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ON EXTREMAL PROBLEMS IN BMOA TYPE ANALYTIC SPACES IN BOUNDED STRONGLY PSEUDOCONVEX DOMAINS WITH SMOOTH BOUNDARY AND IN TUBULAR DOMAINS OVER SYMMETRIC CONES

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Abstract. We provide some new sharp results on extremal problems in new BMOA type spaces in the unit disk and then in bounded pseudoconvex domains with smooth boundary and tubular domains over symmetric cones using so called double Bergman representation formula. These generalize a known one dimensional result to higher dimension in various directions.

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1. Introduction

The goal of this paper is to provide full proof of known sharp theorem on an extremal problem related with the distance function in the unit disk case and then based on it for some values of parameters we provide new results on distance function in more complicated new BMOA-type spaces in the unit ball and in bounded pseudoconvex domains with smooth boundary and in tubular domains over symmetric cones. Such type general analytic BMOA type spaces in such general domains and even in products of such type domains appeared recently in [27] and earlier in the unit ball in [26].

Results we provide in this paper are first sharp results on distance function in analytic BMOA-type spaces in higher dimension. Related results on other spaces were given earlier in [25]. We refer also to this paper for various other recent papers in this direction in higher dimension.

In recent decades many papers appeared where various BMO spaces or BMOA type spaces were studied from various points of views in higher dimension in various domains in $\mathbb{C}^n$. We refer to a series of papers of Krantz and coauthors (see [16],

[15], [17] in particular) and also we indicate [18], [3], [19], [20] where some interesting results also were provided in this direction. In this paper only applying rather simple and unusual double Bergman representation formula, classical Forelly-Rudin estimates and some standard estimates for Bergman kernel we indicate a precise formula for dist function that allows to estimate distances from any function from certain analytic BMOA type class to certain classical weighted Bergman space. The simple nature of our proof in the unit disk allows to consider more general weighted Bergman classes, but we restrict ourselves to standard weights. We assume in advance that inf of any set below and the set itself can not be empty.

Note the unit disk result is known, but we provide it as model case to formulate based on it more general versions of that onedimensional result. These results in higher dimension have the same proofs (in unit ball, pseudoconvex domains, tubular domains). We first add some general discussion concerning some extremal problems in analytic function spaces in the unit disk.

Let D be the unit disk in C, $dA(z)$ or dm_2 be the normalized Lebesgue measure on D so that $A(D) = 1$ and $d\zeta$ be the Lebesgue measure on the ∂D. We denote below everywhere by D^n as usual the unit polydisk (see for example [11] and references there) and use below all standard notations of function theory in (product domains) polydisks that can be found, for example, in [11]. In particular for any natural n we denote by dm_{2n} the Lebegues measure in the unit polydisk. For $f \in H(D)$ and $f(z) = \sum_k a_k z^k$, define the fractional derivative of the function f as usual in the following manner

$$\mathcal{D}^\alpha f(z) = \sum_{k=0}^{\infty} (k+1)^\alpha a_k z^k, \alpha \in \mathbb{R}.$$

We will write $\mathcal{D}f(z)$ if $\alpha = 1$. Obviously, for all $\alpha \in \mathbb{R}$, $\mathcal{D}^\alpha f \in H(D)$ if $f \in H(D)$.

For $a \in D$, let $g(z,a) = log(\frac{1}{\varphi_a(z)})$ be the Green's function for D with pole at a, where $\varphi_a(z) = \frac{a-z}{1-\bar{a}z}$. For $0 < p < \infty$, $-2 < q < \infty$, $0 < s < \infty$, $-1 < q+s < \infty$, we say that $f \in F(p,q,s)$, if $f \in H(D)$ and

$$\|f\|^p_{F(p,q,s)} = \sup_{a\in D} \int_D |\mathcal{D}(f(z))|^p (1-|z|^2)^q g(z,a)^s dA(z) < \infty.$$

As we know [31], if $0 < p < \infty$, $-2 < q < \infty$, $0 < s < \infty$, $-1 < q+s < \infty$, $f \in F(p,q,s)$ if and only if

$$\sup_{a\in D} \int_D |\mathcal{D}(f(z))|^p (1-|z|^2)^q (1-|\varphi_a(z)|)^s dA(z) < \infty.$$

It is known (see [31]) that $F(2,0,1) = BMOA$.

