Acta Universitatis Apulensis No. 47/2016
ISSN: 1582-5329 pp. 73-88
http://www.uab.ro/auajournal/ doi: 10.17114/j.aua.2016.47.06

PROPERTIES OF CERTAIN SUBCLASSES OF CLOSE-TO-CONVEX FUNCTIONS

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Abstract. We define the subclass of close-to-convex functions by using the concept of convolution and subordination and it is shown that functions in this class are univalent. Several interesting properties such as radius problems, coefficient bounds, invariance property under convex convolution and integral representation. We also study these properties of functions belonging to newly defined class under certain univalent integral operators.

2010 *Mathematics Subject Classification*: Primary 30C45 ; Secondary 30C50

Keywords: univalent functions, close-to-convex, starlike functions, convolution operator, subordination.

1. Introduction

Let A denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disc $E = \{z : |z| < 1\}$. Let S, C, S^*, K be the subclasses of A of univalent, convex, starlike, close-to-convex functions. For $f(z)$ given in (1.1) and

$$g(z) = \sum_{n=0}^{\infty} b_n z^n,$$

the Hadamard product (or convolution) is defined by

$$(f * g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n.$$

We define the subclasses C^* and K^* of K as follows.

$$(i)\ \ C^* = \left\{ f \in A : Re\frac{(zf'(z))'}{g'(z)} > 0,\ g \in C,\ z \in E \right\},$$

$$(ii)\ \ K^* = \left\{ f \in A : Re\frac{(zf'(z))'}{G'(z)} > 0,\ G \in S^*,\ z \in E \right\}.$$

In [4], Janowski introduced the class $P[A, B]$ if $p(z)$ subordinate to $\frac{1+Az}{1+Bz}$. When $A = 1, B = -1$, we obtain the class P with positive real part in E. Also for $A = 1 - 2\gamma, B = -1, 0 \leq \gamma < 1$, we have the class $P(\gamma)$. A function $h \in P(\gamma), 0 \leq \gamma < 1$, we have the class $P(\gamma)$, $0 \leq \gamma < 1$ if and only if $Re(h(z) > \beta$, $z \in E$.
Let incomplete beta function $\phi(a, c; z)$, (see [7]), defined by

$$\phi(a, c; z) = z = \sum_{n=2}^{\infty} \frac{(a)_{n-1}}{(c)_{n-1}} z^n, z \in E, \quad (c \neq 0, -1, -2, ..),$$

where $(a)_n$ is Poshhammer symbol defined in term of Gamma function Γ, by

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(n)} \quad = \quad \begin{cases} 1, & n = 0, \\ n(n+1)(n+2)....(a+n-1), & n \in N. \end{cases}$$

For $f \in A$ defined by (1.1), we consider the following linear operators, see [5], [17].

$$L_\beta^\alpha f(z) = \left[\sum_{n=2}^{\infty} \left(\frac{\beta+1}{\beta+n} \right)^\alpha z^n \right] * f(z), \quad (\alpha \geq 0, \beta > -1), \tag{1.2}$$

$$I_\beta^\alpha f(z) = \binom{\alpha+\beta}{\beta} \phi(\beta, \alpha+\beta; z) * f(z), \quad (\alpha \geq 0, \beta > -1). \tag{1.3}$$

where Γ denotes the Gamma function and

$$\binom{\alpha}{\beta} = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-\beta+1)\Gamma(\beta+1)}.$$

By virtue of (1.2) and (1.3), we see that

$$z(L_\beta^\alpha f(z))' = (\beta+1) L_\beta^{\alpha-1} f(z) - \beta L_\beta^\alpha f(z), \quad (\alpha \geq 0) \tag{1.4}$$

$$z(I_\beta^\alpha f(z))' = (\beta+\alpha) I_\beta^{\alpha-1} f(z) - (\alpha+\beta-1) I_\beta^\alpha f(z), \quad (\alpha \geq 0) \tag{1.5}$$

Definition 1. *[9] Let $f \in A$. Then $f \in S^*(h)$, if and only if*

$$\frac{zf'(z)}{f(z)} \prec h(z),$$

where h is analytic, univalent and convex in E with $h(0) = 1$.

The corresponding class of $C(h)$ is defined as follows.

$$f \in C(h) \Leftrightarrow zf' \in S^*(h), \quad z \in E.$$

Now we define the following.

Definition 2. *Let f, g, $\phi \in A$. Let for $F(z) = f(z) * \phi(z)$, $G(z) = g(z) * \phi(z)$ and λ be the complex with $Re(\lambda) \geq 0$, $G \in S^*(h)$ such that*

$$(1-\lambda)\frac{zF'(z)}{G(z)} + \lambda\frac{(zF'(z))'}{G'(z)} \prec h(z), \tag{1.6}$$

where $h(z)$ is analytic, univalent and convex in E with $h(0) = 1$. Then the function f is said to be in the class $Q(\phi, g, h, \lambda)$.

We note that, for $\lambda = 0$, the the class $Q(\phi, g, h, \lambda)$ reduces to the new class $K(\phi, h)$ for close-to-convex univalent functions, when $\lambda = 1$, we have the new class $K^*(\phi, h)$ of univalent functions, and by assigning particular values to the parameters ϕ, g, h, λ in Definition 1.2, we obtain the several known classes investigated by several authers, for example, see [15], [12], [13], [14], [3], [10], [11], [16].

