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INITIAL CHEBYSHEV POLYNOMIAL COEFFICIENT BOUND ESTIMATES FOR BI-UNIVALENT FUNCTIONS

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Abstract. We know that a function is *univalent* if it never takes the same value twice. Also we know that a function is *bi-univalent* if both it and its inverse are univalent. Our goal in the present article is to introduce a new subclass of bi-univalent functions making use of the Chebyshev polynomials. We obtain the bound estimates of initial Chebyshev polynomial coefficients $|a_2|$ and $|a_3|$ of functions in this subclass. Furthermore, we solve the Fekete-Szegö problem in this subclass.

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1. Introduction

Let $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disc on the complex plane. Consider the following well-known function classes:

$$\mathcal{H}(\mathbb{U}) = \{f \in \mathbb{C} : f \textit{ is analytic in the unit disk } \mathbb{U}\},$$

$$\mathcal{A} = \{f \in \mathcal{H}(\mathbb{U}) : f \textit{ is normalized by } f(0) = f'(0) - 1 = 0 \textit{ and } f(z) = z + \sum_{n=2}^{\infty} a_n z^n\}$$

and

$$\mathcal{S} = \{f \in \mathcal{A} : f \textit{ is univalent}\}.$$

If the function f and its inverse $F = f^{-1}$ are univalent in $\mathbb{U}$, we say that the function f in $\mathcal{A}$ is *bi-univalent* in $\mathbb{U}$.

Let Σ define the class of all bi-univalent functions in $\mathbb{U}$. Someone can see a short history and examples of functions in the class Σ in [11]. The Koebe 1/4 Theorem [4] asserts that the image of $\mathbb{U}$ under each univalent function f in $\mathcal{S}$ contains the disk of radius 1/4. According to this, if $F = f^{-1}$ is the inverse of a function $f \in S$, then F has a Taylor-Maclaurin series expansion in some disk about the origin. So, in $\mathcal{S}$ every function

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

has an inverse f^{-1} which satisfies $f^{-1}(f(z)) = z$ for $z \in \mathbb{U}$ and $f(f^{-1}(w)) = w$ for $|w| < r_0(f), r_0(f) \geq 1/4$, where

$$f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots . \tag{2}$$

The following class $\mathcal{M}_\Sigma(\beta, \lambda)$ of bi-univalent functions was introduced by Murugunsundaramoorthy et al. [1]: A function $f \in \Sigma$ is said to be in the class $\mathcal{M}_\Sigma(\beta, \lambda)$ if the conditions

$$f \in \Sigma, \qquad \Re\left(\frac{zf'(z)}{(1-\lambda)f(z) + \lambda z f'(z)}\right) > \beta; 0 \leq \beta < 1, 0 \leq \lambda < 1, z \in \mathbb{U}$$

and

$$\Re\left(\frac{wF'(w)}{(1-\lambda)F(w) + \lambda w F'(w)}\right) > \beta; 0 \leq \beta < 1, 0 \leq \lambda < 1, z \in \mathbb{U}$$

where the function F is inverse of f.

Lewin [9], who obtained the estimate $|a_2| \leq 1.51$, firstly introduced the class of bi-univalent functions. Accordingly, Netanyahu [10] showed that $|a_2| \leq \frac{4}{3}$. Subsequently, Brannan and Clunie [2] developed the bound of $|a_2|$ as $\sqrt{2}$. Brannan and Taha [3] defined certain subclasses of bi-univalent function class Σ similar to the usual subclasses. In fact, the aforementioned work of Srivastava et al. [11] mainly revived the investigation of various subclasses of bi-univalent function class Σ in recent years. Lately, many mathematicians found bounds for several subclasses of bi-univalent functions (see [11],[8],[12]).

Chebyshev polynomials, which are used by us in this paper, play a considerable act in numerical analysis. We know that the Chebyshev polynomials are four kinds. The most of books and articles related to specific orthogonal polynomials of Chebyshev family contain essentially results of Chebyshev polynomials of first and second kinds $T_n(x)$ and $U_n(x)$ and their numerous uses in different applications, see Doha [5] and Mason [6].

The well-known kinds of the Chebyshev polynomials are the first and second kinds. In the case of real variable x on $[-1, 1]$, the first and second kinds are defined by

$$T_n(x) = cosn\theta$$

and

$$U_n(x) = \frac{sin(n+1)\theta}{sin\theta}$$

where the subscript n denotes the polynomial degree and where $x = cos\theta$, respectively.