We recall that the weighted Bloch class $\mathcal{B}^\alpha(D)$, $\alpha > 0$, is the collection of the analytic functions on the D satisfying

$$\|f\|_{\mathcal{B}^\alpha} = \sup_{z\in D} |\mathcal{D}f(z)|(1-|z|^2)^\alpha < \infty.$$

Space $\mathcal{B}^\alpha(D)$ is a Banach space with the norm $\|f\|_{\mathcal{B}^\alpha}$. Note $\mathcal{B}^1(D) = \mathcal{B}(D)$ is a classical Bloch class (see [19] and the references there).

For $k > s$, $0 < p, q \le \infty$, the weighted analytic Besov space $\mathcal{B}^{q,p}_s(D)$ is the class of analytic functions satisfying (see [19])

$$\|f\|^q_{\mathcal{B}^{q,p}_s} = \int_0^1 \Big(\int_T |\mathcal{D}^k f(r\zeta)|^p |d\zeta| \Big)^{\frac{q}{p}} (1-r)^{(k-s)q-1} dr < \infty.$$

Quasinorm $\|f\|^q_{\mathcal{B}^{q,p}_s}$ does not depend on k. If $\min(p,q) \ge 1$, the class $\mathcal{B}^{q,p}_s(D)$ is a Banach space under the norm $\|f\|^q_{\mathcal{B}^{q,p}_s}$. If $\min(p,q) < 1$, then we have quasinormed class.

The well-known so called "duality" approach to extremal problems in theory of analytic functions leads to the following general formula

$$dist_Y(g, X) = \sup_{l\in X^\perp, \|l\|\le 1} |l(g)| = \inf_{\varphi\in X} \|g-\varphi\|_Y,$$

where $g \in Y$, X is subspace of a normed space Y, $Y \in H(D)$ and $X^\perp$ is the orthogonal compliment of X in Y^*, the dual space of Y and l is linear functional on Y (see [13]).

Various extremal problems in H^p Hardy classes in D based on duality approach we mentioned were discussed in [31], Chapter 8. In particular for a function $K \in L^q(T)$ the following equality holds (see [31]), $1 \le p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$,

$$dist_{L^q}(K, H^q) = \inf_{g\in H^q, K\in L^q} \|K-g\|_{H^q} = \sup_{f\in H^p, \|f\|_{H^p}\le 1} \frac{1}{2\pi} \Big| \int_{|\zeta|=1} f(\zeta)K(\zeta)d\zeta \Big|.$$

It is well known that if $p > 1$ then the inf-dual extremal problem in the analytic H^p Hardy classes has a solution, it is unique if an extremal function exists (see [31]). Note also that extremal problems for H^p spaces in multiply connected domains were studied before in [4], [21]. Various new results on extremal problems in A^p Bergman class and in its subspaces were obtained recently by many authors (see [12] and the references there). In this paper we will provide direct proofs for estimation of $dist_Y(f, X) = \inf_{g\in X} \|f-g\|_Y$, $X \subset Y$, $X, Y \subset H(D)$, $f \in Y$.

Let further $\Omega^k_{\alpha,\varepsilon} = \{z \in D : |D^k f(z)|(1-|z|^2)^\alpha \ge \varepsilon\}$, $\alpha \ge 0$, $\varepsilon > 0$, $\Omega^0_{\alpha,\varepsilon} = \Omega_{\alpha,\varepsilon}$.

Applying famous Fefferman duality theorem, P. Jones proved the following

Theorem A. *([31], [5]) Let $f \in \mathcal{B}$. Then the following are equivalent:*

(a) $d_1 = dist_{\mathcal{B}}(f, BMOA)$;

(b) $d_2 = \inf\{\varepsilon > 0 : \mathcal{X}_{\Omega^1_{1,\varepsilon}(f)}(z)\frac{dA(z)}{1-|z|^2}$ *is a Carleson measure*$\}$, *where* $\mathcal{X}$ *denotes the characteristic function of the mentioned set.*

Recently, R. Zhao (see [31]) and then W. Xu (see [30]), repeating arguments of R. Zhao in the unit ball, obtained results on distances from Bloch functions to some Möbius invariant function spaces in one and higher dimensions (unit disk and unit ball) in a relatively direct way. The goal of this paper is to develop further their ideas and present new sharp theorems for analytic spaces in higher dimension even in more complicated domains.

