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OPERATION-SEPARATION AXIOMS VIA α-OPEN SETS

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Abstract. The purpose of this paper is to investigate several types of separation axioms in topological spaces and study some of the essential properties of such spaces. Moreover, we investigate their relationship to some other known separation axioms and some counterexamples.

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Keywords: Operation; α-open set; α_γ-open set; α-γ-T_0 space; α-γ-T_1 space; α-γ-T_2 space; $\alpha_\gamma R_0$ space; $\alpha_\gamma R_1$ space.

1. Introduction

The study of α-open sets was initiated by Njastad [6]. In [2], Ibrahim introduced α_γ-open sets in topological spaces and these α_γ-open sets were used to define three new separation axioms called $\alpha_\gamma T_0$, $\alpha_\gamma T_1$ and $\alpha_\gamma T_2$. Another set of new separation axioms, α-T_i, $i = 0, 1$ were characterized by Maki et al. [5] in 1993. The aim of this paper is to introduce and study some new separation axioms by means operations defined on α-open sets in topological spaces.

2. Preliminaries

Throughout the present paper, for a nonempty set X, (X, τ) always denote a topological space on which no separation axioms are assumed unless explicitly stated. The closure and interior of $A \subseteq X$ will be denoted by $Cl(A)$ and $Int(A)$, respectively. A subset A of a topological space (X, τ) is said to be α-open [6] if $A \subseteq Int(Cl(Int(A)))$. The complement of an α-open set is said to be α-closed. The intersection of all α-closed sets containing A is called the α-closure of A and is denoted by $\alpha Cl(A)$. The family of all α-open sets in a topological space (X, τ) is denoted by $\alpha O(X, \tau)$. An operation $\gamma : \alpha O(X, \tau) \rightarrow P(X)$ [2] is a mapping satisfying the condition, $V \subseteq V^\gamma$ for each $V \in \alpha O(X, \tau)$. We call the mapping γ an operation on $\alpha O(X, \tau)$. A subset

A of X is called an α_γ-open set [2] if for each point $x \in A$, there exists an α-open set U of X containing x such that $U^\gamma \subseteq A$. The complement of an α_γ-open set is said to be α_γ-closed. We denote the set of all α_γ-open (resp., α_γ-closed) sets of (X, τ) by $\alpha O(X, \tau)_\gamma$ (resp., $\alpha C(X, \tau)_\gamma$). The α_γ-closure [2] of a subset A of X with an operation γ on $\alpha O(X)$ is denoted by $\alpha_\gamma Cl(A)$ and is defined to be the intersection of all α_γ-closed sets containing A. A point $x \in X$ is in αCl_γ-closure [2] of a set $A \subseteq X$, if $U^\gamma \cap A \neq \phi$ for each α-open set U containing x. The αCl_γ-closure of A is denoted by $\alpha Cl_\gamma(A)$. An operation γ on $\alpha O(X, \tau)$ is said to be α-regular [2] if for every α-open sets U and V of each $x \in X$, there exists an α-open set W of x such that $W^\gamma \subseteq U^\gamma \cap V^\gamma$. An operation γ on $\alpha O(X, \tau)$ is said to be α-open [2] if for every α-open set U of $x \in X$, there exists an α_γ-open set V of X such that $x \in V$ and $V \subseteq U^\gamma$.

Let (X, τ) be any topological space and γ be an operation defined on $\alpha O(X)$. We recall the following results from [3].

Theorem 2.1. For each $x \in X$, either $\{x\}$ is α_γ-closed or $X \setminus \{x\}$ is α-γ-g.closed in (X, τ).

Theorem 2.2. If a subset A of X is α-γ-g.closed, then $\alpha Cl_\gamma(A) \setminus A$ does not contain any non-empty α_γ-closed set.

Theorem 2.3. Let A be any subset of a topological space (X, τ) . If A is α_γ-g.closed in X, then A is α-γ-g.closed.

Proposition 2.4. For any two distinct points x and y in a topological space X, the following statements are equivalent:

1. $\alpha_\gamma ker(\{x\}) \neq \alpha_\gamma ker(\{y\})$;
2. $\alpha_\gamma Cl(\{x\}) \neq \alpha_\gamma Cl(\{y\})$.

Proposition 2.5. Let $x \in X$, we have $y \in \alpha_\gamma ker(\{x\})$ if and only if $x \in \alpha_\gamma Cl(\{y\})$.

Proposition 2.6. Let A be a subset of X. Then, $\alpha_\gamma ker(A) = \{x \in X \colon \alpha_\gamma Cl(\{x\}) \cap A \neq \phi\}$.

3. α-γ-T_i spaces, where $i = 0, 1/2, 1, 2$

In this section, we introduce some new separation axioms using the notions of operation and α-open sets, also we give some characterization of these types of spaces and study the relationships between them and other well known spaces.

Definition 3.1. A topological space (X, τ) with an operation γ on $\alpha O(X)$ is said to be:

1. An α-γ-T_0 space if for any two distinct points $x, y \in X$, there exists an α-open set U such that either $x \in U$ and $y \notin U^\gamma$ or $y \in U$ and $x \notin U^\gamma$.

2. An $\alpha_\gamma T_0$ [2] (resp., α-T_0 [5]) space if for any two distinct points $x, y \in X$, there exists an α_γ-open (resp., α-open) set U such that either $x \in U$ and $y \notin U$ or $y \in U$ and $x \notin U$.

3. An α-γ-T_1 space if for any two distinct points $x, y \in X$, there exist two α-open sets U and V containing x and y, respectively, such that $y \notin U^\gamma$ and $x \notin V^\gamma$.

4. An $\alpha_\gamma T_1$ [2] (resp., α-T_1 [5]) space if for any two distinct points $x, y \in X$, there exist two α_γ-open (resp., α-open) sets U and V containing x and y, respectively, such that $y \notin U$ and $x \notin V$.

5. An α-γ-T_2 space if for any two distinct points $x, y \in X$, there exist two α-open sets U and V containing x and y, respectively, such that $U^\gamma \cap V^\gamma = \phi$.

6. An $\alpha_\gamma T_2$ [2] (resp., α-T_2 [4]) space if for any two distinct points $x, y \in X$, there exist two α_γ-open (resp., α-open) sets U and V containing x and y, respectively, such that $U \cap V = \phi$.

7. An α-γ-$T_{1/2}$ space if every α-γ-g.closed set of (X, τ) is α_γ-closed.

8. An α-$T_{1/2}$ space [1] if every (α,α)-g-closed set(X, τ) is α-closed.

Theorem 3.2. Suppose that $\gamma : \alpha O(X) \to P(X)$ is α-open. A topological space (X, τ) is α-γ-T_0 if and only if for every pair $x, y \in X$ with $x \neq y$, $\alpha Cl_\gamma(\{x\}) \neq \alpha Cl_\gamma(\{y\})$.

Proof. Necessity: Let x and y be any two distinct points of an α-γ-T_0 space (X, τ). Then, by definition, we assume that there exists an α-open set U such that $x \in U$ and $y \notin U^\gamma$. It follows from assumption that there exists an α_γ-open set S such that $x \in S$ and $S \subseteq U^\gamma$. Hence, $y \in X \setminus U^\gamma \subseteq X \setminus S$. Because $X \setminus S$ is an α_γ-closed set, we obtain that $\alpha Cl_\gamma(\{y\}) \subseteq X \setminus S$ and so $\alpha Cl_\gamma(\{x\}) \neq \alpha Cl_\gamma(\{y\})$.