Now we define our class with the following subordination:

Definition 1. *A function $f \in \Sigma$ is said to be in the class $\mathcal{M}_\Sigma(\lambda, t)$ for $0 \leq \lambda < 1, 0 \leq \beta < 1$ and $t \in \left(\frac{1}{2}, 1\right]$, if the following subordinations hold*

$$\frac{zf'(z)}{(1-\lambda)f(z)+\lambda zf'(z)} \prec H(z,t) = \frac{1}{1-2tz+z^2} \qquad (z \in \mathbb{U}) \qquad (3)$$

and

$$\frac{wF'(w)}{(1-\lambda)F(w)+\lambda wF'(w)} \prec H(w,t) = \frac{1}{1-2tw+w^2} \qquad (w \in \mathbb{U}) \qquad (4)$$

where the function $F(w) = f^{-1}(w)$ is defined by (2).

Letting $t = cos\alpha$, $\alpha \in \left(\frac{-\pi}{3}, \frac{\pi}{3}\right)$, someone can obtain

$$H(z,t) = \frac{1}{1-2tz+z^2} = 1+\sum_{n=1}^{\infty} \frac{sin(n+1)\alpha}{sin\alpha} z^n$$

$$= 1 + 2cos\alpha z + (3cos^2\alpha - sin^2\alpha)z^2 + \cdots \qquad z \in \mathbb{U}.$$

So, we write

$$H(z,t) = 1 + U_1(t)z + U_2(t)z^2 + ... \qquad (z \in \mathbb{U}, t \in (-1,1))$$

where $U_{n-1} = \frac{sin(narccost)}{\sqrt{1-t^2}}$ for $n \in \mathbb{N}$, are the second kind of the Chebyshev polynomials. Furthermore, we know that

$$U_n(t) = 2tU_{n-1}(t) - U_{n-2}(t),$$

and

$$U_1(t) = 2t; U_2(t) = 4t^2 - 1; U_3(t) = 8t^3 - 4t; U_4(t) = 16t^4 - 12t^2 + 1, \cdots . \qquad (5)$$

The Chebyshev polynomials $T_n(t), t \in [-1,1]$, of the first kind have the generating function of the form

$$\sum_{n=0}^{\infty} T_n(t)z^n = \frac{1-tz}{1-2tz+z^2} \qquad (z \in \mathbb{U}).$$

All the same, there is the following relationship between the Chebyshev polynomials of the first kind $T_n(t)$ and the second kind $U_n(t)$:

$$\frac{dT_n(t)}{dt} = nU_{n-1}(t),$$

$$T_n(t) = U_n(t) - tU_{n-1}(t),$$

$$2T_n(t) = U_n(t) - U_{n-2}(t).$$

In this study, by motivating by the earlier work of Dziok et al. [7] and employing the technique used by Srivastava [11] , we use the Chebyshev polynomial expansions to provide bound estimates for initial Chebyshev polynomial coefficients of functions in $\mathcal{M}_\Sigma(\lambda, t)$.

2. Coefficient bound estimates for the function class $\mathcal{M}_\Sigma(\lambda, t)$

Theorem 1. *Let the function $f(z)$ given by (1) be in the class $\mathcal{M}_\Sigma(\lambda, t)$. Then*

$$|a_2| \leq \frac{2t\sqrt{2t}}{1-\lambda} \tag{6}$$

and

$$|a_3| \leq \frac{4t^2}{(1-\lambda)^2} + \frac{t}{1-\lambda}. \tag{7}$$

Proof. Let $f \in \mathcal{M}_\Sigma(\lambda, t)$ given by (1).From (3) and (4), we find

$$\frac{zf'(z)}{(1-\lambda)f(z) + \lambda z f'(z)} = 1 + U_1(t)w(z) + U_2(t)w^2(z) + \cdots , \tag{8}$$

and

$$\frac{wF'(w)}{(1-\lambda)F(w) + \lambda w F'(w)} = 1 + U_1(t)v(w) + U_2(t)v^2(w) + \cdots , \tag{9}$$

for some analytic functions w and v such that $w(0) = v(0) = 0$, $|w(z)| = |c_1 z + c_2 z^2 + \cdots| < 1$ and $|v(w)| = |d_1 w + d_2 w^2 + \cdots| < 1$, for all $z \in \mathbb{U}$. Putting the $w(z)$ and $v(w)$ in the equalities (8) and (9), we have

$$\frac{zf'(z)}{(1-\lambda)f(z) + \lambda z f'(z)} = 1 + U_1(t)c_1 z + [U_1(t)c_2 + U_2(t)c_1^2]z^2 + \cdots , \tag{10}$$

and

$$\frac{wF'(w)}{(1-\lambda)F(w) + \lambda w F'(w)} = 1 + U_1(t)d_1 w + [U_1(t)d_2 + U_2(t)d_1^2]w^2 + \cdots . \tag{11}$$

It is well-known that $|w(z)| < 1$ and $|v(w)| < 1$, $z, w \in \mathbb{U}$, then $|c_j| \leq 1$ and $|d_j| \leq 1$ for all $j \in \mathbb{N}$.

