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ON CERTAIN PROPERTIES OF MEROMORPHIC MULTIVALENT FUNCTIONS DEFINED BY A GENERALIZED DIFFERENTIAL OPERATOR

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Abstract. Inspired by the work of Srivastava and Patel [Applications of differential subordination to certain subclasses of meromorphically multivalent functions, J. Inequal. Pure Appl. Math. 6 (2005), no. 3, Article 88, 15 pp.], in the present manuscript, using the principle of differential subordination, certain interesting results such as subordination properties, coefficient estimates, radius constants and inclusion relationship are discussed. Also the relevant connections of the various results presented in this manuscript with those obtained in earlier works have been pointed out.

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1. Introduction

Let $\Sigma_{m,p}$ denote the set of all analytic and $p-$valent functions of the form

$$f(z) = z^{-p} + \sum_{k=m}^{\infty} a_k z^k \quad (m > -p, p \in \mathbb{N}) \tag{1}$$

in the punctured unit disk $\mathbb{U}^* := \{z : z \in \mathbb{C} \text{ and } 0 < |z| < 1\}$. We define the Hadamard product of functions f, given by (1), and g given by

$$g(z) = z^{-p} + \sum_{k=m}^{\infty} b_k z^k,$$

by

$$(f * g)(z) = z^{-p} + \sum_{k=m}^{\infty} a_k b_k z^k.$$

For a function $f \in \Sigma_{m,p}$, given by (1) and ϕ defined by

$$\phi(z) = z^{-p} + \sum_{k=m}^{\infty} \left(\frac{1}{k+p+1}\right)^{\alpha} z^k \qquad (\alpha \in \mathbb{R}),$$

we define the differential operator $T_{m,p}^{\alpha,n}(\lambda,\mu)$ by the following way:

$$T_{m,p}^{\alpha,0}(\lambda,\mu) = f(z) * \phi(z),$$

$$T_{m,p}^{\alpha,1}(\lambda,\mu) = \lambda\mu\frac{(z^{p+1}f(z)*\phi(z))''}{z^{p-1}} + (\lambda-\mu)\frac{(z^{p+1}f(z)*\phi(z))'}{z^p} + (1-\lambda+\mu)(f(z)*\phi(z)) \quad (2)$$

and in general

$$T_{m,p}^{\alpha,n}(\lambda,\mu)f(z) = T_{m,p}^{\alpha,1}(\lambda,\mu)(T_{m,p}^{\alpha,n-1}(\lambda,\mu)f(z)), \quad (3)$$

where $0 \leq \mu \leq \lambda$ and $m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$. From (2) and (3), it can be verified that

$$T_{m,p}^{\alpha,n}(\lambda,\mu)f(z) = z^{-p} + \sum_{k=m}^{\infty} \Phi_k(\lambda,\mu,n,\alpha,p)a_k z^k, \quad (4)$$

where

$$\Phi_k(\lambda,\mu,n,\alpha,p) = \frac{[1+(k+p)(\lambda-\mu+\lambda\mu(k+p+1))]^n}{[k+p+1]^{\alpha}}. \quad (5)$$

From (4) it is clear that the operator $T_{m,p}^{\alpha,n}(\lambda,\mu)$, by means of convolution, can be written as

$$T_{m,p}^{\alpha,n}(\lambda,\mu)f(z) = (f*h)(z),$$

where

$$h(z) = z^{-p} + \sum_{k=m}^{\infty} \Phi_k(\lambda,\mu,n,\alpha,p)z^k. \quad (6)$$

1. For $\alpha = 0$, the operator $T_{m,p}^{\alpha,n}(\lambda,\mu)$ defined (4) is the operator $D_{\lambda\mu p}^n$ defined by Orhan *et al.*[9].

2. The operator $T_{m,p}^{\alpha,0}(\lambda,\mu)$ is the one parameter family of integral operator

$$\begin{aligned} P_p^{\alpha}(z) &= \frac{1}{z^{p+1}\Gamma\alpha}\int_0^z \left(log\frac{z}{t}\right)^{\alpha-1} t^{\alpha-1}f(t)dt \\ &= z^{-p} + \sum_{k=m}^{\infty}\left(\frac{1}{k+p+1}\right)^{\alpha} z^k \end{aligned}$$

   introduced by Aqlan [3] (see also [2]).