For $g = f$ in Definition 1.2 leads us the new class $Q(\phi, f, h, \lambda)$ of $\alpha-$convex univalent functions defined as:
Let f, g, $\phi \in A$, $F = \phi * f$, $G = \phi * g$, $\lambda \geq 0$. Then f is said be in class $Q(\phi, f, h, \lambda)$ if it satisfies the condition given as:

$$(1-\lambda)\frac{zF'(z)}{F(z)} + \lambda\frac{(zF'(z))'}{F'(z)} \prec h(z),$$

where $h(z)$ is univalent, analytic and convex with $h(0) = 1$ in E.

2. Preliminary Concepts

In order to derive our main results, we require the following lemmas.

Lemma 1. *[8] Let P be a complex function in E, with $Re(P(z)) > 0$ for $z \in E$ and h be a convex function in E. If $p(z)$ be a analytic function in E,with $p(0) = h(0)$ then,*

$$p(z) + P(z)zp'(z) \prec h(z), \tag{2.1}$$

implies that $p(z) \prec h(z)$.

Lemma 2. *[8] Let h be analytic, univalent, convex in E with $h(0) = 1$ and $Re\{\sigma_1 h(z) + \sigma_2\} > 0$, $\sigma_2 \in C$, $z \in E$. If $p(z)$ analytic with $p(0) = h(0)$, then*

$$p(z) + \frac{zp'(z)}{\sigma_1 p(z) + \sigma_2} \prec h(z), \tag{2.2}$$

implies $p(z) \prec q(z) \prec h(z)$, *where* $q(z)$ *is best dominant and is given as*

$$q(z) = \left[\left\{ \int_0^1 exp \int_0^z \frac{h(u)-1}{u} du \quad dt \right\}^{-1} - \frac{\sigma_2}{\sigma_1} \right].$$

Lemma 3. *[20]Let* $p(z)$ *be analytic in* E *with* $p(0) = 1$, *then for any function* F *analytic in* E, *the function* $p * F$ *takes values in the convex hull of image of image of* E *under* F.

Lemma 4. *[19] Let* $f \in C$ *and* $g(z) \in S^*$, *then for analytic function in* E *with* $F(0) = 1$,

$$\frac{f * Fg}{f * g}(E) \subset \overline{C_0}F(E), \quad f \in C, \quad g \in S^*, \tag{2.3}$$

where $\overline{C_0}F(E)$ *denotes the closed convex hull of* $F(E)$ *(the smallest convex set which contains* $F(E)$.

Lemma 5. *[7] Let* $u = u_1 + iu_2$, $v = v_1 + iv_2$, *and let* $\psi(u, v) : D \subset C^2 \to C$ *be complex-valued function satisfying the conditions:*
(i) $\psi(u, v)$ *is continuous in* D,
(ii) $(1, 0) \in D$ *and* $\psi(1, 0) > 0$,
(iii) $Re\{\psi(iu_2, v_1)\} \leq 0$ *whenever* $(iu_2, v_1) \in D$ *and* $v_1 \leq \frac{-1}{2}(1 + iu_2^2)$.
If $h(z) = 1+c_1z+c_2z^2+...$ *is a function that is analytic in* E *such that* $Re\{\psi(h(z), zh'(z)\} > 0$, *for* $z \in E$, *then* $Re\{h(z)\} > 0$.

3. Main Results

Theorem 6. *For* $\alpha \geq 0$

$$Q(\phi, g, h, \lambda) \subset K(\phi, h)$$

Proof. Let $f \in Q(\phi, g, h, \lambda)$. Then by Definition 1.2, for $\lambda \geq 0$, $F(z) = \phi(z) * F(z)$ and $G(z) = \phi(z) * g(z)$, we have

$$(1-\lambda)\frac{zF'(z)}{G(z)} + \lambda\frac{(zF'(z))'}{G'(z)} \prec h(z),$$

where $h(z)$ is analytic, univalent and convex with $h(0) = 1$
Consider

$$\frac{zF'(z)}{G(z)} = p(z) \tag{3.1}$$

where $p(z)$ is analytic with $p(0) = 1$ in E.
Simple calculation yields us

$$(1-\lambda)\frac{zF'(z)}{G(z)} + \lambda\frac{(zF'(z))'}{G'(z)} = p(z) + \lambda\frac{zp'(z)}{p_0(z)}, p_0 = \frac{zG'(z)}{G(z)}. \tag{3.2}$$

Since

$$\frac{zG'(z)}{G(z)} \prec h(z), h(z) \in P,$$

so we have $\frac{zG'(z)}{G(z)} \in P$. Let $h_0(z) = \frac{1}{p_0(z)}$. Thus $h_0(z) \in P$, so we have from (3.1) and (3.2), that

$$(1-\lambda)\frac{zF'(z)}{G(z)} + \lambda\frac{(zF'(z))'}{G'(z)} = p(z) + \lambda h_0(z) z p'(z), \tag{3.3}$$

Now as given $f \in Q(\phi, g, h, \lambda)$ and therefore an application of Lemma 2.1 leads $p(z) \prec h(z)$, which implies that $f \in K(\phi, h)$. This completes the proof.

We note the following special cases.