In next sections of this paper various new sharp assertions for distance function will be given and not only in the unit disk, but also in bounded strongly pseudoconvex domains and in tubular domains over symmetric cones in higher dimension. Since the proof of the unit disk is rather transparent from our point of view we will only indicate proofs of some assertions in details, short sketches of proofs in some cases will be also given.

Throughout the paper, we write C (sometimes with indexes) to denote a positive constant which might be different at each occurrence (even in a chain of inequalities) but is independent of the functions or variables being discussed.

Given two non negative real numbers A, B we will write $A \precsim B$ if there is a positive constant C such that $A < CB$.

2. On new estimates for distances in analytic function spaces of several complex variables and double Bergman representation formula

In this section we prove all main results of this paper.

Let $\alpha > r,\ r > 1,\ \alpha > 1$. Then obviously for A^1_{r-2} Bergman space and $BMOA$ type $Q_{r,\alpha}$ analytic classes we have.

$$\|f\|_{Q_{r,\alpha}} = \sup_{a_j \in D} \int_D \cdots \int_D \frac{|f(z_1,\dots,z_n)| \times \prod_{k=1}^n (1-|z_k|)^{\alpha-2}}{\prod_{k=1}^n (|1-\overline{a_k} z_k|)^{\alpha}} dm_{2n}(z) \times \prod_{k=1}^n (1-|a_k|)^r$$

$$\leq C\|f\|_{A^1_{r-2}} = C\int_{D^n} |f(z_1,...,z_n)| \times \prod_{k=1}^n (1-|z_k|)^{r-2} dm_{2n}(z),$$

where D^n is a unit polydisk, $f \in H(D^n)$, $H(D^n)$ is a space of holomorphic functions in D^n and $dm_{2n}(z)$ is a Lebesgue measure on D^n. So we have a natural extremal problem in polydisk to find

$$dist_{Q_{r,\alpha}}\left(f, A^1_{r-2}\right) = \inf_{g \in A^1_{r-2}} \|f-g\|_{Q_{r,\alpha}}, f \in Q_{r,\alpha}.$$

For $f \in H(D^n)$, we define a set

$$N_f = N^{\alpha,r}_{f,\varepsilon} = \left\{a \in D^n : \int_{D^n} \frac{|f(z_1,\dots,z_n)| \times \prod_{k=1}^n (1-|z_k|)^{\alpha-2}}{\prod_{k=1}^n (|1-\overline{a_k} z_k|)^{\alpha}} dm_{2n}(z) \prod_{k=1}^n (1-|a_k|)^r \geq \varepsilon\right\}$$

$\varepsilon > 0$, $\alpha > 1$, $r > 0$.

Remark 1. *Note A^1_{r-2} is a standard Bergman class in polydisk D^n studied before by various authors (see for example [11]). Classes $Q_{r,\alpha}$ in the unit disk D are so called BMOA type spaces were also under investigation by many authors recently (see [29], [28]).*

Applying classical Bergman representation for the unit disk n times by each variable we get Bergman representation formula in polydisk.(see also [11])

$$f(z_1,\dots,z_n) = C_\gamma \int_{D^n} \frac{f(\omega_1,...,\omega_n) \times \prod_{k=1}^n (1-|\omega_k|)^\gamma}{\prod_{k=1}^n (1-\overline{\omega_k} z_k)^{\gamma+2}} dm_{2n}(\omega),$$

where C_γ is constant, $z_j \in D$, $j = 1,\dots,n$, $\gamma > -1$, $f \in H(D)$. Choose $\gamma_1 > -1$, then applying Bergman representation formula twice we have a double Bergman representation for analytic f function in polydisk

$$f(z_1,...,z_n) = C_\gamma C_{\gamma_1} \int_{D^n} \frac{\prod_{k=1}^n (1-|\omega_k|)^\gamma}{\prod_{k=1}^n (1-\overline{\omega_k} z_k)^{\gamma+2}} \times$$

