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A FIXED POINT THEOREM FOR CYCLIC MAPPINGS SATISFYING AN IMPLICIT RELATION ON PARTIAL METRIC SPACES

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Abstract. A general fixed point theorem for mappings satisfying an implicit relation in partial metric spaces is proved, generalizing the result from [4]. As application, we obtain theorem 2.7 [1] for cyclic mappings.

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1. Introduction

In 2003, Kirk et al. [11] extended Banach's principle to a case of cyclical contractive mappings. In [16], most of the fundamental metrical fixed point theorems in literature (Chatterjea, Reich, Hardy - Rogers, Kannan, Cirić) are extended to cyclical contractive mappings. Other new results are obtained in [14], [15], [17], [22] and in other papers.

In 1994, Matthews [12] introduced the concept of partial metric spaces as a part of the study of denotional semantics of dataflow networks and proved the Banach contraction principle in such spaces. Recently, in [2], [4], [5], [6], [9] and in other papers, some fixed point theorems under various contractive conditions in complete partial metric spaces are proved. Quite recently, in [1], [3], [8], [10], [13], the study of fixed points for cyclical operators on partial metric spaces is initiated.

Several classical fixed point theorems and common fixed point theorems have been unified considering a general condition by an implicit relation in [18], [19] and in other papers. Recently, the method is used in the study of fixed points in metric spaces, symmetric spaces, quasi - metric spaces, b - metric spaces, ultra - metric spaces, convex metric spaces, reflexive spaces, compact metric spaces, in two and three metric spaces, for single - valued mappings, hybrid pairs of mappings and multi - valued mappings. The method is used in the study of fixed points for mappings satisfying a contractive/extensive condition of integral type in fuzzy metric spaces, probabilistic metric spaces, intuitionistic metric spaces and G - metric spaces.

With this method the proofs of some fixed point theorems are more simple. Also, the method allows the study of local and global properties of fixed point structures.

The study of fixed points for mappings satisfying an implicit relation in partial metric spaces is initiated in [23].

The purpose of this paper is prove a general fixed point theorem for mappings satisfying an implicit relation different of those from [23].

As application we obtain a general fixed point theorem, which generalizes and improves theorem 2.7 [1] for cyclical mappings.

2. Preliminaries

Definition 1 ([12])**.** *Let X be a nonempty set. A function $p : X \times X \to \mathbb{R}_+$ is said to be a partial metric on X if for each $x, y, z \in X$, the following conditions hold:*

$(P_1) : p(x,x) = p(y,y) = p(x,y)$ *if and only if* $x = y$,

$(P_2) : p(x,x) \leq p(x,y)$,

$(P_3) : p(x,y) = p(y,x)$,

$(P_4) : p(x,z) \leq p(x,y) + p(y,z) - p(y,y)$.

The pair (X,p) is called a partial metric space.

If $p(x,y) = 0$ then (P_1) and (P_2) imply $x = y$, but the converse does not always hold.

Each partial metric p on X generates a T_0 - topology τ_p which has as base the family of open p - balls $\{B_p(x,\varepsilon) : x \in X,\ \varepsilon > 0\}$, where $B_p(x,\varepsilon) = \{y \in X : p(x,y) \leq p(x,x) + \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$.

A sequence $\{x_n\}$ in a partial metric space converges to a point $x \in X$ with respect to τ_p if and only if $p(x,x) = \lim_{n\to\infty} p(x_n,x)$.

Lemma 1 ([12], [2])**.** *Let $\{x_n\} \to z$ as $n \to \infty$ in a partial metric space (X,p), where $p(z,z) = 0$. Then $\lim_{n\to\infty} p(x_n,y) = p(z,y)$ for every $y \in X$.*

If p is a partial metric on X, then the function $d_p(x,y) = 2p(x,y) - p(x,x) - p(y,y)$ defines a metric on X.

Definition 2 ([12])**.** *a) A sequence $\{x_n\}$ in a partial metric space (X,p) is called Cauchy if $\lim_{n,m\to\infty} p(x_n,x_m)$ exists and is finite.*

b) A partial metric space (X,p) is said to be complete if every Cauchy sequence in X converges with respect to τ_p to a point $x \in X$ such that $p(x,x) = \lim_{n,m\to\infty} p(x_n,x_m)$.

It is evident that every closed subsets of a complete partial metric space is complete.