Sufficiency: Suppose that $x \neq y$ for any $x, y \in X$. Then, we have that $\alpha Cl_\gamma(\{x\}) \neq \alpha Cl_\gamma(\{y\})$. Thus, there exists $z \in \alpha Cl_\gamma(\{x\})$ but $z \notin \alpha Cl_\gamma(\{y\})$. If $x \in \alpha Cl_\gamma(\{y\})$, then we get $\alpha Cl_\gamma(\{x\}) \subseteq \alpha Cl_\gamma(\{y\})$. This implies that $z \in \alpha Cl_\gamma(\{y\})$. This contradiction shows that $x \notin \alpha Cl_\gamma(\{y\})$, by ([2], Definition 2.20), there exists an α-open set W such that $x \in W$ and $W^\gamma \cap \{y\} = \phi$. Consequently, we have that $x \in W$ and $y \notin W^\gamma$. Hence, (X, τ) is an α-γ-T_0.

Theorem 3.3. Suppose that $\gamma : \alpha O(X) \to P(X)$ is α-open. A topological space (X, τ) is α-γ-T_0 if and only if (X, τ) is $\alpha_\gamma T_0$.

Proof. It is obvious that, for any subset A of (X, τ), $\alpha_\gamma Cl(A) = \alpha Cl_\gamma(A)$ holds under the assumption that γ is α-open ([2], Theorem 2.26 (2)). On the other hand, we have Theorem 3.2 and ([2], Theorem 3.9). Consequently, we obtain this proof by using these three facts.

Theorem 3.4. Let (X, τ) be a topological space and γ an operation on $\alpha O(X)$. Then, the following properties are equivalent:

1. A space X is α-γ-$T_{1/2}$;
2. For each $x \in X$, $\{x\}$ is α_γ-closed or α_γ-open.

Proof. (1) $\Rightarrow$ (2): Suppose $\{x\}$ is not α_γ-closed in (X, τ). Then, $X \setminus \{x\}$ is α-γ-g.closed by Theorem 2.1. Since (X, τ) is an α-γ-$T_{1/2}$ space, then $X \setminus \{x\}$ is α_γ-closed and so $\{x\}$ is α_γ-open.

(2) $\Rightarrow$ (1): Let F be an α-γ-g.closed set in (X, τ). We shall prove that $\alpha Cl_\gamma(F) = F$. It is sufficient to show that $\alpha Cl_\gamma(F) \subseteq F$. Assume that there exists a point x such that $x \in \alpha Cl_\gamma(F) \setminus F$. Then by assumption, $\{x\}$ is α_γ-closed or α_γ-open.

Case 1. $\{x\}$ is an α_γ-closed set, for this case, we have an α_γ-closed set $\{x\}$ such that $\{x\} \subseteq \alpha Cl_\gamma(F) \setminus F$. This is a contradiction to Theorem 2.2.

Case 2. $\{x\}$ is an α_γ-open set, we have $x \in \alpha_\gamma Cl(F)$. Since $\{x\}$ is α_γ-open, it implies that $\{x\} \cap F \neq \phi$ by ([2], Theorem 2.23). This is a contradiction. Thus, we have $\alpha Cl_\gamma(F) = F$ and this is shows that F is α_γ-closed.

Theorem 3.5. Let (X, τ) be a topological space and γ an operation on $\alpha O(X)$. Then, the following properties are equivalent:

1. (X, τ) is α-γ-$T_{1/2}$;
2. For each $x \in X$, $\{x\}$ is α_γ-closed or α_γ-open;
3. (X, τ) is α_γ-$T_{1/2}$.

Proof. Follows from ([2], Theorem 3.2) and Theorem 3.4.

Theorem 3.6. Let (X, τ) be a topological space and γ an operation on $\alpha O(X)$. Then, the following properties are equivalent:

1. (X, τ) is α-γ-T_1;
2. For every point $x \in X$, $\{x\}$ is an α_γ-closed set;

3. (X, τ) is $\alpha_\gamma T_1$.

Proof. (1) $\Rightarrow$ (2): Let $x \in X$ be a point. For each point $y \in X \setminus \{x\}$, there exists an α-open set V_y such that $y \in V_y$ and $x \notin V_y^\gamma$. Then $X \setminus \{x\} = \cup\{V_y^\gamma : y \in X \setminus \{x\}\}$. It is shown that $X \setminus \{x\}$ is α_γ-open in (X, τ).

(2) $\Rightarrow$ (3): This follows from ([2], Theorem 3.10).

(3) $\Rightarrow$ (1): It is shown that if $x \in U$, where U is α_γ-open, then there exists an α-open set V such that $x \in V \subseteq V^\gamma \subseteq U$. Using (3), we have that (X, τ) is α-γ-T_1.

Proposition 3.7. The following statements are equivalent for a topological space (X, τ) with an operation γ on $\alpha O(X)$:

1. X is α-γ-T_2;
2. Let $x \in X$. For each $y \neq x$, there exists an α-open set U containing x such that $y \notin \alpha Cl_\gamma(U^\gamma)$;
3. For each $x \in X$, $\cap\{\alpha Cl_\gamma(U^\gamma) : U \in \alpha O(X) \text{ and } x \in U\} = \{x\}$.

Proof. (1) $\Rightarrow$ (2): Since X is α-γ-T_2, there exist α-open set U containing x and α-open set W containing y such that $U^\gamma \cap W^\gamma = \phi$, implies that $y \notin \alpha Cl_\gamma(U^\gamma)$.

(2) $\Rightarrow$ (3): If possible for some $y \neq x$, we have $y \in \alpha Cl_\gamma(U^\gamma)$ for every α-open set U containing x, which then contradicts (2).

(3) $\Rightarrow$ (1): Let $x, y \in X$ and $x \neq y$. Then there exists α-open set U containing x such that $y \notin \alpha Cl_\gamma(U^\gamma)$, implies that $U^\gamma \cap W^\gamma = \phi$ for some α-open set W containing y.

Theorem 3.8. Let X be an α-γ-T_2 space and V^γ be α_γ-open for each $V \in \alpha O(X)$. Then, the following properties hold.

1. For any two distinct points $a, b \in X$, there are α_γ-closed sets C_1 and C_2 such that $a \in C_1$ and $b \notin C_1$ and $a \notin C_2$, $b \in C_2$ and $X = C_1 \cup C_2$.
2. For every point a of X, $\{a\} = \cap C_a$, where C_a is an α_γ-closed set containing α-open set U which contains a.

Proof. 1. Since X is α-γ-T_2 space, then for any $a, b \in X$, there exist α-open sets U and V such that $a \in U$, $b \in V$ and $U^\gamma \cap V^\gamma = \phi$. Therefore, $U^\gamma \subseteq X \setminus V^\gamma$ and $V^\gamma \subseteq X \setminus U^\gamma$. Hence $a \in X \setminus V^\gamma$. Put $X \setminus V^\gamma = C_1$. This gives $a \in C_1$ and $b \notin C_1$. Also $b \in X \setminus U^\gamma$. Put $X \setminus U^\gamma = C_2$. Therefore $b \in C_2$ and $a \notin C_2$. Moreover $C_1 \cup C_2 = (X \setminus U^\gamma) \cup (X \setminus V^\gamma) = X$.