Corollary 7. *Let $\lambda \neq 0$, $f \in Q(\phi, z, h, \lambda)$ it follows from Theorem 3.1 that*

$$\{p(z) + \lambda z p'(z)\} \prec h(z), h \in P,$$

using Lemma 2.1, we obtain $p \prec h$ and it follows that $f'(z) \prec h(z)$ in E.

Corollary 8. *Let $\lambda \geq 1$, Then*

$$Q(\phi, g, h, \lambda) \subset K^*(\phi, h).$$

Proof. For $F = \phi * f, G = \phi * g, G \in S^*(h), \lambda \geq 0$. Consider

$$\lambda\frac{zF(z))}{G'(z)} = \left\{(1-\lambda)\frac{zF'(z)}{G(z)} + \lambda\frac{(zF'(z))'}{G'(z)}\right\} + (1-\lambda)\frac{zF'(z)}{G(z)}.$$

Simple calculation leads us

$$\frac{(zF'(z))'}{G'(z)} = \frac{1}{\lambda}p(z) + (1-\lambda)p_1(z) = H(z),$$

where $p_1(z) \prec h(z)$, by Theorem 3.1 and $p \in P$, since $f \in Q(\phi, g, h, \lambda)$. Thus, it follows that the class P is convex, so $H \prec h$ in E, which is the required result.

Using the similar techniques, we have the following corollary.

Corollary 9. *When $\phi(z) = \psi_\beta^\alpha(z)$, and $h(z) = \frac{1+Az}{1+Bz}$, $(-1 \leq A < B \leq 1)$ in Theorem 3.1, ψ_β^α given by*

$$\psi_\beta^\alpha(z) = \binom{\alpha+\beta}{\beta}\phi(\beta, \alpha+\beta; z), \tag{3.4}$$

we have $F(z) = I_\beta^\alpha f(z) = (\psi_\beta^\alpha * f)(z)$, and $G(z) = I_\beta^\alpha g(z) = (\psi_\beta^\alpha * g)(z) \in S^*\left(\frac{1+Az}{1+Bz}\right)$, $(g \in A, \alpha \geq 0, \beta > -1)$. It follows that, for $\lambda \geq 0$, $f \in Q(\psi_\beta^\alpha, g, \frac{1+Az}{1+Bz}, \lambda)$ implies that

$$\left\{\frac{z(I_\beta^\alpha f(z))'}{I_\beta^\alpha g(z)}\right\} \prec \frac{1+Az}{1+Bz}, z \in E.$$

Proof. Let
$$\frac{z(I_\beta^\alpha f(z))'}{I_\beta^\alpha g(z)} = p_1(z),$$
where $p_1(z)$ is analytic with $p_1(0) = 1$ in E. From (3.2), it follows that
$$(1-\lambda)\frac{z(I_\beta^\alpha f(z))'}{(I_\beta^\alpha g(z))} + \lambda\frac{(z(I_\beta^\alpha f(z))')'}{(I_\beta^\alpha g(z))'} = p_1(z) + \lambda\frac{zp_1{}'(z)}{p_0(z)}, \tag{3.5}$$
where $p_0 = \frac{z(I_\beta^\alpha f(z))'}{I_\beta^\alpha g(z)}$ and since $(I_\beta^\alpha g) \in S^*\left(\frac{1+Az}{1+Bz}\right) \subset S^*$, so $Re(p_0(z)) > 0$. Let $h_2(z) = \frac{1}{p_2(z)}$. Then $h_2 \in P$, so from (3.5), we have
$$H_2(z) = (1-\lambda)\frac{z(I_\beta^\alpha f(z))'}{I_\beta^\alpha g(z)} + \lambda\frac{(z(I_\beta^\alpha f(z))')'}{(I_\beta^\alpha g(z))'} = p_1(z) + \lambda h_2(z)zp_1{}'(z).$$
Since $f \in Q(\psi_\beta^\alpha, g, h, \lambda)$, so $H_2(z) \prec \frac{1+Az}{1+Bz}$. Hence Lemma 2.1 implies that $p_1(z) \prec \frac{1+Az}{1+Bz}$, that is
$$\frac{z(I_\beta^\alpha f(z))'}{I_\beta^\alpha g(z)} \prec \frac{1+Az}{1+Bz}, \quad z \in E,$$
which is the required result.

Theorem 10. *Let $f, g, \phi \in A$, $\lambda \geq 0$, for $F = \phi * f$ and $G = \phi * g, G \in S^*(h)$. Then $f \in Q(\phi, g, h, \lambda)$, if there exists a function $R = \phi * \mu, \mu \in A$, $R \in C(h)$ such that a function $\eta(z)$ defined by*
$$\eta'(z) = \frac{(zF'(z))'}{1 + \dfrac{zR''(z)}{R'(z)}},$$
belongs to the class $K(\phi, h)$, for $z \in E$.