Lemma 2 ([17], [5])**.** *Let (X,p) be a partial metric space.*

(*a*) *A sequence in (X,p) is a Cauchy sequence if and only if is a Cauchy sequence in (X,d_p).*

(*b*) *A partial metric space (X,p) is complete if and only if the metric space (X,d_p) is complete.*

Further more, $\lim_{n\to\infty} d_p(x_n,x)=0$ if and only if

$$p(x,x)=\lim_{n\to\infty} p(x_n,x)=\lim_{n.m\to\infty} p(x_n,x_m). \tag{1}$$

Lemma 3. (3) *Every 0 - Cauchy sequence in (X,p) is Cauchy in (X,d_p).*

(4) *If (X,p) is complete, then is 0 - complete.*

Let (X,d) be a metric space, $m\in\mathbb{N}$, $m>1$, T a self mapping of X and $\{A_i\}_{i=1}^m$ nonempty closed sets of X. The mapping T is said to be cyclical [11] if

$$T(A_i)\subset A_{i+1}, i=\overline{1,m}, \text{ where } A_{m+1}=A_1. \tag{2}$$

In [11] the following theorem is proved.

Theorem 4 ([11])**.** *Let $\{A_i\}_{i=1}^m$ be nonempty closed subsets of a complete metric space (X,d) and suppose $T:\bigcup\limits_{i=1}^m A_i\to\bigcup\limits_{i=1}^m A_i$ satisfy condition (2) and there exists $\alpha\in(0,1)$ such that*

$$d(Tx,Ty)\le\alpha d(x,y) \tag{3}$$

for all $x\in A_i$, $y\in A_{i+1}, i=\overline{1,m}$.

Then T has an unique fixed point in $\bigcap\limits_{i=1}^m A_i$.

In [14], [15], [16], [22] some fixed point theorems for cyclical contraction mappings in metric spaces extend the results of Kannan, Reich, Rus, Hardy - Rogers and of other authors.

The following theorem is proved in [4].

Theorem 5 ([4])**.** *Let (X,p) be a complete partial metric space and let $f:X\to X$ be a mapping such that*

$$p(fx,fy)\le a_1p(x,y)+a_2p(x,fx)+a_3p(y,fy)+a_4p(x,fy)+a_5p(y,fx) \tag{4}$$

for all $x,y\in X$, where $a_i\ge0$, $i=\overline{1,5}$ such that if $a_4\ge a_5$, then $a_1+a_2+a_3+a_4+a_5<1$ and if $a_4<a_5$, then $a_1+a_2+a_3+a_4+2a_5<1$. Then f has an unique fixed point.

In [1] the authors extend this theorem for cyclical mappings in partial metric spaces.

Theorem 6 ([1]). *Let (X, p) be a complete partial metric space, A_i, $i = \overline{1, m}$ nonempty closed subsets of (X, d_p) and $Y = \bigcup_{i=1}^{m} A_i$. Suppose that f satisfies condition (2) and condition (3) from Theorem 5. Then, f has an unique fixed point.*

3. Implicit relations

Definition 3. *Let $\mathfrak{Fp}_6$ be the set of continuous functions $F : \mathbb{R}_+^6 \to \mathbb{R}$ satisfying*

(F_1) : *F is nonincreasing in variables t_3, t_4, t_5, t_6;*

(F_2) : *there exists $h \in [0, 1)$ such that for all $u, v \geq 0$, $F(u, v, v, u, u + v, v) \leq 0$ implies $u \leq hv$.*

In the following examples, the property (F_1) is obviously.

Example 1. *$F(t_1, ..., t_6) = t_1 - a_1 t_2 - a_2 t_3 - a_3 t_4 - a_4 t_5 - a_5 t_6$, where $a_1, ..., a_5 \geq 0$ and $a_1 + a_2 + a_3 + a_4 + 2a_5 < 1$.*

(F_2) : *Let $u, v \geq 0$ be and $F(u, v, v, u, u + v, v) = u - a_1 v - a_2 v - a_3 u - a_4 (u + v) - a_5 v \leq 0$. Then $u \leq hv$, where $0 \leq h = \dfrac{a_1 + a_2 + a_4 + a_5}{1 - (a_3 + a_4)} < 1$.*

Example 2. *$F(t_1, ..., t_6) = t_1 - k \max\{t_2, ..., t_6\}$, where $k \in \left[0, \dfrac{1}{2}\right)$.*

(F_2) : *Let $u, v \geq 0$ be and $F(u, v, v, u, u + v, v) = u - k(u + v) \leq 0$, which implies $u \leq hv$, where $0 \leq h = \dfrac{k}{1 - k} < 1$.*