2. Since X is α-γ-T_2 space, therefore for any a, b, $a \neq b$, there exist α-open sets U and V such that $a \in U, b \in V$ and $U^\gamma \cap V^\gamma = \phi$. This gives $U^\gamma \subseteq X \setminus V^\gamma$. Since $X \setminus V^\gamma$ is α_γ-closed and $U^\gamma \subseteq X \setminus V^\gamma = C_a$, α_γ-closed containing a and does not contain b. Since b is an arbitrary point of X different from a, $b \notin \cap C_a$. Thus a is the only point which is in every α_γ-closed containing a, that is, $\{a\} = \cap C_a$.

Theorem 3.9. For a topological space (X, τ) and γ an operation on $\alpha O(X)$, the following properties hold.

1. Every $\alpha_\gamma T_i$ space is α-γ-T_i, where $i \in \{2, 0\}$.
2. Every α-γ-T_2 space is α-γ-T_1.
3. Every α-γ-T_1 space is α-γ-$T_{1/2}$.
4. Every α-γ-T_1 space is α_γ-$T_{1/2}$.
5. Every α-γ-$T_{1/2}$ space is $\alpha_\gamma T_0$.
6. Every γ-T_i space is α-γ-T_i, where $i \in \{2, 1, 1/2, 0\}$.
7. Every α-γ-T_i space is α-T_i, where $i \in \{2, 1, 1/2, 0\}$.

Proof. (1), (2): The proofs are obvious by Definition 3.1.

(3): This follows from Theorems 3.6 and 3.4.

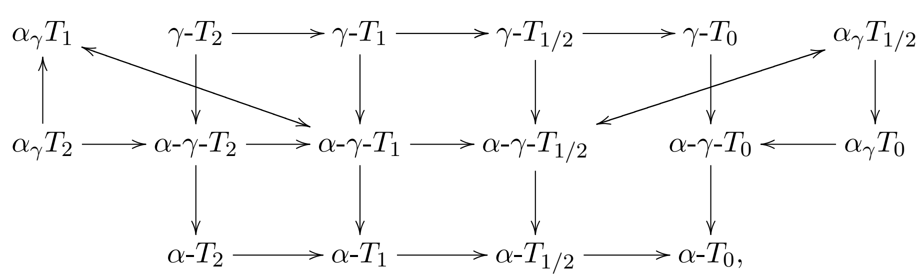
(4): This follows from Theorems 3.6 and 3.5.

(5): This follows from Theorem 3.4 and Definition 3.1 (2).

(6): For any open set U of (X, τ), we have $U \in \alpha O(X, \tau)$ holds. Thus, the proofs of (6) for $i \in \{2, 1, 0\}$ are obvious from ([7], Definitions 4.1, 4.2, 4.3) and Definition 3.1. The proof for $i = 1/2$, is obtained by ([7], Proposition 4.10 (i)), ([2], Theorem 2.8) and Theorem 3.4.

(7): The proof is obvious by Definition 3.1 and ([2], Definition 2.1).

Remark 3.10. From Theorems 3.5, 3.6 and 3.9, we obtain the following diagram of implications:



where $A \to B$ represents that A implies B.

Example 3.11. The converse of Theorem 3.9 (1) for $i = 0$ is not true in general. Consider $X = \{a, b, c\}$ and $\tau = \{\phi, X, \{a\}, \{b\}, \{a, b\}\}$ be a topology on X. For each $A \in \alpha O(X)$ we define γ on $\alpha O(X)$ by

$$A^\gamma = \begin{cases} \{a, c\} & \text{if } A = \{a\}, \\ \{a, b\} & \text{if } A = \{b\}, \\ \{a, b\} & \text{if } A = \{a, b\}, \\ X & \text{if } A = X, \\ \phi & \text{if } A = \phi. \end{cases}$$

Then, X is not $\alpha_\gamma T_0$. Indeed, for every α_γ-open set V_a containing a, we have $b \in V_a$, for every α_γ-open set V_b containing b, we have $a \in V_b$. By Definition 3.1 (2) the space X is not $\alpha_\gamma T_0$. Moreover, the space X is α-γ-T_0.

Example 3.12. The converse of Theorem 3.9 (2) is not true in general. Consider $X = \{a, b, c\}$ with the discrete topology τ on X. For each $A \in \alpha O(X)$ we define γ on $\alpha O(X)$ by

$$A^\gamma = \begin{cases} A & \text{if } A = \{a, b\} \text{ or } \{a, c\} \text{ or } \{b, c\}, \\ X & \text{otherwise.} \end{cases}$$

Then, it is shown directly that each singleton is α_γ-closed in (X, τ). By Theorem 3.6, X is α-γ-T_1. But, we can show that $U^\gamma \cap V^\gamma \neq \phi$ holds for any α-open sets U and V. This implies X is not α-γ-T_2.

Example 3.13. The converse of Theorem 3.9 (3) and (4) are not true in general. Consider $X = \{a, b, c\}$ and $\tau = \{\phi, X, \{a\}, \{a, b\}, \{a, c\}\}$ be a topology on X. For each $A \in \alpha O(X)$ we define γ on $\alpha O(X)$ by $A^\gamma = A$. Then, it is shown directly that each singleton is α_γ-closed or α_γ-open in (X, τ). By Theorem 3.5, X is both α-γ-$T_{1/2}$ and α_γ-$T_{1/2}$. However, by Theorem 3.6, X is not α-γ-T_1, in fact, a singleton $\{a\}$ is not α_γ-closed.

Example 3.14. The converse of Theorem 3.9 (5) is not true in general. Consider $X = \{a, b, c\}$ and $\tau = \{\phi, X, \{a\}, \{a, b\}\}$ be a topology on X. For each $A \in \alpha O(X)$ we define γ on $\alpha O(X)$ by

$$A^\gamma = \begin{cases} A & \text{if } A = \{a\} \text{ or } \{a, b\}, \\ X & \text{otherwise.} \end{cases}$$

Then, X is not α-γ-$T_{1/2}$ because a singleton $\{b\}$ is neither α_γ-open nor α_γ-closed. It is shown directly that X is $\alpha_\gamma T_0$.

Example 3.15. Some converses of Theorem 3.9 (6) are not true in general. Consider $X = \{a, b, c\}$ and $\tau = \{\phi, X, \{c\}\}$ be a topology on X. For each $A \in \alpha O(X)$, we define γ on $\alpha O(X)$ by $A^\gamma = A$. Then, X is α-γ-T_i but it is not γ-T_i for $i = 0, 1/2$.

Example 3.16. The converse of Theorem 3.9 (7) is not true in general. Consider $X = \{a, b, c\}$ with the discrete topology τ on X. For each $A \in \alpha O(X)$ we define γ on $\alpha O(X)$ by $A^\gamma = A$. Then, X is α-T_i but it is not α-γ-T_i for $i = 0, 1/2, 1, 2$.

Proposition 3.17. If (X, τ) is $\alpha_\gamma D_0$, then $\alpha_\gamma T_0$.