It follows from (10) and (11) that

$$(1-\lambda)a_2 = U_1(t)c_1, \tag{12}$$

$$(\lambda^2-1)a_2^2 + 2(1-\lambda)a_3 = U_1(t)c_2 + U_2(t)c_1^2, \tag{13}$$

and

$$-(1-\lambda)a_2 = U_1(t)d_1, \tag{14}$$

$$(\lambda^2-4\lambda+3)a_2^2 - 2(1-\lambda)a_3 = U_1(t)d_2 + U_2(t)d_1^2. \tag{15}$$

From (12) and (14), we have

$$c_1 = -d_1, \tag{16}$$

and

$$2(1-\lambda)^2a_2^2 = U_1^2(t)(c_1^2+d_1^2). \tag{17}$$

Now by summing (13) and (15), we obtain

$$2(\lambda^2-2\lambda+1)a_2^2 = U_1(t)(c_2+d_2) + U_2(t)(c_1^2+d_1^2). \tag{18}$$

By putting (17) in (18), we have

$$\left[2(\lambda^2-2\lambda+1) - 2\frac{U_2(t)}{U_1^2(t)}(1-\lambda)^2\right] a_2^2 = U_1(t)(c_2+d_2). \tag{19}$$

By considering (5) and (19) together, we obtain

$$|a_2| \leq \frac{2t\sqrt{2t}}{1-\lambda}. \tag{20}$$

Now, so as to find the bound on $|a_3|$, let's subtract from (13) to (15). So, we find

$$4(1-\lambda)a_3 - 4(1-\lambda)a_2^2 = U_1(t)(c_2-d_2) + U_2(t)(c_1^2-d_1^2). \tag{21}$$

Then, in view of (16) and (17), we obtain from (21)

$$a_3 = \frac{U_1^2(t)}{2(1-\lambda)^2}(c_1^2+d_1^2) + \frac{U_1(t)}{4(1-\lambda)}(c_2-d_2). \tag{22}$$

By considering (5), we get

$$|a_3| \leq \frac{4t^2}{(1-\lambda)^2} + \frac{t}{1-\lambda}. \tag{23}$$

3. Fekete-Szegö inequality for the function class $\mathcal{M}_\Sigma(\lambda, t)$

The following theorem is the solution of the Fekete-Szegö problem in $\mathcal{M}_\Sigma(\lambda, t)$.

Theorem 2. *Let f given by (1) be in the class $\mathcal{M}_\Sigma(\lambda, t)$ and $\mu \in \mathbb{R}$. Then*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{t}{1-\lambda}, & |\mu - 1| \leq \frac{1-\lambda}{8t^2} \\ \frac{8|1-\mu|t^3}{(1-\lambda)^2}, & |\mu - 1| \geq \frac{1-\lambda}{8t^2} \end{cases}$$

Proof. From (19) and (21)

$$a_3 - \mu a_2^2 = (1-\mu)\frac{U_1^3(t)(c_2 + d_2)}{2(1-\lambda)^2 U_1^2(t) - 2(1-\lambda)^2 U_2(t)} + \frac{U_1(t)(c_2 - d_2)}{4(1-\lambda)} \tag{24}$$

$$= U_1(t)\left[\left(h(\mu) + \frac{1}{4(1-\lambda)}\right)c_2 + \left(h(\mu) - \frac{1}{4(1-\lambda)}\right)d_2\right] \tag{25}$$

where

$$h(\mu) = \frac{(1-\mu)U_1^2(t)}{2(1-\lambda)^2 U_1^2(t) - 2(1-\lambda)^2 U_2(t).} \tag{26}$$

Then, by taking modulus of (25) and considering (5) , we have

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{t}{1-\lambda}, & 0 \leq |h(\mu)| \leq \frac{1}{4(1-\lambda)} \\ 4t|h(\mu)|, & |h(\mu)| \geq \frac{1}{4(1-\lambda)} \end{cases}$$

Taking $\mu = 1$, we have the following corollary.

Corollary 3. *If $f \in \mathcal{M}_\Sigma(\lambda, t)$, then*

$$|a_3 - a_2^2| \leq \frac{t}{1-\lambda}. \tag{27}$$

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