3. The differential operator $T^{0,n}_{m,p}(1,0) = D^n$ was introduced and studied by Srivatava and Patel [10].

4. For $m = 0, \lambda = 1, \mu = 0$ and $\alpha = 0$ the operator $T^{0,n}_{0,p}(1,0)$ was introduce by Liu and Srivastava [4].

5. For $p = 1, m = 0, \alpha = 0, \lambda = 1, \mu = 0$ the operator $T^{0,n}_{0,1}(1,0)$ was considered earlier by Uralegaddi and Somamatha [11].

6. A special case of the operator $T^{\alpha,n}_{m,p}(\lambda,\mu)$, when $m = 0, \alpha = 0, \lambda = 1, \mu = 0, p \in \mathbb{N}$ was considered by Aouf and Hossen [1].

The recent paper of Mostafa and Aouf [8] and that of Noor and Riaz [**?**] are also a useful reference in this direction.

From the definition of $T^{\alpha,n}_{m,p}(\lambda,\mu)f(z)$ it can be easily verified that

$$\lambda z(T^{\alpha,n}_{m,p}(\lambda,\mu)f(z))' = T^{\alpha-1,n}_{m,p}(\lambda,\mu)f(z) - (p+1)T^{\alpha,n}_{m,p}(\lambda,\mu)f(z) \tag{7}$$

$$\mu z(T^{\alpha,n}_{m,p}(0,\mu)f(z))' = T^{\alpha,n+1}_{m,p}(0,\mu)f(z) \tag{8}$$

$$\lambda z(T^{\alpha,n}_{m,p}(\lambda,0)f(z))' = T^{\alpha,n+1}_{m,p}(\lambda,0)f(z) - (\lambda p+1)T^{\alpha,n}_{m,p}(\lambda,0)f(z) \tag{9}$$

Definition 1.1. Let $\Sigma^{\alpha,\lambda,\mu,n}_{m,p}(A,B)$ be the class of functions $f \in \Sigma_{p,m}$ which satisfy the following differential subordination:

$$-\frac{z^{p+1}(T^{\alpha,n}_{m,p}(\lambda,\mu)f(z))'}{p} \prec \frac{1+Az}{1+Bz} \quad (-1 \le B < A \le 1, z \in \mathbb{U}) \tag{10}$$

In view of the differential subordination (10) can be written as

$$\left|\frac{z^{p+1}(T^{\alpha,n}_{m,p}(\lambda,\mu)f(z))' + p}{B(T^{\alpha,n}_{m,p}(\lambda,\mu)f(z))' + pA}\right| < 1 \quad (z \in \mathbb{U}). \tag{11}$$

For convenience, let $\Sigma^{\alpha,\lambda,\mu,n}_{m,p}(1-2\eta/p,-1) =: \Sigma^{\alpha,\lambda,\mu,n}_{m,p}(\eta)$ $(0 \le \eta < p)$.

Definition 1.2. Let $\Sigma^{\alpha,\lambda,\mu,n}_{m,p}(\eta)$ be the class of functions $f \in \Sigma_{p,m}$ satisfying

$$-\operatorname{Re}(T^{\alpha,n}_{m,p}(\lambda,\mu)f(z)) > \eta.$$

Let $\Sigma_{0,p} =: \Sigma_p$ and $\Sigma^{\alpha,\lambda,\mu,n}_{0,p}(A,B) =: \Sigma^{\alpha,\lambda,\mu,n}_{p}(A,B)$.

The following existing results: Let $P(\gamma)$ be the class of functions $p(z)$ of the form $p(z) = 1 + b_1 z + b_2 z^2 + b_3 z^3 + \cdots$, which are analytic in $\mathbb{U}$ and satisfy $\operatorname{Re}(p(z)) > \gamma, (0 \le \gamma < 1, z \in \mathbb{U})$.

Lemma 1.3. [6, 7] Let the function $h(z)$ be analytic and convex (univalent) in $\mathbb{U}$ with $h(0) = 1$. Suppose also that the function p given by

$$p(z) = 1 + c_{p+m}z^{p+m} + c_{p+m+1}z^{p+m+1} + \cdots \tag{12}$$

is analytic in $\mathbb{U}$. If

$$p(z) + \frac{zp'(z)}{\gamma} \prec h(z) \quad (\mathrm{Re}(\gamma) \geq 0, \gamma \neq 0, z \in \mathbb{U}), \tag{13}$$

then

$$p(z) \prec q(z) = \frac{\gamma}{p+m} z^{-\frac{\gamma}{p+m}} \int_0^z t^{-\frac{\gamma}{p+m}} h(t)dt$$

and $q(z)$ is the best dominant of (13).