Proof. Suppose that X is $\alpha_\gamma D_0$. Then for each distinct pair $x, y \in X$, at least one of x, y, say x, belongs to an $\alpha_\gamma D$-set G but $y \notin G$. Let $G = U_1 \setminus U_2$ where $U_1 \neq X$ and $U_1, U_2 \in \alpha O(X, \tau)_\gamma$. Then $x \in U_1$, and for $y \notin G$ we have two cases: (a) $y \notin U_1$, (b) $y \in U_1$ and $y \in U_2$.
In case (a), $x \in U_1$ but $y \notin U_1$.
In case (b), $y \in U_2$ but $x \notin U_2$.
Thus in both the cases, we obtain that X is $\alpha_\gamma T_0$.

Proposition 3.18. If (X, τ) is $\alpha_\gamma D_0$, then α-γ-T_0.

Proof. Follows from Proposition 3.17 and Theorem 3.9 (1).

Corollary 3.19. If (X, τ) is $\alpha_\gamma D_1$, then it is α-γ-T_0.

Proof. Follows from ([2], Remark 3.7 (3)) and Proposition 3.18.

Proposition 3.20. Let (X, τ) be an α-γ-$T_{1/2}$ topological space and γ be an α-regular operation on $\alpha O(X)$. If $\alpha_\gamma ker(\{x\}) \neq X$ for a point $x \in X$, then $\{x\}$ is an $\alpha_\gamma D$-set of (X, τ).

Proof. Since $\alpha_\gamma ker(\{x\}) \neq X$ for a point $x \in X$, then there exists a subset $U \in \alpha O(X, \tau)_\gamma$ such that $\{x\} \subseteq U$ and $U \neq X$. Using Proposition 3.4, for the point x, we have $\{x\}$ is α_γ-open or α_γ-closed in (X, τ). When the singleton $\{x\}$ is α_γ-open, $\{x\}$ is an $\alpha_\gamma D$-set of (X, τ). When the singleton $\{x\}$ is α_γ-closed, then $X \setminus \{x\}$ is α_γ-open in (X, τ). Put $U_1 = U$ and $U_2 = U \cap (X \setminus \{x\})$. Then, $\{x\} = U_1 \setminus U_2$, $U_1 \in \alpha O(X, \tau)_\gamma$ and $U_1 \neq X$. It follows from the hypothesis that $U_2 \in \alpha O(X, \tau)_\gamma$ and so $\{x\}$ is an $\alpha_\gamma D$-set.

Proposition 3.21. For an α-γ-$T_{1/2}$ topological space (X, τ) with at least two points, (X, τ) is an $\alpha_\gamma D_1$ space if and only if $\alpha_\gamma ker(\{x\}) \neq X$ holds for every point $x \in X$.

Proof. Necessity: Let $x \in X$. For a point $y \neq x$, there exists an $\alpha_\gamma D$-set U such that $x \in U$ and $y \notin U$. Say $U = U_1 \setminus U_2$, where $U_i \in \alpha O(X,\tau)_\gamma$ for each $i \in \{1,2\}$ and $U_1 \neq X$. Thus, for the point x, we have an α_γ-open set U_1 such that $\{x\} \subseteq U_1$ and $U_1 \neq X$. Hence, $\alpha_\gamma ker(\{x\}) \neq X$.

Sufficiency: Let x and y be a pair of distinct points of X. We prove that there exist $\alpha_\gamma D$-sets A and B containing x and y, respectively, such that $y \notin A$ and $x \notin B$. Using Proposition 3.4, we can take the subsets A and B for the following four cases for two points x and y.
Case1. $\{x\}$ is α_γ-open and $\{y\}$ is α_γ-closed in (X,τ). Since $\alpha_\gamma ker(\{y\}) \neq X$, then there exists an α_γ-open set V such that $y \in V$ and $V \neq X$. Put $A = \{x\}$ and $B = \{y\}$. Since $B = V \setminus (X \setminus \{y\})$, then V is an α_γ-open set with $V \neq X$ and $X \setminus \{y\}$ is α_γ-open, and B is a required $\alpha_\gamma D$-set containing y such that $x \notin B$. Obviously, A is a required $\alpha_\gamma D$-set containing x such that $y \notin A$.
Case 2. $\{x\}$ is α_γ-closed and $\{y\}$ is α_γ-open in (X,τ). The proof is similar to Case 1.
Case 3. $\{x\}$ and $\{y\}$ are α_γ-open in (X,τ). Put $A = \{x\}$ and $B = \{y\}$.
Case 4. $\{x\}$ and $\{y\}$ are α_γ-closed in (X,τ). Put $A = X \setminus \{y\}$ and $B = X \setminus \{x\}$.
For each case of the above, the subsets A and B are the required $\alpha_\gamma D$-sets. Therefore, (X,τ) is an α_γ-D_1 space.

Definition 3.22. A point $x \in X$ which has only X as the α_γ-neighbourhood is called an α_γ-neat point.

Proposition 3.23. For an $\alpha_\gamma T_0$ topological space (X,τ), the following are equivalent:

1. (X,τ) is $\alpha_\gamma D_1$;
2. (X,τ) has no α_γ-neat point.

Proof. (1) $\Rightarrow$ (2): Since (X,τ) is $\alpha_\gamma D_1$, then each point x of X is contained in an $\alpha_\gamma D$-set $A = U \setminus V$ and thus in U. By definition $U \neq X$. This implies that x is not an α_γ-neat point.

(2) $\Rightarrow$ (1): If X is $\alpha_\gamma T_0$, then for each distinct pair of points $x, y \in X$, at least one of them, x (say) has an α_γ-neighbourhood U containing x and not y. Thus, U which is different from X is an $\alpha_\gamma D$-set. If X has no α_γ-neat point, then y is not an α_γ-neat point. This means that there exists an α_γ-neighbourhood V of y such that $V \neq X$. Thus, $y \in V \setminus U$ but not x and $V \setminus U$ is an $\alpha_\gamma D$-set. Hence, X is $\alpha_\gamma D_1$.

Corollary 3.24. An $\alpha_\gamma T_0$ space X is not $\alpha_\gamma D_1$ if and only if there is a unique α_γ-neat point in X.

Proof. We only prove the uniqueness of the α_γ-neat point. If x and y are two α_γ-neat points in X, then since X is $\alpha_\gamma T_0$, at least one of x and y, say x, has an α_γ-neighbourhood U containing x but not y. Hence $U \neq X$. Therefore x is not an α_γ-neat point which is a contradiction.

Definition 3.25. A topological space (X, τ) with an operation γ on $\alpha O(X)$, is said to be α_γ-symmetric if for x and y in X, $x \in \alpha_\gamma Cl(\{y\})$ implies $y \in \alpha_\gamma Cl(\{x\})$.

Proposition 3.26. If (X, τ) is a topological space with an operation γ on $\alpha O(X)$, then the following are equivalent:

1. X is an α_γ-symmetric space;
2. $\{x\}$ is α_γ-g.closed, for each $x \in X$.

Proof. (1) $\Rightarrow$ (2): Assume that $\{x\} \subseteq U \in \alpha O(X)_\gamma$, but $\alpha_\gamma Cl(\{x\}) \nsubseteq U$. Then $\alpha_\gamma Cl(\{x\}) \cap X \setminus U \neq \phi$. Now, we take $y \in \alpha_\gamma Cl(\{x\}) \cap X \setminus U$, then by hypothesis $x \in \alpha_\gamma Cl(\{y\}) \subseteq X \setminus U$ and $x \notin U$, which is a contradiction. Therefore $\{x\}$ is α_γ-g.closed, for each $x \in X$.