Proof. Since $f \in Q(\phi, g, h, \lambda)$, for $F = \phi * f$, $G = \phi * g$, $G \in S^*(h)$, we have from Corollary 3.2 that
$$\left\{\frac{(zF'(z))'}{G'(z)}\right\} \prec h(z), G \in S^*(h), \tag{3.6}$$
$h(z)$ is analytic, convex univalent with $h(0) = 1$ in E. Let $G(z) = zR'(z)$. Then we have $R(z) \in S^*(h)$. Now consider
$$\begin{aligned} G'(z) &= (zR'(z))' \\ &= R'(z)\left\{1 + \frac{zR''(z)}{R'(z)}\right\}. \end{aligned}$$
Thus, we have
$$\begin{aligned} \frac{(zF'(z))'}{G'(z)} &= \frac{(zF'(z))'}{R'(z)\left\{1 + \dfrac{zR''(z)}{R'(z)}\right\}} \\ &= \frac{\eta'(z)}{R'(z)} \end{aligned} \tag{3.7}$$

from (3.6) and (3.7), we have $\eta(z) \in K$, that is $\left\{ \dfrac{\eta'(z)}{R'(z)} \right\} \prec h(z)$ in E. This completes the proof.

Corollary 11. *[14] We take* $h(z) = \dfrac{1+Az}{1+Bz} \quad (-1 \leq B \leq A \leq 1)$, $\phi(z) = \frac{z}{1-z}$, *and* $\lambda = 1$ *in Theorem 3.2. It follows that* $f \in Q\left(\frac{z}{1-z}, g, \frac{1+Az}{1+Bz}, 1\right) \equiv K^*[A,B]$ *then there exists a function* $\mu \in C\left(\frac{1+Az}{1+Bz}\right)$ *such that* $\eta_1(z)$ *defined by*

$$\eta_1'(z) = \frac{(zf'(z))'}{1 + \dfrac{z\mu''(z)}{\mu(z)}}$$

belongs to class K *in* E.

Corollary 12. *For* $\phi(z) = \psi_\beta^\alpha(z)$, $h(z) = \frac{1+Az}{1+Bz}$, $(-1 \leq A < B \leq 1)$, ψ_β^α *is given by (3.4), in Theorem 3.2, it follows that, for* $f, g \in A$, $f \in Q(\psi_\beta^\alpha, g, \frac{1+Az}{1+Bz}, \lambda)$, *then there exists a function* $I_\beta^\alpha \mu(z) = \psi_\beta^\alpha(z) * \mu(z), \mu \in A$, $I_\beta^\alpha \mu(z) \in C\left(\frac{1+Az}{1+Bz}\right)$ *such that* $I_\beta^\alpha \eta_1(z)$ *defined by*

$$I_\beta^\alpha {\eta_1}'(z) = \frac{(z(I_\beta^\alpha f'(z)))'}{1 + \dfrac{I_\beta^\alpha \mu''(z)}{I_\beta^\alpha \mu'(z)}}$$

belongs to the class $K\left(\psi_\beta^\alpha, \frac{1+Az}{1+Bz}\right)$ *in* E.

Theorem 13. *Let* $\lambda > \lambda_1 \geq 0$. *Then for* $f, g, \phi \in A$,

$$Q(\phi, g, h, \lambda) \subset Q(\phi, g, h, \lambda_1).$$

Proof. If $\lambda_1 = 0$ then we have

$$Q(\phi, g, h, \lambda) \subset Q(\phi, g, h, 0)$$

We consider $\lambda_1 > 0$ and let $f \in Q(\phi, g, h, \lambda)$ implies that, for $F(z) = \phi(z) * f(z)$ and $G(z) = \phi(z) * g(z)$, we have $f \in K(\phi, h)$, by Theorem 3.1, that is

$$\frac{zF'(z)}{G(z)} \prec h(z), z \in E,$$

by Theorem 3.3. By Definition 1.2, it follows that

$$(1-\lambda_1)\frac{zF'(z)}{G(z)} + \lambda_1 \frac{(zF'(z))'}{G'(z)} = \frac{\lambda_1}{\lambda}\left[\left(\frac{\lambda_1}{\lambda} - 1\right)\frac{zF'(z)}{G(z)} + (1-\lambda)\frac{zF'(z)}{G(z)} + \lambda \frac{(zF'(z))'}{G'(z)}\right] \prec h(z)$$

for $z \in E$, which is the required result.

Now we have the following special cases.

Corollary 14. *[3] We take* $h(z) = \dfrac{1+z}{1-z}, \phi(z) = \dfrac{z}{1-z}$, *in Theorem 3.3, it follows that, for* $\lambda > \lambda_1 \geq 0$, *we have*

$$Re\left\{(1-\lambda)\frac{zf'(z)}{g(z)} + \lambda\frac{(zf'(z))'}{g'(z)}\right\} > 0,$$

implies that

$$Re\left\{(1-\lambda_1)\frac{zf'(z)}{g(z)} + \lambda_1\frac{(zf'(z))'}{g'(z)}\right\} > 0,$$

in E.

Corollary 15. *When* $h(z) = \dfrac{1+Az}{1+Bz}, (-1 \leq B < A \leq 1)$ $\phi(z) = \psi_\beta^\alpha(z)$, ψ_β^α *is given by (3.4), we have for* $\lambda > \lambda_1 \geq 0$,

$$Q\left(\psi_\beta^\alpha, g, \frac{1+Az}{1+Bz}, \lambda\right) \subset Q\left(\psi_\beta^\alpha, g, \frac{1+Az}{1+Bz}, \lambda_1\right)$$