Example 3. *$F(t_1, ..., t_6) = t_1^2 - t_1(at_2 + bt_3 + ct_4) - dt_5 t_6$, where $a, b, c, d \geq 0$ and $a + b + c + 2d < 1$.*

(F_2) : *Let $u, v \geq 0$ be and $F(u, v, v, u, u + v, v) = u^2 - u(av + bv + cu) - d(u + v) \leq 0$. If $u > v$, then $u^2[1 - (a + b + c + 2d)] \leq 0$, a contradiction. Hence $u \leq v$, which implies $u \leq hv$, where $0 \leq h = a + b + c + 2d < 1$.*

Example 4. *$F(t_1, ..., t_6) = t_1^2 - a \max\{t_2^2, t_3^2, t_4^2\} - bt_5 t_6$, where $a, b \geq 0$ and $a + 2b < 1$.*

(F_2) : *Let $u, v \geq 0$ be and $F(u, v, v, u, u + v, v) = u^2 - a \max\{u^2, v^2\} - bv(u + v) \leq 0$. If $u > v$, then $u^2[1 - (a + 2b)] \leq 0$, a contradiction. Hence $u \leq v$, which implies $u \leq hv$, where $0 \leq h = \sqrt{a + 2b} < 1$.*

Example 5. *$F(t_1,...,t_6) = t_1^3 - at_1^2t_2 - bt_1t_2^2 - ct_2t_3t_4 - dt_1t_5t_6$, where $a,b,c,d \geq 0$ and $a+b+c+2d < 1$.*

(F_2) : *Let $u,v \geq 0$ be and $F(u,v,v,u,u+v,v) = u^3 - auv^2 - buv^2 - cv^2u - duv(u+v) \leq 0$. If $u > v$, then $u^3[1-(a+b+c+2d)] \leq 0$, a contradiction. Hence $u \leq v$, which implies $u \leq hv$, where $0 \leq h = \sqrt[3]{a+b+c+2d} < 1$.*

Example 6. *$F(t_1,...,t_6) = t_1 - at_2 - bt_3 - c\max\{2t_4, t_5+t_6\}$, where $a,b,c \geq 0$ and $a+b+3c < 1$.*

(F_2) : *Let $u,v \geq 0$ be and $F(u,v,v,u,u+v,v) = u - av - bv - c\max\{2u, u+2v\} \leq 0$. If $u > v$, then $u[1-(a+b+3c)] \leq 0$, a contradiction. Hence $u \leq v$, which implies $u \leq hv$, where $0 \leq h = a+b+3c < 1$.*

Example 7. *$F(t_1,...,t_6) = t_1^2 + \dfrac{t_1}{1+t_5+t_6} - \left(at_2^2 + bt_3^2 + ct_4^2\right)$, where $a,b,c \geq 0$ and $a+b+c < 1$.*

(F_2) : *Let $u,v \geq 0$ be and $F(u,v,v,u,u+v,v) = u^2 + \dfrac{u}{1+u+2v} - (av^2 + bv^2 + cu^2) \leq 0$, which implies $u^2 - \left(av^2 + bv^2 + cu^2\right) \leq 0$. Hence $u \leq hv$, where $0 \leq h = \sqrt{\dfrac{a+b}{1-c}} < 1$.*

Example 8. *$F(t_1,...,t_6) = t_1 - at_2 - bt_3 - c\max\{2t_4, t_5\} - d\max\{2t_3, t_6\}$, where $a,b,c,d \geq 0$ and $a+b+2c+2d < 1$.*

(F_2) : *Let $u,v \geq 0$ be and $F(u,v,v,u,u+v,v) = u - av - bv - c\max\{2u, u+v\} - 2dv \leq 0$. If $u > v$, then $u[1-(a+b+2c+2d)] \leq 0$, a contradiction. Hence $u \leq v$, which implies $u \leq hv$, where $0 \leq h = a+b+2c+2d < 1$.*

4. Main results

Theorem 7. *Let (X,d) be a complete partial metric space and $T : X \to X$ be a self mapping. If*

$$F(p(Tx,Ty), p(x,y), p(x,Tx), p(y,Ty), p(x,Ty), p(y,Tx)) \leq 0 \qquad (5)$$

for all $x,y \in X$ and $F \in \mathfrak{F}_{p6}$, then T has an unique fixed point z such that $p(z,z) = 0$.