(2) $\Rightarrow$ (1): Assume that $x \in \alpha_\gamma Cl(\{y\})$, but $y \notin \alpha_\gamma Cl(\{x\})$. Then $\{y\} \subseteq X \setminus \alpha_\gamma Cl(\{x\})$ and hence $\alpha_\gamma Cl(\{y\}) \subseteq X \setminus \alpha_\gamma Cl(\{x\})$. Therefore $x \in X \setminus \alpha_\gamma Cl(\{x\})$, which is a contradiction and hence $y \in \alpha_\gamma Cl(\{x\})$.

Proposition 3.27. If a topological space (X, τ) is α_γ-symmetric, then $\{x\}$ is α-γ-g.closed, for each $x \in X$.

Proof. Follows from Theorem 2.3 and Proposition 3.26.

Corollary 3.28. If a topological space (X, τ) with an operation γ on $\alpha O(X)$ is an α-γ-T_1 space, then it is α_γ-symmetric.

Proof. Since every singleton is α_γ-closed according to Theorem 3.6, we have it is α_γ-g.closed. Then by Proposition 3.26, (X, τ) is α_γ-symmetric.

Corollary 3.29. For a topological space (X, τ) with an operation γ on $\alpha O(X)$, the following statements are equivalent:

1. (X, τ) is α_γ-symmetric and $\alpha_\gamma T_0$;
2. (X, τ) is α-γ-T_1.

Proof. By Theorem 3.6, Corollary 3.28 and ([2], Remark 3.7 (1)), it suffices to prove only (1) $\Rightarrow$ (2):

Let $x \neq y$ and as (X, τ) is $\alpha_\gamma T_0$, we may assume that $x \in U \subseteq X \setminus \{y\}$ for some $U \in \alpha O(X)_\gamma$. Then $x \notin \alpha_\gamma Cl(\{y\})$ and hence $y \notin \alpha_\gamma Cl(\{x\})$. There exists an

α_γ-open set V such that $y \in V \subseteq X \setminus \{x\}$ and thus by Theorem 3.6, X is an α-γ-T_1 space.

Proposition 3.30. If (X, τ) is an α_γ-symmetric space with an operation γ on $\alpha O(X)$, then the following statements are equivalent:

1. (X, τ) is an $\alpha_\gamma T_0$ space;
2. (X, τ) is an $\alpha_\gamma T_{\frac{1}{2}}$ space;
3. (X, τ) is an α-γ-T_1 space.

Proof. (1) $\Leftrightarrow$ (3) : Obvious from Corollary 3.29.
(3) $\Rightarrow$ (2) and (2) $\Rightarrow$ (1): Directly from Theorem 3.5 and Theorem 3.9 (4) and (5).

Corollary 3.31. For an α_γ-symmetric space (X, τ), the following are equivalent:

1. (X, τ) is $\alpha_\gamma T_0$;
2. (X, τ) is $\alpha_\gamma D_1$;
3. (X, τ) is α-γ-T_1.

Proof. (1) $\Rightarrow$ (3). Follows from Corollary 3.29.
(3) $\Rightarrow$ (2) $\Rightarrow$ (1): Follows from Theorem 3.6, ([2], Remark 3.7 (2) and (3)) and Proposition 3.17.

Definition 3.32. A topological space (X, τ) with an operation γ on $\alpha O(X)$, is said to be $\alpha_\gamma R_0$ if U is an α_γ-open set and $x \in U$ then $\alpha_\gamma Cl(\{x\}) \subseteq U$.

Proposition 3.33. For a topological space (X, τ) with an operation γ on $\alpha O(X)$, the following properties are equivalent:

1. (X, τ) is $\alpha_\gamma R_0$;
2. For any $F \in \alpha C(X)_\gamma$, $x \notin F$ implies $F \subseteq U$ and $x \notin U$ for some $U \in \alpha O(X)_\gamma$;
3. For any $F \in \alpha C(X)_\gamma$, $x \notin F$ implies $F \cap \alpha_\gamma Cl(\{x\}) = \phi$;
4. For any distinct points x and y of X, either $\alpha_\gamma Cl(\{x\}) = \alpha_\gamma Cl(\{y\})$ or $\alpha_\gamma Cl(\{x\}) \cap \alpha_\gamma Cl(\{y\}) = \phi$.

Proof. (1) ⇒ (2): Let $F \in \alpha C(X)_\gamma$ and $x \notin F$. Then by (1), $\alpha_\gamma Cl(\{x\}) \subseteq X \setminus F$. Set $U = X \setminus \alpha_\gamma Cl(\{x\})$, then U is an α_γ-open set such that $F \subseteq U$ and $x \notin U$.

(2) ⇒ (3): Let $F \in \alpha C(X)_\gamma$ and $x \notin F$. There exists $U \in \alpha O(X)_\gamma$ such that $F \subseteq U$ and $x \notin U$. Since $U \in \alpha O(X)_\gamma$, $U \cap \alpha_\gamma Cl(\{x\}) = \phi$ and $F \cap \alpha_\gamma Cl(\{x\}) = \phi$.

(3) ⇒ (4): Suppose that $\alpha_\gamma Cl(\{x\}) \neq \alpha_\gamma Cl(\{y\})$ for distinct points $x, y \in X$. There exists $z \in \alpha_\gamma Cl(\{x\})$ such that $z \notin \alpha_\gamma Cl(\{y\})$ (or $z \in \alpha_\gamma Cl(\{y\})$ such that $z \notin \alpha_\gamma Cl(\{x\})$). There exists $V \in \alpha O(X)_\gamma$ such that $y \notin V$ and $z \in V$, hence $x \in V$. Therefore, we have $x \notin \alpha_\gamma Cl(\{y\})$. By (3), we obtain $\alpha_\gamma Cl(\{x\}) \cap \alpha_\gamma Cl(\{y\}) = \phi$.

(4) ⇒ (1): Let $V \in \alpha O(X)_\gamma$ and $x \in V$. For each $y \notin V$, $x \neq y$ and $x \notin \alpha_\gamma Cl(\{y\})$. This shows that $\alpha_\gamma Cl(\{x\}) \neq \alpha_\gamma Cl(\{y\})$. By (4), $\alpha_\gamma Cl(\{x\}) \cap \alpha_\gamma Cl(\{y\}) = \phi$ for each $y \in X \setminus V$ and hence $\alpha_\gamma Cl(\{x\}) \cap (\bigcup_{y \in X \setminus V} \alpha_\gamma Cl(\{y\})) = \phi$. On other hand, since $V \in \alpha O(X)_\gamma$ and $y \in X \setminus V$, we have $\alpha_\gamma Cl(\{y\}) \subseteq X \setminus V$ and hence $X \setminus V = \bigcup_{y \in X \setminus V} \alpha_\gamma Cl(\{y\})$. Therefore, we obtain $(X \setminus V) \cap \alpha_\gamma Cl(\{x\}) = \phi$ and $\alpha_\gamma Cl(\{x\}) \subseteq V$. This shows that (X, τ) is an $\alpha_\gamma R_0$ space.