For real or complex number a, b and c ($c \neq 0, -1, -2, -3, \cdots$), the Gauss hypergeometric function [12] is defined by

$${}_2F_1(a, b; c; z) = 1 + \frac{ab}{c}\frac{z}{1!} + \frac{a(a+1)b(b+1)}{c(c+1)}\frac{z^2}{2!} + \cdots .$$

Lemma 1.4. [12] For real or complex parameters a, b and c ($c \neq 0, -1, -2, -3, \cdots$),

$$\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}dt = \frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)}\ {}_2F_1(a, b; c; z) \quad (\mathrm{Re}(c) > \mathrm{Re}(b) > 0); \tag{14}$$

$${}_2F_1(a, b; c; z) = (1-z)^{-a}\ {}_2F_1(a, c-b; c; \frac{z}{1-z}) \tag{15}$$

$$(b+1)\ {}_2F_1(1, b; b+1; z) = (b+1) + bz\ {}_2F_1(1, b+1; b+2; z) \tag{16}$$

and

$${}_2F_1(a, b; c; z) = {}_2F_1(b, a; c; z). \tag{17}$$

2. Main Results

Note that throughout this section, unless otherwise mentioned specifically, the parameters A, B, P, λ, δ and γ are constrained as follows:

$$-1 \leq B < A \leq 1, n, p \in \mathbb{N}, \alpha \in \mathbb{R}, 0 \leq \lambda \leq 1, 0 \leq \mu \leq \lambda, \leq \delta \leq 1 \text{ and } 0 \leq \eta < p.$$

Theorem 2.1. Let $\gamma = \frac{1+p(1-\lambda)}{\delta\lambda}$ and the function f given by (1) satisfy the following differential subordination

$$-\frac{z^{p+1}((1-\delta)(1+p(1-\lambda))[T_{m,p}^{\alpha,n}(\lambda,\mu)f(z)]' + \delta[T_{m,p}^{\alpha-1,n}(\lambda,\mu)f(z)]')}{p} \prec \frac{1+Az}{1+Bz},$$

then

$$-\frac{z^{p+1}(T_{m,p}^{\alpha,n}(\lambda,\mu)f(z))'}{p} \prec Q^*(z) \prec \frac{1+Az}{1+Bz}, \tag{18}$$

where

$$Q^*(z) = \begin{cases} \frac{A}{B} + (1-\frac{A}{B})(1+Bz)^{-1}\ {}_2F_1(1,1;\frac{\gamma}{p+m}+1;\frac{Bz}{1+Bz}), & B \neq 0; \\ 1 + \left(\frac{\gamma}{p+m+\gamma}\right) Az, & \text{B=0} \end{cases}$$

is the best dominant of (18). Furthermore

$$-\operatorname{Re}\left(\frac{z^{p+1}(T_{m,p}^{\alpha,n}(\lambda,\mu)f(z))'}{p}\right) > \rho, \tag{19}$$

where

$$\rho = \begin{cases} \frac{A}{B} + (1-\frac{A}{B})(1-B)^{-1}\ {}_2F_1(1,1;\frac{\gamma}{p+m}+1;\frac{B}{B-1}), & B \neq 0; \\ 1 - \frac{\gamma A}{p+m+\gamma}, & \text{B=0.} \end{cases}$$

The inequality in (19) is the best possible.

Proof. Define the function φ by

$$\varphi(z) = -\frac{z^{p+1}(T_{m,p}^{\alpha,n}(\lambda,\mu)f(z))'}{p}. \tag{20}$$

Then it can be easily seen that $\varphi(z)$ is of the form (12) and is analytic in $\mathbb{U}$. Using the identity (7) in (20) and differentiating, we obtain

$$-\frac{1}{1+p(1-\lambda)}\frac{z^{p+1}}{p}(T_{m,p}^{\alpha-1,n}(\lambda,\mu)f(z))' = \varphi(z) + \frac{\lambda}{1+p(1-\lambda)}z\varphi'(z). \tag{21}$$