Proof. Let $x_0 \in X$ be and $x_n = Tx_{n-1}$ for $n = 1,2,....$

If there exists $n_0 \in \mathbb{N}$ such that $x_{n_0} = x_{n_0+1}$, then $x_{n_0} = Tx_{n_0}$ and x_{n_0} is a fixed point of T. We suppose that $x_n \neq x_{n+1}$ for each $n \in \mathbb{N}$. Then by (5) we have successively

$$F\begin{pmatrix} p(Tx_n, Tx_{n+1}), p(x_n, x_{n+1}), p(x_n, Tx_n), \\ p(x_{n+1}, Tx_{n+1}), p(x_n, Tx_{n+1}), p(x_{n+1}, Tx_n) \end{pmatrix} \leq 0,$$

$$F\left(\begin{array}{c} p\left(x_{n+1},x_{n+2}\right),p\left(x_{n},x_{n+1}\right),p\left(x_{n},x_{n+1}\right), \\ p\left(x_{n+1},x_{n+2}\right),p\left(x_{n},x_{n+2}\right),p\left(x_{n+1},x_{n+1}\right) \end{array}\right)\leq 0.$$

Since by (P_4) and (P_2) we have

$$p\left(x_{n},x_{n+2}\right)\leq p\left(x_{n},x_{n+1}\right)+p\left(x_{n+1},x_{n+2}\right)-p\left(x_{n+1},x_{n+1}\right)$$

and by (F_2) we obtain

$$\begin{array}{c} F(p\left(x_{n+1},x_{n+2}\right),p\left(x_{n},x_{n+1}\right),p\left(x_{n},x_{n+1}\right),p\left(x_{n+1},x_{n+2}\right), \\ p\left(x_{n},x_{n+1}\right)+p\left(x_{n+1},x_{n+2}\right),p\left(x_{n},x_{n+1}\right))\leq 0. \end{array}$$

By (F_2) we have

$$p\left(x_{n+1},x_{n+2}\right)\leq hp\left(x_{n},x_{n+1}\right)$$

which implies that

$$p\left(x_{n},x_{n+1}\right)\leq hp\left(x_{n-1},x_{n}\right)\leq ...\leq h^{n}p\left(x_{0},x_{1}\right).$$

For $m,n\in\mathbb{N}$, $m>n$ by (P_4) we obtain

$$\begin{array}{rcl} p\left(x_{n},x_{m}\right) & \leq & p\left(x_{n},x_{n+1}\right)+p\left(x_{n+1},x_{n+2}\right)+...+p\left(x_{m-1},x_{m}\right) \\ & \leq & h^{n}\left(1+h+...+h^{m-1}\right)p\left(x_{0},x_{1}\right) \\ & \leq & \dfrac{h^{n}}{1-h}p\left(x_{0},x_{1}\right). \end{array}$$

Hence

$$\lim_{n,m\to\infty}p\left(x_{n},x_{m}\right)=0.$$

Because

$$d_{p}\left(x_{n},x_{m}\right)\leq 2p\left(x_{n},x_{m}\right)\to 0,$$

then

$$\lim_{n,m\to\infty}d_{p}\left(x_{n},x_{m}\right)=0.$$

Then by Lemma 2 (a), $\{x_n\}$ is a Cauchy sequence, both in (X,p) and in (X,d_p). Since (X,p) is complete, by Lemma 2 (b), (X,d_p) is complete and $\lim_{n\to\infty}d_p\left(x_n,z\right)=0$ for some $z\in X$. Again, by Lemma 2 and relation (1) we get

$$p\left(z,z\right)=\lim_{n\to\infty}p\left(x_{n},z\right)=\lim_{n,m\to\infty}p\left(x_{n},x_{m}\right)=0. \tag{6}$$

We prove that z is a fixed point of T.

Again, by (5) we have successively

$$F\left(p\left(Tx_{n},Tz\right),p\left(x_{n},z\right),p\left(x_{n},Tx_{n}\right),p\left(z,Tz\right),p\left(x_{n},Tz\right),p\left(z,Tx_{n}\right)\right)\leq 0,$$

$$F\left(p\left(x_{n+1},Tz\right),p\left(x_n,z\right),p\left(x_n,x_{n+1}\right),p\left(z,Tz\right),p\left(x_n,Tz\right),p\left(z,x_{n+1}\right)\right)\leq 0.$$

Letting n tends to infinity, by (6) and Lemma 1 we obtain

$$F\left(p\left(z,Tz\right),0,0,p\left(z,Tz\right),p\left(z,Tz\right),0\right)\leq 0,$$

which implies by (F_2) that $p\left(z,Tz\right)=0$, i.e. $z=Tz$ and z is a fixed point of T.