Theorem 16. *For* $f \in A$,

$$Q(\psi_\beta^{\alpha-1}, f, \frac{1+Az}{1+Bz}, 0) \subset Q(\psi_\beta^\alpha, f, \frac{1+Az}{1+Bz}, 0),\ \beta \geq 0,\ z \in E.$$

where ψ_β^α *is given by (3.4).*

Proof. Let

$$\frac{z(I_\beta^\alpha f(z))'}{I_\beta^\alpha f(z)} = p_2(z),\ I_\beta^\alpha f = \psi_\beta^\alpha * f,\ \beta \geq 0,\ z \in E. \tag{3.8}$$

Using identity given by (1.5), we have

$$(\alpha+\beta)\frac{I_\beta^{\alpha-1} f(z))}{I_\beta^\alpha f(z))} - (\alpha+\beta-1) = p_2(z).$$

Logarithmic differentiation and some computation we have

$$\frac{z(I_\beta^{\alpha-1} f(z))'}{I_\beta^{\alpha-1} f(z)} = \frac{zp_2'(z)}{p_2(z) + (\alpha+\beta-1)} + p_2(z).$$

Since $\frac{z(I_\beta^{\alpha-1} f(z))'}{I_\beta^{\alpha-1} f(z)} \prec \frac{1+Az}{1+Bz}$. For $\beta > 0$, $Re(h(z) + (\alpha+\beta-1)) > 0$, $p_2(z) \in A$, $p_2(0) = h(0)$, so Lemma 2.7, we have $p_2(z) \prec \frac{1+Az}{1+Bz}$ in E which completes the proof.

Corollary 17. *For* $f, g \in A$, $\lambda, \beta \geq 0$, $\phi = \psi_\beta^\alpha$ *given by (3.4), we have*

$$Q(\psi_\beta^{\alpha-1}, f, \frac{1+Az}{1+Bz}, 1) \subset Q(\psi_\beta^\alpha, f, \frac{1+Az}{1+Bz}, 1).$$

Proof. One can easily proof above corollary by using the Alxender relation and Theorem 3.4.

Theorem 18. *For $\lambda \geq 0$, $\beta \geq 0$, f, g, $\phi \in A$,*

$$Q\left(\psi_\beta^{\alpha-1}, g, \frac{1+Az}{1+Bz}, \lambda\right) \subset Q\left(\psi_\beta^{\alpha}, g, \frac{1+Az}{1+Bz}, \lambda\right),$$

where ψ_β^α is given by (3.4).

Proof. Let $f \in Q(\psi_\beta^{\alpha-1}, g, h, \lambda)$, for $\beta, \lambda \geq 0$, $I_\beta^\alpha f = \psi_\beta^\alpha * f$, $I_\beta^\alpha g = \psi_\beta^\alpha * g$, $I_\beta^\alpha g \in S^*\left(\frac{1+Az}{1+Bz}\right)$, that is,

$$(1-\lambda)\frac{z(I_\beta^{\alpha-1}f(z))'}{I_\beta^{\alpha-1}g(z)} + \lambda\frac{(z(I_\beta^{\alpha-1}f(z))')'}{(I_\beta^{\alpha-1}g(z))'} \prec \frac{1+Az}{1+Bz}, \tag{3.9}$$

for $-1 \leq B < A \leq 1$, $z \in E$.
First we consider $\lambda = 0$ in (3.9), we have

$$\frac{z(I_\beta^{\alpha-1}f(z))'}{I_\beta^{\alpha-1}g(z)} \prec \frac{1+Az}{1+Bz}, \ z \in E. \tag{3.10}$$

Consider

$$\frac{z(I_\beta^{\alpha}f(z))'}{I_\beta^{\alpha}g(z)} = H_1(z),$$

where $H_1(z)$ is analytic with $H_1(0) = 1$ in E.
Simple calculation and identity given by (1.5) leads us

$$\frac{z(I_\beta^{\alpha-1}f(z))'}{I_\beta^{\alpha-1}g(z)} = \frac{zH_1'(z)}{p_2(z) + (\alpha+\beta-1)} + H_1(z), \ p_2 = \frac{z(I_\beta^{\alpha}g(z))'}{I_\beta^{\alpha}g(z)}. \tag{3.11}$$

Since $I_\beta^{\alpha-1}g(z) \in S^*\left(\frac{1+Az}{1+Bz}\right)$, so by Theorem 3.4, we have $I_\beta^{\alpha}g(z) \in S^*\left(\frac{1+Az}{1+Bz}\right)$. Thus $p_2 \in P(A,B) \subset P$. Let $H_2(z) = \frac{1}{p_2(z)}$, then $H_2(z) \in P$ in E and for $\alpha \geq 1$, $\beta \geq 0$, we have $\{H_2(z) + (\alpha+\beta-1)\} \in P$.

Let $H_0(z) = \frac{1}{h_0(z)+(\alpha+\beta+1)}$, $Re\{H_0(z)\} > 0$ in E.