Proposition 3.34. A topological space (X, τ) with an operation γ on $\alpha O(X)$ is α-γ-T_1 if and only if (X, τ) is $\alpha_\gamma T_0$ and $\alpha_\gamma R_0$.

Proof. Necessity: Let U be any α_γ-open set of (X, τ) and $x \in U$. Then by Proposition 3.6, we have $\alpha_\gamma Cl(\{x\}) \subseteq U$ and so by Proposition 3.9 , it is clear that X is $\alpha_\gamma T_0$ and an $\alpha_\gamma R_0$ space.

Sufficiency: Let x and y be any distinct points of X. Since X is $\alpha_\gamma T_0$, there exists an α_γ-open set U such that $x \in U$ and $y \notin U$. As $x \in U$ implies that $\alpha_\gamma Cl(\{x\}) \subseteq U$. Since $y \notin U$, so $y \notin \alpha_\gamma Cl(\{x\})$. Hence $y \in V = X \setminus \alpha_\gamma Cl(\{x\})$ and it is clear that $x \notin V$. Hence it follows that there exist α_γ-open sets U and V containing x and y respectively, such that $y \notin U$ and $x \notin V$. Therefore, by Theorem 3.6 implies that X is α-γ-T_1.

Proposition 3.35. For a topological space (X, τ) with an operation γ on $\alpha O(X)$, the following properties are equivalent:

1. (X, τ) is $\alpha_\gamma R_0$;
2. $x \in \alpha_\gamma Cl(\{y\})$ if and only if $y \in \alpha_\gamma Cl(\{x\})$, for any points x and y in X.

Proof. (1) ⇒ (2): Assume that X is $\alpha_\gamma R_0$. Let $x \in \alpha_\gamma Cl(\{y\})$ and V be any α_γ-open set such that $y \in V$. Now by hypothesis, $x \in V$. Therefore, every α_γ-open set which contain y contains x. Hence, $y \in \alpha_\gamma Cl(\{x\})$.

(2) ⇒ (1): Let U be an α_γ-open set and $x \in U$. If $y \notin U$, then $x \notin \alpha_\gamma Cl(\{y\})$ and hence $y \notin \alpha_\gamma Cl(\{x\})$. This implies that $\alpha_\gamma Cl(\{x\}) \subseteq U$. Hence (X, τ) is $\alpha_\gamma R_0$.

Remark 3.36. From Definition 3.25 and Proposition 3.35, the notions of α_γ-symmetric and $\alpha_\gamma R_0$ are equivalent.

Proposition 3.37. A topological space (X, τ) with an operation γ on $\alpha O(X)$ is $\alpha_\gamma R_0$ if and only if for every x and y in X, $\alpha_\gamma Cl(\{x\}) \neq \alpha_\gamma Cl(\{y\})$ implies $\alpha_\gamma Cl(\{x\}) \cap \alpha_\gamma Cl(\{y\}) = \phi$.

Proof. Necessity: Suppose that (X, τ) is $\alpha_\gamma R_0$ and $x, y \in X$ such that $\alpha_\gamma Cl(\{x\}) \neq \alpha_\gamma Cl(\{y\})$. Then, there exists $z \in \alpha_\gamma Cl(\{x\})$ such that $z \notin \alpha_\gamma Cl(\{y\})$ (or $z \in \alpha_\gamma Cl(\{y\})$ such that $z \notin \alpha_\gamma Cl(\{x\})$). There exists $V \in \alpha O(X)_\gamma$ such that $y \notin V$ and $z \in V$, hence $x \in V$. Therefore, we have $x \notin \alpha_\gamma Cl(\{y\})$. Thus $x \in [X \setminus \alpha_\gamma Cl(\{y\})] \in \alpha O(X)_\gamma$, which implies $\alpha_\gamma Cl(\{x\}) \subseteq [X \setminus \alpha_\gamma Cl(\{y\})]$ and $\alpha_\gamma Cl(\{x\}) \cap \alpha_\gamma Cl(\{y\}) = \phi$.

Sufficiency: Let $V \in \alpha O(X)_\gamma$ and $x \in V$. We show that $\alpha_\gamma Cl(\{x\}) \subseteq V$. Let $y \notin V$, that is $y \in X \setminus V$. Then $x \neq y$ and $x \notin \alpha_\gamma Cl(\{y\})$. This shows that $\alpha_\gamma Cl(\{x\}) \neq \alpha_\gamma Cl(\{y\})$. By assumption, $\alpha_\gamma Cl(\{x\}) \cap \alpha_\gamma Cl(\{y\}) = \phi$. Hence $y \notin \alpha_\gamma Cl(\{x\})$ and therefore $\alpha_\gamma Cl(\{x\}) \subseteq V$.

Proposition 3.38. A topological space (X, τ) with an operation γ on $\alpha O(X)$ is $\alpha_\gamma R_0$ if and only if for any points x and y in X, $\alpha_\gamma ker(\{x\}) \neq \alpha_\gamma ker(\{y\})$ implies $\alpha_\gamma ker(\{x\}) \cap \alpha_\gamma ker(\{y\}) = \phi$.

Proof. Suppose that (X, τ) is an $\alpha_\gamma R_0$ space. Thus by Proposition 2.4, for any points x and y in X if $\alpha_\gamma ker(\{x\}) \neq \alpha_\gamma ker(\{y\})$ then $\alpha_\gamma Cl(\{x\}) \neq \alpha_\gamma Cl(\{y\})$. Now we prove that $\alpha_\gamma ker(\{x\}) \cap \alpha_\gamma ker(\{y\}) = \phi$. Assume that $z \in \alpha_\gamma ker(\{x\}) \cap \alpha_\gamma ker(\{y\})$. By $z \in \alpha_\gamma ker(\{x\})$ and Proposition 2.5, it follows that $x \in \alpha_\gamma Cl(\{z\})$. Since $x \in \alpha_\gamma Cl(\{x\})$, by Proposition 3.33, $\alpha_\gamma Cl(\{x\}) = \alpha_\gamma Cl(\{z\})$. Similarly, we have $\alpha_\gamma Cl(\{y\}) = \alpha_\gamma Cl(\{z\}) = \alpha_\gamma Cl(\{x\})$. This is a contradiction. Therefore, we have $\alpha_\gamma ker(\{x\}) \cap \alpha_\gamma ker(\{y\}) = \phi$.