From (20) and (21), we have

$$-\frac{z^{p+1}((1-\delta)(1+p(1-\lambda))[T_{m,p}^{\alpha,n}(\lambda,\mu)f(z)]' + \delta[T_{m,p}^{\alpha-1,n}(\lambda,\mu)f(z)]')}{p}$$
$$= \varphi(z) + \frac{\delta\lambda z\varphi'(z)}{1+p(1-\lambda)} \prec \frac{1+Az}{1+Bz}$$

Using Lemma 1.3 for $\gamma = (1+p(1-\lambda))/(\delta\lambda)$ accompanied by change of variables and using identities (14), (15), (16) and (17), we deduce that

$$
\begin{aligned}
-\frac{z^{p+1}(T^{\alpha,n}_{m,p}(\lambda,\mu)f(z))'}{p} &\prec Q^*(z)\\
&= \frac{\gamma}{p+m} z^{-\frac{\gamma}{p+m}} \int_0^z t^{\frac{\gamma}{p+m}-1}\left(\frac{1+At}{1+Bt}\right) dt\\
&= \begin{cases} \frac{A}{B} + (1-\frac{A}{B})(1+Bz)^{-1}\ {}_2F_1(1,1;\frac{\gamma}{p+m}+1;\frac{Bz}{1+Bz}), & B \neq 0;\\ 1 + \left(\frac{\gamma}{p+m+\gamma}\right) Az, & \text{B=0} \end{cases}
\end{aligned}
$$

This completes the proof of the claim (18). The assertion (19) follows from (18) which can be seen as follows. To prove the assertion (19) is sharp, it is sufficient to prove that

$$\inf_{|z|<1}\{\mathrm{Re}(Q^*(z))\} = Q^*(-1).$$

It is known that for $|z| \leq r < 1$,

$$\mathrm{Re}\left(\frac{1+Az}{1+Bz}\right) \geq \frac{1-Ar}{1-Br} \quad (|z| \leq r < 1).$$

Setting $g(s,z) = \frac{1+Asz}{1+Bsz}$ and $d\nu(s) = \frac{\gamma}{p+m} t^{\frac{\gamma}{p+m}-1} ds \quad (0 \leq s \leq 1)$, with positive measure on the closed interval $[0,1]$, we have

$$Q^*(z) = \int_0^1 g(s,z) d\nu(s),$$

therefore

$$Q^*(z) = \int_0^1 g(s,z) d\nu(s)$$

$$\mathrm{Re}(Q^*(z)) \geq \int_0^1 \frac{1-Asr}{1-Bsr} d\nu(s) = Q^*(-r)\ (|z| \leq r < 1). \tag{22}$$

The assertion (19) follows from inequality (22) by letting $r \to 1^-$. The result is the best possible because $Q^*(z)$ is the best dominant of (18). Setting $\delta = 1$ and $m = 0$, we deduce the following inclusion property from Theorem 2.1:

Corollary 2.2. Let $f \in \Sigma_p$. Then

$$\Sigma_p^{\alpha-1,\lambda,\mu,n}(A,B) \subset \Sigma_p^{\alpha,\lambda,\mu,n}(1-2\varrho,-1) \subset \Sigma_p^{\alpha,\lambda,\mu,n}(A,B),$$

where
$$\varrho = \frac{A}{B} + \left(1 - \frac{A}{B}\right)(1-B)^{-1}\ {}_2F_1\left(1,1;\frac{\gamma}{p}+1;\frac{B}{B-1}\right)$$
and $\gamma = \frac{1+p(1-\lambda)}{\lambda}$. The result is best possible.

Remark 2.3. Corollary 2.2 generalizes the result [10, Corollary 1]. In fact when $\lambda = 1, \mu = 0$ and $\alpha = 0$, the above Corollary 2.2 reduces to the result [10, Corollary 1] proved by Srivastava and Patel.

Theorem 2.4. Let $0 \le \eta < p$ and $f \in \Sigma_p^{\alpha,\lambda,\mu,n}(\eta)$, then
$$-\operatorname{Re}[z^{p+1}((1-\delta)[T_{m,p}^{\alpha,n}(\lambda,\mu)f(z)]' + \delta[T_{m,p}^{\alpha-1,n}(\lambda,\mu)f(z)]')] > \eta[\delta(1+p(1-\lambda))+(1-\delta)] \quad (|z|<R),$$
where
$$R = \left(\frac{\sqrt{\lambda^2\delta^2(p+m)^2 + [\delta(1+p(1-\lambda))+(1-\delta)]^2} - \lambda\delta(p+m)}{\delta(1+p(1-\lambda))+(1-\delta)}\right)^{\frac{1}{p+m}}. \tag{23}$$
The result is best possible.

