big)\\ &= g^{-1}\big(\int_0^1 g((1-t)^{s-1})g(f^s(t))dt\big).\end{aligned}$$

By classic Chebyshev's integral inequality ([14]), we have;

$$\begin{aligned}g^{-1}\big(\int_0^1 g((1-t)^{s-1})g(f^s(t))dt\big) &\geq g^{-1}\big[\big(\int_0^1 g((1-t)^{s-1})dt\big(\int_0^1 g(f^s(t))dt\big)\big]\\ &= g^{-1}\big[gg^{-1}\big(\int_0^1 g((1-t)^{s-1})dt\big)gg^{-1}\big(\int_0^1 g(f^s(t))dt\big)\big]\\ &= g^{-1}\big[g\big(\int_{[0,1]}^{\oplus}(1-t)^{s-1}dt\big)g\big(\int_{[0,1]}^{\oplus}f^s(t)dt\big)\big]\\ &= \big(\int_{[0,1]}^{\oplus}(1-t)^{s-1}dt\big)\odot\big(\int_{[0,1]}^{\oplus}f^s(t)dt\big).\end{aligned}$$

By using the Theorem 5,

$$\int_{[0,1]}^{\oplus}(1-t)^{s-1}\odot f^s(t)dt \geq \big(\int_{[0,1]}^{\oplus}(1-t)^{s-1}dt\big)\odot\big(\int_{[0,1]}^{\oplus}f(t)dt\big)^s, \tag{3}$$

in the other hand by using the classic Jensen inequality ([13]), we can show that

$$\begin{aligned}\int_{[0,1]}^{\oplus}(1-t)^{s-1}dt &= g^{-1}\big(\int_0^1 g((1-t)^{s-1})dt\big)\\ &\geq g^{-1}\big(g\int_0^1 (1-t)^{s-1}dt\big) \qquad = \int_0^1 (1-t)^{s-1}dt = \frac{1}{s},\end{aligned}$$

so by (3) and (4) we obtain that:

$$\int_{[0,1]}^{\oplus}(1-t)^{s-1}\odot f^s(t)dt \geq \frac{1}{s}\odot\big(\int_{[0,1]}^{\oplus}f(t)dt\big)^s.$$

Thereby, the theorem is proved.

Example 1. *Let $g(x)=e^x$. The corresponding pseudo-operations are $x\oplus y = \ln(e^x+e^y)$ and $x\odot y = x+y$, the Theorem 8 reduces on the following inequality,*

$$\ln\big(\int_0^1 e^{(1-t)^{s-1}+f^s(t)}dt\big) \geq \frac{1}{s}+\big(\ln\big(\int_0^1 e^{f(t)}dt\big)\big)^s.$$

Theorem 9. *(Pseudo Bushell-Okrasiaski inequality: increasing case) Let $f : [0,1] \to]a,b[$ be a continuous and increasing function. If a generator $g :]a,b[\to]a,b[$ of the pseudo-addition $\oplus$ and the pseudo-multiplication $\odot$ is a convex and increasing function, then*

$$\int_{[0,1]}^{\oplus} t^{s-1} \odot f^s(t)dt \geq \frac{1}{s} \odot \big(\int_{[0,1]}^{\oplus} f(t)dt \big)^s,$$

holds for all $s \geq 2$.

Proof. The proof is similar to Theorem 8.

Theorem 10. *(Pseudo Bushell-Okrasiaski inequality: decreasing case) Let $f : [0,1] \to]a,b[$ be a continuous and decreasing function, and $\odot$ is represented by a convex and increasing multiplication generator g and m be the same as in Theorem 3, then*

$$\int_{[0,1]}^{\sup} (1-t)^{s-1} \odot f^s(t) \odot dm \geq \frac{1}{s} \odot \big(\int_{[0,1]}^{\sup} f(t)dt \big)^s,$$

holds for all $s \geq 2$.

Proof. By Theorem 4 we have:

$$\begin{aligned}\int_{[0,1]}^{\sup} (1-t)^{s-1} \odot f^s(t) \odot dm &= \lim_{\lambda\to\infty} \int_{[0,1]}^{\oplus_\lambda} (1-t)^{s-1} \odot f^s(t) \odot dm_\lambda \\ &= \lim_{\lambda\to\infty} (g^\lambda)^{-1} \big(\int_0^1 g^\lambda((1-t)^{s-1} \odot f^s(t))dt \big).\end{aligned}$$

Using the Theorem 7 so we have

$$\begin{aligned}\int_{[0,1]}^{\sup} (1-t)^{s-1} \odot f^s(t) \odot dm &\geq \lim_{\lambda\to\infty} [(g^\lambda)^{-1}(\int_0^1 g^\lambda((1-t)^{s-1})dt) \odot (g^\lambda)^{-1}(\int_0^1 g^\lambda(f^s(t))dt)] \\ &= [\lim_{\lambda\to\infty} (g^\lambda)^{-1} \int_0^1 g^\lambda((1-t)^{s-1})dt) \odot (\lim_{\lambda\to\infty} (g^\lambda)^{-1} \int_0^1 g^\lambda((f^s(t))dt] \\ &= (\int_{[0,1]}^{\sup} (1-t)^{s-1} \odot dm) \odot (\int_{[0,1]}^{\sup} f^s(t) \odot dm).\end{aligned}$$

Applying the Theorem 6, we obtain that:

$$\int_{[0,1]}^{\sup} (1-t)^{s-1} \odot f^s(t) \odot dm \geq \big(\int_{[0,1]}^{\sup} (1-t)^{s-1} \odot dm \big) \odot \big(\int_{[0,1]}^{\sup} f(t) \odot dm \big)^s. \quad (4)$$