Conversely, let (X, τ) be a topological space such that for any points x and y in X, $\alpha_\gamma ker(\{x\}) \neq \alpha_\gamma ker(\{y\})$ implies $\alpha_\gamma ker(\{x\}) \cap \alpha_\gamma ker(\{y\}) = \phi$. If $\alpha_\gamma Cl(\{x\}) \neq \alpha_\gamma Cl(\{y\})$, then by Proposition 2.4, $\alpha_\gamma ker(\{x\}) \neq \alpha_\gamma ker(\{y\})$. Hence, $\alpha_\gamma ker(\{x\}) \cap \alpha_\gamma ker(\{y\}) = \phi$ which implies $\alpha_\gamma Cl(\{x\}) \cap \alpha_\gamma Cl(\{y\}) = \phi$. Because $z \in \alpha_\gamma Cl(\{x\})$ implies that $x \in \alpha_\gamma ker(\{z\})$ and therefore $\alpha_\gamma ker(\{x\}) \cap \alpha_\gamma ker(\{z\}) \neq \phi$. By hypothesis, we have $\alpha_\gamma ker(\{x\}) = \alpha_\gamma ker(\{z\})$. Then $z \in \alpha_\gamma Cl(\{x\}) \cap \alpha_\gamma Cl(\{y\})$ implies that $\alpha_\gamma ker(\{x\}) = \alpha_\gamma ker(\{z\}) = \alpha_\gamma ker(\{y\})$. This is a contradiction. Therefore, $\alpha_\gamma Cl(\{x\}) \cap \alpha_\gamma Cl(\{y\}) = \phi$ and by Proposition 3.33, (X, τ) is an $\alpha_\gamma R_0$ space.

Proposition 3.39. For a topological space (X, τ) with an operation γ on $\alpha O(X)$, the following properties are equivalent:

1. (X, τ) is $\alpha_\gamma R_0$;
2. For any non-empty set A and $G \in \alpha O(X)_\gamma$ such that $A \cap G \neq \phi$, there exists $F \in \alpha C(X)_\gamma$ such that $A \cap F \neq \phi$ and $F \subseteq G$;

3. For any $G \in \alpha O(X)_\gamma$, we have $G = \cup\{F \in \alpha C(X)_\gamma \colon F \subseteq G\}$;

4. For any $F \in \alpha C(X)_\gamma$, we have $F = \cap\{G \in \alpha O(X)_\gamma \colon F \subseteq G\}$;

5. For every $x \in X$, $\alpha_\gamma Cl(\{x\}) \subseteq \alpha_\gamma ker(\{x\})$.

Proof. (1) $\Rightarrow$ (2): Let A be a non-empty subset of X and $G \in \alpha O(X)_\gamma$ such that $A \cap G \neq \phi$. There exists $x \in A \cap G$. Since $x \in G \in \alpha O(X)_\gamma$ implies that $\alpha_\gamma Cl(\{x\}) \subseteq G$. Set $F = \alpha_\gamma Cl(\{x\})$, then $F \in \alpha C(X)_\gamma$, $F \subseteq G$ and $A \cap F \neq \phi$.

(2) $\Rightarrow$ (3): Let $G \in \alpha O(X)_\gamma$, then $G \supseteq \cup\{F \in \alpha C(X)_\gamma \colon F \subseteq G\}$. Let x be any point of G. There exists $F \in \alpha C(X)_\gamma$ such that $x \in F$ and $F \subseteq G$. Therefore, we have $x \in F \subseteq \cup\{F \in \alpha C(X)_\gamma \colon F \subseteq G\}$ and hence $G = \cup\{F \in \alpha C(X)_\gamma \colon F \subseteq G\}$.

(3) $\Rightarrow$ (4): Obvious.

(4) $\Rightarrow$ (5): Let x be any point of X and $y \notin \alpha_\gamma ker(\{x\})$. There exists $V \in \alpha O(X)_\gamma$ such that $x \in V$ and $y \notin V$, hence $\alpha_\gamma Cl(\{y\}) \cap V = \phi$. By (4), $(\cap\{G \in \alpha O(X)_\gamma \colon \alpha_\gamma Cl(\{y\}) \subseteq G\}) \cap V = \phi$ and there exists $G \in \alpha O(X)_\gamma$ such that $x \notin G$ and $\alpha_\gamma Cl(\{y\}) \subseteq G$. Therefore $\alpha_\gamma Cl(\{x\}) \cap G = \phi$ and $y \notin \alpha_\gamma Cl(\{x\})$. Consequently, we obtain $\alpha_\gamma Cl(\{x\}) \subseteq \alpha_\gamma ker(\{x\})$.

(5) $\Rightarrow$ (1): Let $G \in \alpha O(X)_\gamma$ and $x \in G$. Let $y \in \alpha_\gamma ker(\{x\})$, then $x \in \alpha_\gamma Cl(\{y\})$ and $y \in G$. This implies that $\alpha_\gamma ker(\{x\}) \subseteq G$. Therefore, we obtain $x \in \alpha_\gamma Cl(\{x\}) \subseteq \alpha_\gamma ker(\{x\}) \subseteq G$. This shows that X is an $\alpha_\gamma R_0$ space.

Corollary 3.40. For a topological space (X, τ) with an operation γ on $\alpha O(X)$, the following properties are equivalent:

1. (X, τ) is $\alpha_\gamma R_0$;

2. $\alpha_\gamma Cl(\{x\}) = \alpha_\gamma ker(\{x\})$ for all $x \in X$.

Proof. (1) $\Rightarrow$ (2): Suppose that X is an $\alpha_\gamma R_0$ space. By Proposition 3.39, $\alpha_\gamma Cl(\{x\}) \subseteq \alpha_\gamma ker(\{x\})$ for each $x \in X$. Let $y \in \alpha_\gamma ker(\{x\})$, then $x \in \alpha_\gamma Cl(\{y\})$ and by Proposition 3.33, $\alpha_\gamma Cl(\{x\}) = \alpha_\gamma Cl(\{y\})$. Therefore, $y \in \alpha_\gamma Cl(\{x\})$ and hence $\alpha_\gamma ker(\{x\}) \subseteq \alpha_\gamma Cl(\{x\})$. This shows that $\alpha_\gamma Cl(\{x\}) = \alpha_\gamma ker(\{x\})$.

(2) $\Rightarrow$ (1): Follows from Proposition 3.39.

Proposition 3.41. For a topological space (X, τ) with an operation γ on $\alpha O(X)$, the following properties are equivalent:

1. (X, τ) is $\alpha_\gamma R_0$;

2. If F is α_γ-closed, then $F = \alpha_\gamma ker(F)$;

3. If F is α_γ-closed and $x \in F$, then $\alpha_\gamma ker(\{x\}) \subseteq F$;

4. If $x \in X$, then $\alpha_\gamma ker(\{x\}) \subseteq \alpha_\gamma Cl(\{x\})$.

Proof. (1) $\Rightarrow$ (2): Let F be an α_γ-closed and $x \notin F$. Thus $(X \setminus F)$ is an α_γ-open set containing x. Since (X, τ) is $\alpha_\gamma R_0$, $\alpha_\gamma Cl(\{x\}) \subseteq (X \setminus F)$. Thus $\alpha_\gamma Cl(\{x\}) \cap F = \phi$ and by Proposition 2.6, $x \notin \alpha_\gamma ker(F)$. Therefore $\alpha_\gamma ker(F) = F$.

(2) $\Rightarrow$ (3): In general, $A \subseteq B$ implies $\alpha_\gamma ker(A) \subseteq \alpha_\gamma ker(B)$. Therefore, it follows from (2), that $\alpha_\gamma ker(\{x\}) \subseteq \alpha_\gamma ker(F) = F$.

(3) $\Rightarrow$ (4): Since $x \in \alpha_\gamma Cl(\{x\})$ and $\alpha_\gamma Cl(\{x\})$ is α_γ-closed, by (3), $\alpha_\gamma ker(\{x\}) \subseteq \alpha_\gamma Cl(\{x\})$.