Proof. Let
$$-z^{p+m}(T_{m,p}^{\alpha,n}(\lambda,\mu)f(z))' = \eta + (p-\eta)u(z). \tag{24}$$
Then $u(z)$ is of the form (12) and has positive real part in the unit disk $\mathbb{U}$. Using the identity (14) in (24) and differentiating, we have
$$-z^{p+m}(T_{m,p}^{\alpha-1,n}(\lambda,\mu)f(z))' = \eta[p(1-\eta)+1] + (p-\eta)[p(1-\eta)+1]u(z) + (p-\eta)zu'(z). \tag{25}$$
From (24) and (25), we have
$$-\frac{[z^{p+1}((1-\delta)[T_{m,p}^{\alpha,n}(\lambda,\mu)f(z)]' + \delta[T_{m,p}^{\alpha-1,n}(\lambda,\mu)f(z)]')] + \eta[(1-\delta)(1+p(1-\lambda))+\delta]}{[\delta(1+p(1-\lambda))+(1-\delta)](p-\eta)} = u(z) + \frac{zu'(z)}{\delta(1+p(1-\lambda))+(1-\delta)} \tag{26}$$
It is well known that [5]:
$$\frac{|zu'(z)|}{\operatorname{Re}\{u(z)\}} \le \frac{2(p+m)r^{p+m}}{1-r^{2(p+m)}} \quad (|z| = r < 1),$$
using this in (26), we get
$$-\operatorname{Re}\left\{\frac{[z^{p+1}((1-\delta)[T_{m,p}^{\alpha,n}(\lambda,\mu)f(z)]' + \delta[T_{m,p}^{\alpha-1,n}(\lambda,\mu)f(z)]')] + \eta[\delta(1+p(1-\lambda))+(1-\delta)]}{[\delta(1+p(1-\lambda))+(1-\delta)](p-\eta)}\right\} \ge \operatorname{Re}(u(z))\left(1 - \frac{(p+m)r^{p+m}}{(1-r^{2(p+m)})[\delta(1+p(1-\lambda))+(1-\delta)]}\right). \tag{27}$$

It is easy to see that the right hand side of (27) is positive if $r < R$, where R is given by (23). We now show that the value of R is best possible for that let us consider the function defined by

$$-z^{p+m}(T_{m,p}^{\alpha,n}(\lambda,\mu)f(z))' = \eta + (p-\eta)\frac{1+z^{p+m}}{1-z^{p+m}}.$$

Note that

$$-\frac{[z^{p+1}((1-\delta)[T_{m,p}^{\alpha,n}(\lambda,\mu)f(z)]' + \delta[T_{m,p}^{\alpha-1,n}(\lambda,\mu)f(z)]')] + \eta[\delta(1+p(1-\lambda)) + (1-\delta)]}{[\delta(1+p(1-\lambda)) + (1-\delta)](p-\eta)}$$
$$= \frac{[\delta(1+p(1-\lambda)) + (1-\delta)] - [\delta(1+p(1-\lambda)) + (1-\delta)]z^{2(p+m)}}{[(1-\delta)(1+p(1-\lambda)) + \delta](1-z)^{2(p+m)}} = 0,$$

for $z = R.\exp\{\frac{i\pi}{p+m}\}$. The proof is complete at this juncture.

Remark 2.5. Theorem 2.4 generalizes [10, Theorem 2]. In fact when $\lambda = 1, \mu = 0$ and $\alpha = 0$, the above Theorem 2.4 reduces to the result [10, Theorem 2] proved by Srivastava and Patel.

Theorem 2.6. A function $f \in \Sigma_{p,m}$ of the form (1) is in the class $\Sigma_{m,p}^{\alpha,\lambda,\mu,n}(A,B)$ if and only if

$$\sum_{k=m}^{\infty} \varphi_k(\lambda,\mu,n,\alpha,p)|a_k| \leq \frac{p(A-B)}{(1-B)}, \tag{28}$$

where $\varphi_k(\lambda,\mu,n,p)$ is as defined in (5).