Also we have:

$$\begin{aligned}\int_{[0,1]}^{\sup}(1-t)^{s-1}\odot dm &= \lim_{\lambda\to\infty}\Big(\int_{[0,1]}^{\oplus_\lambda}(1-t)^{s-1}\odot dm_\lambda\Big)\\ &= \lim_{\lambda\to\infty}(g^\lambda)^{-1}\Big(\int_0^1 g^\lambda((1-t)^{s-1})dt\Big)\\ &\geq \lim_{\lambda\to\infty}(g^\lambda)^{-1}\Big(g^\lambda\int_0^1((1-t)^{s-1})dt\Big)\\ &= \lim_{\lambda\to\infty}\int_0^1((1-t)^{s-1})dt = \frac{1}{s} \qquad (5)\end{aligned}$$

from (5) and (6) we have;

$$\int_{[0,1]}^{\sup}(1-t)^{s-1}\odot f^s(t)\odot dm \geq \frac{1}{s}\odot\Big(\int_{[0,1]}^{\sup}f(t)dt\Big)^s.$$

Example 2. *Let $g^\lambda = e^{\lambda x}$ and $\psi(x)$ be from Theorem 3, then*

$$x\odot_\lambda y = x+y \qquad and \qquad \lim_{\lambda\to\infty}\Big(\frac{1}{\lambda}\ln(e^{\lambda x}+e^{\lambda y})\Big) = max(x,y).$$

Therefore B-O type inequality from Theorem 10 reduces on

$$\sup_{x\in[0,1]}\big[\big((1-x)^{s-1}+f^s(x)\big)+\psi(x)\big] \geq \frac{1}{s}+\big[\sup_{x\in[0,1]}\big(f(x)+\psi(x)\big)\big]^s.$$

Theorem 11. *(Pseudo Bushell-Okrasiaski inequality: increasing case) Let $f:[0,1]\to\,]a,b[$ be a continuous and increasing function, and $\odot$ is represented by a convex and increasing multiplication generator g and m be the same as in Theorem 3, then*

$$\int_{[0,1]}^{\sup}t^{s-1}\odot f^s(t)\odot dm \geq \frac{1}{s}\odot\Big(\int_{[0,1]}^{\sup}f(t)dt\Big)^s,$$

holds for all $s\geq 2$.

Proof. The proof is similar to Theorem 10.

Note that third important case $\oplus$ =max and $\odot$ =min has been studied in [11] and the Pseudo-integrals in such a case yields the Sugeno integral.

References

[1] B. Daraby, *Genralization of the Stolaresky type inequality for pseudo-integrals*, Fuzzy Sets and Systems, 194 (2012) 90-96.

[2] B. Daraby, A. Shafiloo, A. Rahimi, *Generalization of the Feng Qi type inequality for pesudo-integral*, Gazi University Journal of Science 28 (4) (2015), 695-702.

[3] B. Daraby, F. Rostampour, A. Rahimi, *Hardy's type inequality for pesudo-integrals*, Acta Universitatis Apulensis, 42 (2015), 53-65.

[4] P. J. Bushell, W.Okrasinski, *Nonlinear volterra integral equations with convolution kernel*, J. London Math. Soc. 41 (2) (1990) 503-510.

[5] W. Kuich, Semiring, Automata, *Languages*, Springer-verlag, Berlin, 1986.

[6] S. M. Malamud, *Som complements to the Jensen and Chebyshev inequalities and a problem of W. Walter*, pro. Am. Math. Soc. 129 (2001) 2671-2678.

[7] M. Meehan, D. O'Regan, *positive solutions of voltera integral equations using integral inequalities*, J. Inequal. Appl. 7 (2) (2002) 285-307.

[8] E. Pap, *An integral generated by decomposable measure*, Univ, u Novom sadu zb. Rad. prirod. Mat. Fak. Ser. Mat, 20 (1) (1990) 135-144.

[9] E. Pap, *g-calculus*, u Novom sadu zb. Rad. prirod. Mat. Fak. Ser. Mat. 23 (1) (1993) 145-156.

[10] E. Pap, M. Š trboja, *Generalizationof the Jensen inequality for Pseudo-integral*, Inform. Sci. 180 (2010) 543-548.

[11] H. Román-Flores, A. Flores-Franulič, Y. Chalco-Cano, *A convolution tyoe inequality for fuzzy integrals*, Applied Mathematics and Computation, 195 (2008) 94-99.

[12] H. Román-Flores, A. Flores-Franulič, Y. Chalco-Cano, *Sugeno integral and geometric inequalities*, International journal of Uncertainly Fuzziness and knowledge-baced Systems 15 (2007) 1-11.

[13] H. L Royden, *Real Analysis*, Macmillan, New York, 1988.

[14] Y. L. Tong, *Relationship between stochastic inequalities and some classical mathematical inequalities*, J. Inequal. Appl. 1 (1997) 85-98.

[15] Y. Ouyang, R. Mesiar, *On the Chebyshev type inequality for seminormed fuzzy integral*, Applied Mathematical Letter 22 (2009) 1810-1815.

[16] W. Walter, V. Weckesser, *An integral inequality of convolution type*, Aequationes Math. 46 (1993) 200-202.

Bayaz Daraby
Department of Mathematics, University of Maragheh,
Maragheh, Iran
email: *bdaraby@maragheh.ac.ir*