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$$\times \int_{D^n} \frac{\prod_{k=1}^n (1-|\widetilde{\omega}_k|)^{\gamma_1} f(\widetilde{\omega}_1,\dots,\widetilde{\omega}_n)}{\prod_{k=1}^n \left(1-\overline{\widetilde{\omega}}_k \omega_k\right)^{\gamma_1+2}} dm_{2n}(\widetilde{\omega})\, dm_{2n}(\omega) = f_1(z_1,\dots,z_n) + f_2(z_1,\dots,z_n)$$

and

$$f_1(z_1,\dots,z_n) = C \int_{D^n} \int_{D^n \setminus N_f} \frac{\prod_{k=1}^n (1-|\omega_k|)^\gamma}{\prod_{k=1}^n (1-\overline{\omega_k} z_k)^{\gamma+2}} \frac{\prod_{k=1}^n (1-|\widetilde{\omega}_k|)^{\gamma_1} f(\widetilde{\omega}_1,\dots,\widetilde{\omega}_n)}{\prod_{k=1}^n \left(1-\overline{\widetilde{\omega}}_k \omega_k\right)^{\gamma_1+2}}$$

$$\times dm_{2n}(\widetilde{\omega})\, dm_{2n}(\omega)$$

$$f_2(z_1,\ldots,z_n) = C\int_{D^n}\int_{N_f}\frac{\prod_{k=1}^n(1-|\omega_k|)^\gamma}{\prod_{k=1}^n(1-\overline{\omega_k}z_k)^{\gamma+2}}\frac{\prod_{k=1}^n(1-|\widetilde{\omega}_k|)^{\gamma_1}f(\widetilde{\omega}_1,...,\widetilde{\omega}_n)}{\prod_{k=1}^n\left(1-\overline{\widetilde{\omega}_k}\omega_k\right)^{\gamma_1+2}}dm_{2n}(\widetilde{\omega})\,dm_{2n}(\omega),$$

where $dm_{2n}(\omega) = dm_2(\omega_1)\cdots dm_2(\omega_n)$, $C = C_{\gamma,\gamma_1}$.

Fix some $\varepsilon > 0$. Our task to show $f_1 \in Q_{r,\alpha}$, $\|f_1\|_{Q_{r,\alpha}} \le C\varepsilon$, $f_2 \in A^1_{r-2}$. Then we will have

$$dist_{Q_{r,\alpha}}\left(f, A^1_{r-2}\right) \le C\|f-f_2\|_{Q_{r,\alpha}} = C\|f_1\|_{Q_{r,\alpha}} \le C\varepsilon,\ f \in Q_{r,\alpha}.$$

Moreover it turns out the reverse estimate is also true and we have the following sharp theorem. We assume also that all inf everywhere in paper are taken from not empty sets.

Theorem 1. *Let $r < \alpha-1$, $r > 1$, $\alpha > 1$, $f \in Q_{r,\alpha}$, then the following are equivalent:*

(1) $dist_{Q_{r,\alpha}}\left(f,\ A^1_{r-2}\right)$;

(2) $\inf\left\{\varepsilon > 0,\ \int_{D^n}\left(\chi_{N_f}\right)(a_1,\ldots,a_n)\prod_{k=1}^n(1-|a_k|)^{-2}dm_{2n}(a) < \infty\right\}$, *where χ_M is characteristic function of a set M, $M \subset D^n$.*

Proof. Let us show first the implication (2) → (1) using arguments we provided above. We have for $\gamma > r-2$

$$\|f_2\|_{A^1_{r-2}} = \int_{D^n}\prod_{k=1}^n(1-|z_k|)^{r-2}|f(z_1,...,z_n)|\,dm_{2n}(z)$$

$$\le C\int_{D^n}\prod_{k=1}^n(1-|z_k|)^{r-2}\int_{D^n}\int_{N_f}\frac{\prod_{k=1}^n(1-|\omega_k|)^\gamma}{\left|\prod_{k=1}^n(1-\overline{\omega_k}z_k)^{\gamma+2}\right|}\times$$

$$\times\int_{D^n}\frac{\prod_{k=1}^n(1-|\widetilde{\omega}_k|)^{\gamma_1}|f(\widetilde{\omega}_1,...,\widetilde{\omega}_n)|}{\left|\prod_{k=1}^n\left(1-\overline{\widetilde{\omega}_k}\omega_k\right)^{\gamma_1+2}\right|}dm_{2n}(\widetilde{\omega})\,dm_{2n}(\omega)\,dm_{2n}(z),$$