Proof. Let the inequality (28) holds true. Then, for $z \in \partial\mathbb{U} : \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$, from (1) and (28), we deduce that

$$\begin{aligned}\left|\frac{z^{P+1}(T_{m,p}^{\alpha,n}(\lambda,\mu)f(z))'}{B(T_{m,p}^{\alpha,n}(\lambda,\mu)f(z))' + pA}\right| &\leq \frac{\sum_{k=m}^{\infty}\varphi_k(\lambda,\mu,n,\alpha,p)k|a_k|}{p(A-B) + B\sum_{k=0}^{\infty}\varphi_k(\lambda,\mu,n,\alpha,p)k|a_k|} \\ &< 1.\end{aligned}$$

An application of maximum modulus theorem completes the proof of the sufficient condition. Conversely suppose f be given by (1) and that $\Sigma_{m,p}^{\alpha,\lambda,\mu,n}(A,B)$. Then from (1) and (11), we have

$$\begin{aligned}\left|\frac{z^{P+1}(T_{m,p}^{\alpha,n}(\lambda,\mu)f(z))'}{B(T_{m,p}^{\alpha,n}(\lambda,\mu)f(z))' + pA}\right| &\leq \left|\frac{\sum_{k=m}^{\infty}\varphi_k(\lambda,\mu,n,\alpha,p)k|a_k z^{k+p}}{p(A-B) + B\sum_{k=0}^{\infty}\varphi_k(\lambda,\mu,n,\alpha,p)ka_k z^{k+p}}\right| \\ &< 1.\end{aligned}$$

Since $\mathrm{Re}(z) \leq z,\ z \in \mathbb{C}$, it follows that

$$\mathrm{Re}\left\{\frac{\sum_{k=m}^{\infty}\varphi_k(\lambda,\mu,n,\alpha,p)ka_kz^{k+p}}{p(A-B)+B\sum_{k=0}^{\infty}\varphi_k(\lambda,\mu,n,\alpha,p)ka_kz^{k+p}}\right\} < 1. \tag{29}$$

Letting $z \to 1^-$ through the real axis in the above inequality (29) we have the inequality (28).

From the above Theorem 2.6, we deduce the following result:

Corollary 2.7. If the function $f \in \Sigma_{p,m}$ given by (1) is in the class $\Sigma_{m,p}^{\alpha,\lambda,\mu,n}(A,B)$, then

$$|a_k| \leq \frac{p(A-B)}{(1-B)\varphi_k(\lambda,\mu,n,\alpha,p)} \quad (k > -p).$$

The result is sharp for the function defined by

$$f(z) = z^{-p} + \frac{p(A-B)}{(1-B)\varphi_k(\lambda,\mu,n,\alpha,p)}z^k$$

For the function $f \in \Sigma_{p,m}$ given by (1), the integral operator

$$F_{c,p} : \Sigma_{p,m} \to \Sigma_{p,m}$$

is defined by

$$\begin{aligned} F_{c,p}(z) &= \frac{c}{z^{c+p}}\int_0^z t^{c+p-1}f(t)dt & (30)\\ &= z^{-p} + \sum_{k=m}^{\infty}\left(\frac{c}{k+p+c}\right)a_kz^k, \quad c > 0. & (31) \end{aligned}$$

From (31) it is clear that $F_{c,p} \in \Sigma_{p,m}$. From (30), after a simple computation, we obtain that

$$\lambda z(F_{c,p}(f)(z))' = cf(z) - (c+p)F_{c,p}(f)(z). \tag{32}$$

Taking convolution with $h(z) = z^{-p} + \sum_{k=m}^{\infty}\varphi_k(\lambda,\mu,n,\alpha,p)z^k$, given by (6), on both sides of (32) and using the fact that $(f*g)'(z) = f(z) * zg'(z)$, it is easy to verify that

$$\lambda z(T_{m,p}^{\alpha,n}(\lambda,\mu)F_{c,p}(f)(z))' = cT_{m,p}^{\alpha,n}(\lambda,\mu)f(z) - (c+p)T_{m,p}^{\alpha,n}(\lambda,\mu)F_{c,p}(f)(z) \tag{33}$$