Suppose that u is another fixed point of T. Then by (5) we have successively

$$F\left(p\left(Tu,Tz\right),p\left(u,z\right),p\left(u,Tu\right),p\left(z,Tz\right),p\left(u,Tz\right),p\left(z,Tu\right)\right)\leq 0,$$

$$F\left(p\left(u,z\right),p\left(u,z\right),p\left(u,u\right),p\left(z,z\right),p\left(u,z\right),p\left(u,z\right)\right)\leq 0.$$

Since by (P_2),

$$p\left(u,u\right)\leq p\left(u,z\right) \text{ and } p\left(z,z\right)\leq p\left(u,z\right),$$

by (F_1) we obtain

$$F\left(p\left(u,z\right),p\left(u,z\right),p\left(u,z\right),p\left(u,z\right),2p\left(u,z\right),p\left(u,z\right)\right)\leq 0,$$

which implies by (F_2) that

$$p\left(u,z\right)\leq hp\left(u,z\right),$$

therefore

$$p\left(u,z\right)\left(1-h\right)\leq 0.$$

Hence, $p\left(u,z\right)=0$ and $u=z$. Hence, z is the unique fixed point of T.

By Theorem 7 and Example 1 we obtain

Corollary 8. *Let (X,p) be a complete partial metric space and let $f:X\to X$ be a mapping such that*

$$p\left(fx,fy\right)\leq a_1p\left(x,y\right)+a_2p\left(x,fx\right)+a_3p\left(y,fy\right)+a_4p\left(x,fy\right)+a_5p\left(y,fx\right)$$

for all $x,y\in X$, where $a_i\geq 0$, $i=\overline{1,5}$ and $a_1+a_2+a_3+a_4+2a_5<1$. Then f has an unique fixed point.

Remark 1. *1. This is a new form of Theorem 5.*

2. By Theorem 7 and Examples 2 - 8 we obtain new particular results.

Theorem 9. *Let (X, p) be a complete partial metric space, $A_1, A_2, ..., A_m$ nonempty closed subsets of (X, d_p). Let $Y = \bigcup_{i=1}^{m} A_i$ and $T : Y \to Y$ be a mapping satisfying the following conditions:*

$$T(A_i) \subset A_{i+1}, i = \overline{1, m}, \text{ where } A_{m+1} = A_1, \tag{7}$$

$$F(p(Tx, Ty), p(x, y), p(x, Tx), p(y, Ty), p(x, Ty), p(y, Tx)) \leq 0 \tag{8}$$

for all $x \in A_i, y \in A_{i+1}$. Then T has an unique fixed point z in Y and $z \in \bigcap_{i=1}^{m} A_i$.

Proof. Let $x_0 \in A_1$, by (7)

$$\begin{gathered} x_1 = Tx_0 \in A_2, \\ \vdots \\ x_{m-1} = Tx_{m-2} \in A_m, \\ x_m = Tx_{m-1} \in A_{m+1} = A_1. \end{gathered}$$

If $x_{n_0} = x_{n_0+1} = Tx_{n_0}$, then x_{n_0} is a fixed point of T. Suppose that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$.

By (8) we have successively

$$F(p(Tx_0, Tx_1), p(x_0, x_1), p(x_0, Tx_0), p(x_1, Tx_1), p(x_0, Tx_1), p(x_1, Tx_0)) \leq 0,$$

$$F(p(x_1, x_2), p(x_0, x_1), p(x_0, x_1), p(x_1, x_2), p(x_0, x_2), p(x_1, x_1)) \leq 0.$$

By (P_4) and (P_2) we have

$$\begin{aligned} p(x_0, x_2) &\leq p(x_0, x_1) + p(x_1, x_2) - p(x_1, x_1) \\ &\leq p(x_0, x_1) + p(x_1, x_2) \end{aligned}$$

and

$$p(x_1, x_1) \leq p(x_0, x_1).$$

By (F_2) we obtain

$$p(x_1, x_2) \leq hp(x_0, x_1).$$

Similarly, we obtain

$$\begin{gathered} p(x_2, x_3) \leq hp(x_1, x_2), \\ \vdots \\ p(x_{m-2}, x_{m-1}) \leq hp(x_{m-3}, x_{m-2}). \end{gathered}$$