Therefore from (3.11) we have

$$\{H_1(z) + H_2(z)(zH_1'(z))\} \prec \frac{1+Az}{1+Bz}, \ -1 \leq A < B \leq 1.$$

Lemma 2.1 leads us

$$H_1(z) \prec \frac{1+Az}{1+Bz}, \ -1 \leq A < B \leq 1, \ z \in E. \tag{3.12}$$

Now we consider $\lambda = 1$, it follows from (3.9) that

$$\frac{(z(I_\beta^{\alpha-1}f(z))')'}{(I_\beta^{\alpha-1}g(z))'} \prec \frac{1+Az}{1+Bz} \text{ in } E. \tag{3.13}$$

Let

$$\frac{(zI_\beta^{\alpha}f(z))')'}{(I_\beta^{\alpha}g(z))'} = H_3(z) \tag{3.14}$$

where $H_3(z)$ is analytic with $H_3(0) = 1$ in E.
Some simple calculations leads us

$$\frac{(z(I_\beta^{\alpha}f(z))')'}{(I_\beta^{\alpha}g(z))'} = \frac{zH_3'(z)}{H_4(z)+(\alpha+\beta-1)} + H_3(z), \tag{3.15}$$

where

$$\frac{(z(I_\beta^{\alpha}g(z))')'}{(I_\beta^{\alpha}g(z))'} = H_4(z) \prec \frac{1+Az}{1+Bz},$$

by Corollary 3.8, it follows that $H_4(z) \in P(A,B) \subset P$ and for $\alpha \geq 0$, $\beta \geq 0$, $Re\,\{H_4(z) + (\alpha+\beta-1)\} > 0$.
From (3.15), it follows that

$$\{H_3(z) + \rho(z)(zH_3'(z))\} \prec \frac{1+Az}{1+Bz},\ -1 \leq A < B \leq 1,$$

where $\rho(z) = \frac{1}{H_4(z)+(\alpha+\beta-1)}$, $Re\,\{H_4(z) + (\alpha+\beta-1)\} > 0$, $z \in E$.
Using Lemma 2.1, we have

$$H_3(z) \prec \frac{1+Az}{1+Bz},\ -1 \leq A < B \leq 1,\ z \in E. \tag{3.16}$$

From (3.12) and (3.16), it follows that

$$(1-\lambda)H_1(z) + \lambda H_3(z) \prec \frac{1+Az}{1+Bz},$$

for $-1 \leq A < B \leq 1$, $z \in E$. This completes the proof.

Theorem 19. *For $f, g, \phi \in A$, $f \in Q(\phi, g, h, \lambda)$, $\lambda \geq 0$, if and only if there exist a function $G = \phi * g$, $G \in S^*(h)$ and function $F_1(z) \in Q(\phi, g, h, 0)$ such that for $F = \phi * f$,*

$$zF'(z) = \frac{1}{\lambda}(G(z))^{1-\frac{1}{\lambda}} \int_0^z (G(z))^{1-\frac{1}{\lambda}} F_1(t)dt, \tag{3.17}$$

where all powers are meant as principle value.

Proof. Let $f \in Q(\phi, g, h, \lambda)$. Then for $F = \phi * f$, $G \in S^*(h)$,

$$(1-\lambda)\frac{zF'(z)}{G(z)} + \lambda\frac{(zF'(z))'}{G'(z)} = \rho_1(z) \quad \rho_1 \prec h(z), \quad G \in S^*(h), \quad \lambda \geq 0$$

Multiplying both sides by $\frac{1}{\lambda}(G(z))^{1-\frac{1}{\lambda}}G'(z)$, it follows that

$$\frac{1}{\lambda}zF'(z)\left[(G(z))^{\frac{1}{\lambda}-2}G'(z)\right] + (zF'(z))'(G(z))^{\frac{1}{\lambda}-1} = \frac{1}{\lambda}\rho_1(z)\left[(G(z))^{\frac{1}{\lambda}-2}G'(z)\right]$$

which implies that

$$(zF'(z)(G(z))^{\frac{1}{\lambda}-1})' = \frac{1}{\lambda}\rho_1(z)\left[(G(z))^{\frac{1}{\lambda}-1}G'(z)\right] \tag{3.18}$$

From integration and substituting $\rho_1(z)G(z) = 2F_1'(z)$, $F_1 \in Q(\phi, g, h, 0)$, we get the required result.
The converse follows immediately from (3.17).

Remark 3.1. Let $f \in Q(\phi, g, h, \lambda)$ with $g(z) = z$ and $0 < \lambda \leq 1$. Then f can be expressed by convolution of the function

$$k(z) = \frac{1}{\lambda}z^{1-\frac{1}{\lambda}}\int_0^z \frac{t^{\frac{1}{\lambda}-1}}{1-t}dt,$$

with the function,

$$J(z) = \int_0^z p(t)dt, \qquad p \prec h(z).$$

Corollary 20. *When $\phi = \psi_\beta^\alpha$, ψ_β^α given by (3.4), in the Theorem 3.5, we have, $I_\beta^\alpha f = \psi_\beta^\alpha * f$, then $f \in Q(\psi_\beta^\alpha, g, h, \lambda)$ if and only if there exist $I_\beta^\alpha g = \psi_\beta^\alpha * g \in S^*\left(\frac{1+Az}{1+Bz}\right)$, and $F_2(z) \in Q(\psi_\beta^\alpha, g, h, 0)$ such that*

$$z(I_\beta^\alpha f(z))' = \frac{1}{\lambda}(I_\beta^\alpha g(z))^{1-\frac{1}{\lambda}}\int_0^z (I_\beta^\alpha g(z))^{\frac{1}{\lambda}-1}F_2(t)dt, \tag{3.19}$$

where $\lambda \geq 0$, $z \in E$.

Now we have the convolution property of the class $Q(\phi, f, h, \lambda)$ and we note that following result improves the convolution results given in [11].