(4) $\Rightarrow$ (1): We show the implication by using Proposition 3.35. Let $x \in \alpha_\gamma Cl(\{y\})$. Then by Proposition 2.5, $y \in \alpha_\gamma ker(\{x\})$. Since $x \in \alpha_\gamma Cl(\{x\})$ and $\alpha_\gamma Cl(\{x\})$ is α_γ-closed, by (4), we obtain $y \in \alpha_\gamma ker(\{x\}) \subseteq \alpha_\gamma Cl(\{x\})$. Therefore $x \in \alpha_\gamma Cl(\{y\})$ implies $y \in \alpha_\gamma Cl(\{x\})$. The converse is obvious and (X, τ) is $\alpha_\gamma R_0$.

Definition 3.42. A topological space (X, τ) with an operation γ on $\alpha O(X)$, is said to be $\alpha_\gamma R_1$ if for x, y in X with $\alpha_\gamma Cl(\{x\}) \neq \alpha_\gamma Cl(\{y\})$, there exist disjoint α_γ-open sets U and V such that $\alpha_\gamma Cl(\{x\}) \subseteq U$ and $\alpha_\gamma Cl(\{y\}) \subseteq V$.

Proposition 3.43. A topological space (X, τ) with an operation γ on $\alpha O(X)$ is $\alpha_\gamma R_1$ if it is $\alpha_\gamma T_2$.

Proof. Let x and y be any points of X such that $\alpha_\gamma Cl(\{x\}) \neq \alpha_\gamma Cl(\{y\})$. By ([2], Remark 3.7 (1)), every $\alpha_\gamma T_2$ space is $\alpha_\gamma T_1$. Therefore, by Theorem 3.6, $\alpha_\gamma Cl(\{x\}) = \{x\}$, $\alpha_\gamma Cl(\{y\}) = \{y\}$ and hence $\{x\} \neq \{y\}$. Since (X, τ) is $\alpha_\gamma T_2$, there exist disjoint α_γ-open sets U and V such that $\alpha_\gamma Cl(\{x\}) = \{x\} \subseteq U$ and $\alpha_\gamma Cl(\{y\}) = \{y\} \subseteq V$. This shows that (X, τ) is $\alpha_\gamma R_1$.

Proposition 3.44. For a topological space (X, τ) with an operation γ on $\alpha O(X)$, the following are equivalent:

1. (X, τ) is $\alpha_\gamma T_2$;
2. (X, τ) is $\alpha_\gamma R_1$ and $\alpha_\gamma T_1$;
3. (X, τ) is $\alpha_\gamma R_1$ and $\alpha_\gamma T_0$.

Proof. Straightforward.

Proposition 3.45. For a topological space (X, τ) with an operation γ on $\alpha O(X)$, the following statements are equivalent:

1. (X, τ) is $\alpha_\gamma R_1$;

2. If $x, y \in X$ such that $\alpha_\gamma Cl(\{x\}) \neq \alpha_\gamma Cl(\{y\})$, then there exist α_γ-closed sets F_1 and F_2 such that $x \in F_1$, $y \notin F_1$, $y \in F_2$, $x \notin F_2$ and $X = F_1 \cup F_2$.

Proof. Obvious.

Proposition 3.46. If (X, τ) is $\alpha_\gamma R_1$, then (X, τ) is $\alpha_\gamma R_0$.

Proof. Let U be α_γ-open such that $x \in U$. If $y \notin U$, since $x \notin \alpha_\gamma Cl(\{y\})$, we have $\alpha_\gamma Cl(\{x\}) \neq \alpha_\gamma Cl(\{y\})$. So, there exists an α_γ-open set V such that $\alpha_\gamma Cl(\{y\}) \subseteq V$ and $x \notin V$, which implies $y \notin \alpha_\gamma Cl(\{x\})$. Hence $\alpha_\gamma Cl(\{x\}) \subseteq U$. Therefore, (X, τ) is $\alpha_\gamma R_0$.

Remark 3.47. The converse of the above proposition need not be ture in general as shown in the following example.

Example 3.48. Consider $X = \{1, 2, 3\}$ with the discrete topology τ on X. Define an operation γ on $\alpha O(X)$ by

$$A^\gamma = \begin{cases} A & \text{if } A = \{1,2\} \text{ or } \{1,3\} \text{ or } \{2,3\} \\ X & \text{otherwise.} \end{cases}$$

Then, X is an $\alpha_\gamma R_0$ space but not $\alpha_\gamma R_1$.

Corollary 3.49. A topological space (X, τ) with an operation γ on $\alpha O(X)$ is $\alpha_\gamma R_1$ if and only if for $x, y \in X$, $\alpha_\gamma ker(\{x\}) \neq \alpha_\gamma ker(\{y\})$, there exist disjoint α_γ-open sets U and V such that $\alpha_\gamma Cl(\{x\}) \subseteq U$ and $\alpha_\gamma Cl(\{y\}) \subseteq V$.

Proof. Follows from Proposition 2.4.

Proposition 3.50. A topological space (X, τ) is $\alpha_\gamma R_1$ if and only if $x \in X \setminus \alpha_\gamma Cl(\{y\})$ implies that x and y have disjoint α_γ-open neighbourhoods.

Proof. Necessity: Let $x \in X \setminus \alpha_\gamma Cl(\{y\})$. Then $\alpha_\gamma Cl(\{x\}) \neq \alpha_\gamma Cl(\{y\})$, so, x and y have disjoint α_γ-open neighbourhoods.

Sufficiency: Firstly, we show that (X, τ) is $\alpha_\gamma R_0$. Let U be an α_γ-open set and $x \in U$. Suppose that $y \notin U$. Then, $\alpha_\gamma Cl(\{y\}) \cap U = \phi$ and $x \notin \alpha_\gamma Cl(\{y\})$. There exist α_γ-open sets U_x and U_y such that $x \in U_x$, $y \in U_y$ and $U_x \cap U_y = \phi$. Hence, $\alpha_\gamma Cl(\{x\}) \subseteq \alpha_\gamma Cl(U_x)$ and $\alpha_\gamma Cl(\{x\}) \cap U_y \subseteq \alpha_\gamma Cl(U_x) \cap U_y = \phi$. Therefore, $y \notin \alpha_\gamma Cl(\{x\})$. Consequently, $\alpha_\gamma Cl(\{x\}) \subseteq U$ and (X, τ) is $\alpha_\gamma R_0$. Next, we show that (X, τ) is $\alpha_\gamma R_1$. Suppose that $\alpha_\gamma Cl(\{x\}) \neq \alpha_\gamma Cl(\{y\})$. Then, we can assume that there exists $z \in \alpha_\gamma Cl(\{x\})$ such that $z \notin \alpha_\gamma Cl(\{y\})$. There exist α_γ-open sets V_z and V_y such that $z \in V_z$, $y \in V_y$ and $V_z \cap V_y = \phi$. Since $z \in \alpha_\gamma Cl(\{x\})$, $x \in V_z$. Since (X, τ) is $\alpha_\gamma R_0$, we obtain $\alpha_\gamma Cl(\{x\}) \subseteq V_z$, $\alpha_\gamma Cl(\{y\}) \subseteq V_y$ and $V_z \cap V_y = \phi$. This shows that (X, τ) is $\alpha_\gamma R_1$.

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