Theorem 2.8. Let the function f given by (1) is in the class $\Sigma_{m,p}^{\alpha,\lambda,\mu,n}(A,B)$ and the function $F_{c,p}$ be defined by (30). Then

$$-\frac{z^{p+m}(T_{m,p}^{\alpha,n}(\lambda,\mu)F_{c,p}(f)(z))'}{p} \prec Q^*(z) \prec \frac{1+Az}{1+Bz}, \tag{34}$$

where

$$Q^*(z) = \begin{cases} \frac{A}{B} + (1 - \frac{A}{B})(1 + Bz)^{-1} \ {}_2F_1(1, 1; \frac{c}{p+m} + 1; \frac{Bz}{1+Bz}), & B \neq 0; \\ 1 + \left(\frac{Ac}{p+m+c}\right) Az, & \text{B=0} \end{cases}$$

is the best dominant of (34). Furthermore

$$-\operatorname{Re}\left\{\frac{z^{p+1}(T^{\alpha,n}_{m,p}(\lambda,\mu)F_{c,p}(f)(z))'}{p}\right\} > \rho, \tag{35}$$

where

$$\rho = \begin{cases} \frac{A}{B} + (1 - \frac{A}{B})(1 - B)^{-1} \ {}_2F_1(1, 1; \frac{c}{p+m} + 1; \frac{B}{B-1}), & B \neq 0; \\ 1 - \frac{cA}{p+m+c}, & \text{B=0.} \end{cases}$$

The value of ρ in inequality (35) is the best possible.

Proof. Define the function φ by

$$\varphi(z) = -\frac{z^{p+1}(T^{\alpha,n}_{m,p}(\lambda,\mu)F_{c,p}(f)(z))'}{p}, \tag{36}$$

then $\varphi(z)$ is analytic in $\mathbb{U}$. Using the identity (33) in (36) and then differentiation once, we have

$$-\frac{z^{p+1}(T^{\alpha,n}_{m,p}(\lambda,\mu)(f)(z))'}{p} = \varphi(z) + \frac{z\varphi'(z)}{c}. \tag{37}$$

Now an application of Lemma 1.3, similar to that in the proof of Theorem 2.1, completes the proof.

Remark 2.9. The above Theorem 2.8 gives a generalization to the result [10, Theorem 3]. In fact when $\lambda = 1, \mu = 0$ and $\alpha = 0$ it reduces to [10, Theorem 3].

Theorem 2.10. Let the function $F_{c,p}$ be defined by (30) satisfies

$$z^p\left\{(1-\delta)T^{\alpha,n}_{m,p}(\lambda,\mu)F_{c,p}(f)(z) + \delta T^{\alpha,n}_{m,p}(\lambda,\mu)F_{c,p}(f)(z)\right\} \prec \frac{1+Az}{1+Bz}, \tag{38}$$

then

$$-\operatorname{Re}\left\{\delta z^p T^{\alpha,n}_{m,p}(\lambda,\mu)F_{c,p}(f)(z)\right\} > \rho, \tag{39}$$

where

$$\rho = \begin{cases} \frac{A}{B} + (1 - \frac{A}{B})(1 - B)^{-1} \ {}_2F_1(1, 1; \frac{c}{\delta(p+m)} + 1; \frac{B}{B-1}), & B \neq 0; \\ 1 - \frac{cA}{\delta(p+m)+c}, & \text{B=0.} \end{cases}$$

The value of ρ in inequality (39) is the best possible.

Proof. Let us define the function φ by

$$\varphi(z) = -z^p T_{m,p}^{\alpha,n}(\lambda,\mu)F_{c,p}(f)(z), \tag{40}$$

then $\varphi(z)$ is and analytic in $\mathbb{U}$. Differentiating (40) and using the identity (40), we obtain

$$z^p T_{m,p}^{\alpha,n}(\lambda,\mu)(f)(z) = \varphi(z) + \frac{z\varphi'(z)}{c}. \tag{41}$$

From (40) and (41), have

$$z^p \left\{(1-\delta)T_{m,p}^{\alpha,n}(\lambda,\mu)F_{c,p}(f)(z) + \delta T_{m,p}^{\alpha,n}(\lambda,\mu)F_{c,p}(f)(z)\right\} = \varphi(z) + \frac{\delta}{c}z\varphi'(z).$$

Now an application of Lemma 1.3, similar to that in the proof of Theorem 2.1, completes the proof.

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