Theorem 21. *The class $Q(\phi, f, h, \lambda)$ is closed under convex convolution.*

Proof. Let $\sigma \in C$ and $f \in Q(\phi, f, h, \lambda)$. Then we have for $F = \phi * f$, $\lambda \geq 0$, we have

$$(1-\lambda)\frac{zF'(z)}{F(z)} + \lambda\frac{(zF'(z))'}{F'(z)} \prec h(z), h \in P, z \in E. \tag{3.20}$$

We will first show that $\sigma * f \in Q(\phi, f, h, 0)$. For this purpose, consider $\lambda = 1$ in (3.20), we have

$$\frac{zF'(z)}{F(z)} \prec h(z), h \in P, z \in E. \tag{3.21}$$

Consider

$$\begin{aligned}\frac{z(\phi * (\sigma * f))'(z)}{\phi * (\sigma * f)(z)} &= \frac{\sigma * \frac{z(\phi * f)'(z)}{(\phi * f)(z)}(\phi * f)(z)}{\sigma * (\phi * f)(z)} \\ &= \frac{\sigma * \rho_0(\phi * f)(z)}{\sigma * (\phi * f)(z)},\end{aligned} \tag{3.22}$$

where $\phi * f = F \in S^*(h) \subset S^*$, $\rho_0 = \frac{z(\phi*f)'(z)}{(\phi*f)(z)}$, $\rho_0 \prec h(z)$, therefore Lemma 2.4 leads us $\sigma * f \in Q(\phi, f, h, 0)$.
Now consider $\lambda = 1$ in (3.20), we have

$$\frac{(zF'(z))'}{F'(z)} \prec h(z), h \in P, z \in E. \tag{3.23}$$

Let

$$\begin{aligned}\frac{z(\phi * (\sigma * f)')'}{(\phi * (\sigma * f))'} &= \frac{\sigma * \frac{(zF'(z))'}{F'(z)}(zF'(z))}{\sigma * (zF'(z)}, F = \phi * f, \\ &= \frac{\sigma * \rho_1(zF'(z))}{\sigma * (zF'(z))}, \rho_1 = \frac{(zF'(z))'}{F'(z)}.\end{aligned} \tag{3.24}$$

since $\rho_1 \prec h(z)$, $h \in P$, by (3.23), and $zF'(z) \in S^*(h) \subset S^*$, by Lemma 2.4, it follows that $\sigma * f \in Q(\phi, f, h, 1)$ and from (3.22) and (3.24), we have our required result.

We note the following special cases.

Corollary 22. *[16] The class $Q(\phi, z, h, \lambda)$ is closed under convex convolution.*

Corollary 23. *The class $Q(\psi_\beta^\alpha, f, h, \lambda)$ are closed under convex convolution, where ψ is given by (3.4).*

Corollary 24. *The classes $Q(\frac{z}{1-z}, f, h, 0) \equiv S^*(h)$ and $Q(\frac{z}{1-z}, f, h, 1) \equiv K(h)$ are invariant under convolution with convex function.*

As an application of Theorem 3.7, we have the integral preserving properties of the class $Q(\phi, f, h, \lambda)$, given in the following corollaries.

Corollary 25. *Let $f \in Q(\phi, f, h, \lambda)$. Then the class $Q(\phi, f, h, \lambda)$ is invariant under the integral operators listed below.*

(i) $f_1(z) = \int_0^z \frac{f(t)}{t} dt,$

(ii) $f_2(z) = \frac{2}{z}\int_0^z f(t)dt$, *(Libera's operator [6]),*

(iii) $f_3(z) = \int_0^z \frac{f(t)-f(xt)}{t-xt}dt$, $|x| \leq 1$, $x \neq 1$,

(iv) $f_4(z) = \frac{1+c}{z^c}\int_0^z t^{c-1}f(t)dt$, $Re(c) \leq 0$ *(Generalized Bernardi operator [1]).*

Proof. The proof is immediate by observing that $f_i = \theta(z) * f(z)$, see [2], where $\theta_i \in C$, $i = 1, 2, 3, 4$, where
$\theta_1(z) = -log(1-z) = \sum_{n=1}^{\infty} \frac{1}{n}z^n$,
$\theta_2(z) = \frac{-2[z+log(1-z)]}{z} = \sum_{n=1}^{\infty} \frac{2}{n+1}z^n$,
$\theta_3 = \frac{1}{1-x}\log\frac{1-xz}{1-z} = \sum_{n=1}^{\infty} \frac{1-x^n}{(1-x)^n}z^n$, $|x| \leq 1$, $x \neq 1$,
$\theta_4 = \sum_{n=1}^{\infty} \frac{1+c}{n+c}z^n$, $Re\{c\} \geq 0$,
θ_1, θ_2 and θ_3 are easily verified to be convex. For $\theta_4 \in C$, we refer [18].

Remark 3.2 Let D_1 and D_2 be linear operators given by

$$D_1 f(z) = zf'(z), \quad D_2 f(z) = \frac{f(z) + zf'(z)}{2}.$$

Both of these operators can be written as convolution operator [2] given by

$$D_i f(z) = \mu_i(z) * f(z), \quad i = 1, 2, \tag{3.25}$$

where,

$$\mu_1(z) = \sum_{n=1}^{\infty} nz^n = \frac{z}{(1-z)^2},$$

$$\mu_2(z) = \sum_{n=1}^{\infty} \frac{n+1}{z}z^n. \tag{3.26}$$

It can easily be verified that radius of convexity $r_c(\mu_1) = 2 - \sqrt{3}$ and $r_c(\mu_2) = \frac{1}{2}$.