Again, by (8) we have

$$p\left(x_{m-1},x_m\right) \leq hp\left(x_{m-2},x_{m-1}\right).$$

Therefore,

$$p\left(x_n,x_{n+1}\right) \leq p\left(x_{n-1},x_n\right) \leq ... \leq h^n p\left(x_0,x_1\right).$$

For $m,n \in \mathbb{N}$ and $m > n$, by (P_4) we obtain

$$\begin{aligned} p\left(x_n,x_m\right) &\leq p\left(x_n,x_{n+1}\right) + p\left(x_{n+1},x_{n+2}\right) + ... + p\left(x_{m-1},x_m\right) \\ &\leq h^n\left(1+h+...+h^{m-1}\right) p\left(x_0,x_1\right) \\ &\leq \frac{h^n}{1-h} p\left(x_0,x_1\right) \to 0 \text{ for } m,n \to \infty. \end{aligned}$$

Hence $\{x_n\}$ is a Cauchy sequence in (X,p). Since $d_p\left(x_n,x_m\right) \leq 2p\left(x_n,x_m\right) \to 0$ as $n,m \to \infty$ it follows that $\{x_n\}$ is a Cauchy sequence in (Y,p).

Since Y is closed in (Y,d_p), is also complete. Thus there exists $y_0 \in Y$ such that $x_n \to y_0$ in (X,d_p), equivalently

$$p\left(y_0,y_0\right) = \lim_{n\to\infty} p\left(y_0,x_n\right) = \lim_{n,m\to\infty} p\left(x_n,x_m\right) = 0.$$

Notice that the iterative sequence $\{x_n\}$ has an infinite number of terms in A_i for each $i = \overline{1,m}$. Hence, in each A_i we can construct a subsequence of $\{x_n\}$, that converges to y_0 since A_i is closed. Hence $\bigcap\limits_{i=1}^{m} A_i \neq \varnothing$.

Let $Z = \bigcap\limits_{i=1}^{m} A_i$. Clearly, Z is also closed and (Z,p) is complete. Consider the restriction of T in Z, $T|Z$. Then $T|Z$ satisfies the assumptions of Theorem 7 and thus $T|Z$ has an unique point in Z.

Suppose that there exists $u \in Y$ and $u = Tu$. Then by (8) we obtain

$$F\left(p\left(Tu,Tz\right), p\left(u,z\right), p\left(u,Tu\right), p\left(z,Tz\right), p\left(u,Tz\right), p\left(z,Tu\right)\right) \leq 0,$$

$$F\left(p\left(u,z\right), p\left(u,z\right), p\left(u,u\right), p\left(z,z\right), p\left(u,z\right), p\left(u,z\right)\right) \leq 0.$$

Since by (P_2), $p\left(u,u\right) \leq p\left(u,z\right)$ and $p\left(z,z\right) \leq p\left(u,z\right)$, then by (F_1) we obtain

$$F\left(p\left(u,z\right), p\left(u,z\right), p\left(u,z\right), p\left(u,z\right), 2p\left(u,z\right), p\left(u,z\right)\right) \leq 0,$$

which implies by (F_2) that

$$p\left(u,z\right) \leq hp\left(u,z\right).$$

Hence $p\left(u,z\right)\left(1-h\right) \leq 0$,which implies $p\left(u,z\right) = 0$, i.e. $u = z$.

Corollary 10. *Let (X, p) be a complete partial metric space, $A_1, A_2, ..., A_m$ nonempty closed subsets of (X, d_p) and $Y = \bigcup_{i=1}^{m} A_i$. Suppose that $T : Y \to Y$ is a function such that*

$$p(Tx, Ty) \leq a_1 p(x, y) + a_2 p(x, Tx) + a_3 p(y, Ty) + a_4 p(x, Ty) + a_5 p(y, Tx)$$

for all $x \in A_i$, $y \in A_{i+1}$, $i = \overline{1, m}$ and $A_{m+1} = A_1$, where $a_i \geq 0$, $i = \overline{1, 5}$ and $a_1 + a_2 + a_3 + a_4 + 2a_5 < 1$. Then T has an unique fixed point z in Y and $z \in \bigcap_{i=1}^{m} A_i$.

Proof. The proof it follows by Theorem 9 and Example 1.

Remark 2. *1. This is a new form of Theorem 2.7 [1].*
2. By Theorem 7 and Examples 2 - 8 we obtain new particular results.

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