Using the Remark 3.2, we have he following corollary.

Corollary 26. *Let $f \in Q(\phi, f, h, \lambda)$. Then $D_1(f) = \mu_1(z) * f(z) \in Q(\phi, f, h, \lambda)$ for $|z| < 2 - \sqrt{3}$ and $D_2(f) = \mu_2(z) * f(z) \in Q(\phi, f, h, \lambda)$ for $|z| < \frac{1}{2}$.*

Now we will investigate some radius problems for $Q(\phi, g, h, \lambda)$ as follows.

Theorem 27. *Let f, g, $\phi \in A$, $F = \phi * f$, $G = \phi * g$, $f \in Q\left(\phi, g, \frac{1+(1-2\gamma)z}{1-z}, 0\right)$. Then for $G \in S^*\left(\phi, \frac{1+z}{1-z}\right)$ such that*

$$\frac{(zF'(z))'}{G'(z)} \prec \frac{1+(1-2\gamma)z}{1-z}, \quad 0 \leq \gamma < 1,$$

for $|z| < r_\gamma$, r_γ is given as

$$r_\gamma = \begin{cases} \frac{2-\gamma-\sqrt{\gamma^2-2\gamma+3}}{1-2\gamma}, & \gamma \neq \frac{1}{2}, \\ \frac{1}{3}, & \gamma = \frac{1}{2}, \end{cases} \tag{3.27}$$

Proof. For $z \in E$, we can write

$$zF'(z) = G(z)q_0(z), \quad q_0 \in P(\gamma), 0 \leq \gamma < 1$$

where, $F = \phi * f$, $G = \phi * g$, $G \in S^*\left(\phi, \frac{1+z}{1-z}\right)$.
Simple calculation leads us

$$Re\left\{\frac{(zF'(z))'}{G'(z)} - \gamma\right\} > Re\{q_0(z)\} - \gamma - \left|\frac{G(z)}{G'(z)}\right| |q_0'(z)|.$$

It is well known that

$$\left|\frac{G(z)}{G'(z)}\right| \leq \frac{r(r+1)}{1-(1-2\gamma)r},$$

$$|q_3(z)| \leq \frac{2[Re\{q_0(z)\} - \gamma}{1-r^2}.$$

It follows that

$$Re\left\{\frac{(zF'(z))'}{G'(z)} - \gamma\right\} \geq \{Re(q_0(z)) - \gamma\}\left\{\frac{1-(4-\gamma)r+(1-2\gamma)r^2}{(1-r)(1-(1-2\gamma)r}\right\}. \tag{3.28}$$

The right hand side of (3.28) is positive for $|z| < r_\gamma$, r_γ is given by (3.27) and which is the required result.

Theorem 28. *Let f, g, $\phi \in A$, $f \in Q(\phi, g, h, 0)$, $0 < \lambda \leq 1$. Then $f \in Q(\phi, z, h, \lambda)$ for $|z| < r_\lambda$, where*

$$r_\lambda = \frac{1}{2\lambda + \sqrt{4r^2 - 2\lambda + 1}} \tag{3.29}$$

Proof. Let $f \in Q(\phi, g, h, 0)$, $0 < \lambda \leq 1$. For $F = \phi * f$, $G = \phi * g$, consider

$$\begin{aligned} \upsilon_\lambda(z) &= F'(z) + \lambda z F''(z) \\ &= \frac{k_\lambda(z)}{z} * f'(z), \end{aligned} \tag{3.30}$$

where $k_\lambda(z) = z + \sum_{n=2}^{\infty}(1+(n-1)\alpha)z^n$ and it is known that k_λ is convex for $|z| < r_\lambda$. Consequently, for $Re\left(\frac{k_\lambda}{z}\right) > \frac{1}{2}$, $|z| < r_\lambda$. Hence from (3.30), we have

$$\upsilon_\lambda(z) = \frac{k_\lambda(z)}{z} * p(z), \quad f'(z) = p(z) \prec h(z). \tag{3.31}$$

Thus using Lemma 2.3, $\upsilon_\lambda \prec h(z)$ for $|z| < r_\lambda$ which implies that $f \in Q(\phi, z, h, \lambda)$ for $|z| < r_\lambda$, r_λ is given by (3.29). This completes the proof.

Corollary 29. *[16] Let $f \in Q(\phi, z, h, \lambda)$, $Re(h) > 0$. Then f maps $|z| < \sqrt{2}-1$ into convex domain for $|z| < \sqrt{2}-1$.*

Remark 3.3. All mentioned theorems are also holds for $\phi = \phi_\beta^\alpha$, where ϕ_β^α is given by

$$\phi_\beta^\alpha(z) = \binom{\alpha+\beta}{\beta}\phi(\beta, \alpha+\beta; z), \tag{3.32}$$

and we note that the results derived in this research paper improves the results proved by Noor in [11].

Acknowledgements. The author would like to express their sincere gratitude to Dr. M. Junaid Zaidi, Rector, COMSATS Institute of Information Technology, Pakistan, for providing excellent research facilities.

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