Hence using Fubini's theorem and putting $\gamma_1 = \alpha-2$, we will have

$$\|f_2\|_{A^1_{r-2}} = C\int_{D^n}\int_{N_f}\frac{|f(\widetilde{\omega_1},\ldots,\widetilde{\omega_n})|\prod_{k=1}^n(1-|\widetilde{\omega_k}|)^{\gamma_1}}{\left|\prod_{k=1}^n\left(1-\overline{\widetilde{\omega_k}}\omega_k\right)^{\gamma_1+2}\right|}\times\prod_{k=1}^n(1-|\omega_k|)^\gamma\times$$

$$\times\prod_{k=1}^n(1-|\omega_k|^{r-\gamma-2}dm_{2n}(\widetilde{\omega})dm_{2n}(\omega) \le \sup_{\omega_k}C\int_{D^n}\frac{|f(\widetilde{\omega}_1,\ldots,\widetilde{\omega_n})|\prod_{k=1}^n(1-|\widetilde{\omega_k}|)^{\alpha-2}}{\prod_{k=1}^n(|1-\widetilde{\omega_k}\overline{\omega}_k|)^\alpha}\times$$

$$\times \prod_{k=1}^{n} (1-|\omega_k|)^r \left(\int_{D^n} (\chi N_f)(\omega_1, \dots, \omega_n) \prod_{k=1}^{n} (1-|\omega_k|)^{-2} \, dm_{2n}(\omega) \right)$$

We used that

$$\int_{D^n} \frac{\prod_{k=1}^{n} (1-|\omega_k|)^{t_1} \, dm_{2n}(\omega)}{\prod_{k=1}^{n} |(1-\widetilde{\omega_k} z_k)|^{t_2}} \le C \prod_{k=1}^{n} (1-|z_k|)^{t_1+2-t_2}, t_1 > -1, t_2 > t_1+2, z \in D^n \tag{1}$$

Now we show $\|f_1\|_{Q_{r,\alpha}} \le C\varepsilon$. We have using (1) and Fubini's theorem for $\gamma > r-1$

$$\|f_1\| \, Q_{r,a} = (\sup_{a_j \in D}) \int_{D^n} \frac{|f_1(z_1, ..., z_n)| \prod_{k=1}^{n} (1-|z_k|)^{\alpha-2}}{\prod_{k=1}^{n} |(1-\bar{a}_k z_k)|^{\alpha}} dm_{2n}(z) \times \prod_{k=1}^{n} (1-|a_k|)^r$$

$$\le (C \sup_{a_j \in D}) \int_{D^n} \int_{D^n} \int_{D^n \setminus N_f} \frac{\prod_{k=1}^{n} (1-|\omega_k|)^{\gamma} \prod_{k=1}^{n} (1-|z_k|)^{\alpha-2}}{\prod_{k=1}^{n} |(1-\bar{\omega}_k z_k)|^{\gamma+2}} \times \prod_{k=1}^{n} (1-|a_k|)^r \times$$

$$\times \frac{\prod_{k=1}^{n} (1-|\widetilde{\omega_k}|)^{\alpha-2} |f(\widetilde{\omega_1}, ..., \widetilde{\omega_n})|}{\prod_{k=1}^{n} \left| \left(1-\overline{\widetilde{\omega_k}} \omega_k\right)^{\alpha} \right|} dm_{2n}(z) \, dm_{2n}(\widetilde{\omega}) \, dm_{2n}(\omega)$$

$$\le (C\varepsilon) \sup_{a_j \in D} \frac{\prod_{k=1}^{n} (1-|a_k|)^r \prod_{k=1}^{n} (1-|\omega_k|)^{\alpha-2-r}}{\prod_{k=1}^{n} |(1-\overline{a_k} \omega_k)|^{\alpha}} dm_{2n}(\omega) \le C_1 \varepsilon.$$

So we showed one part of our theorem. To show the reverse we assume the reverse to that assertion is true. Hence by assumption there are ε, ε_1, $\varepsilon > 0, \varepsilon_1 > 0$ and there is $f_{\varepsilon_1} \in A^1_{r-2}$, $\varepsilon > \varepsilon_1$, $\|f - f_{\varepsilon_1}\| \le \varepsilon_1$ and

$$K = \int_{D^n} \left(\chi_{N_f}(a)\right)(a_1, \dots, a_n) \prod_{k=1}^{n} (1-|a_k|)^{-2} \, dm_{2n}(a) = \infty.$$

Using this we arrive easily to contradiction. Indeed we have $\widetilde{f} = f - f_{\varepsilon_1}, \tau > \rho$, $\beta - \tau + \rho = r-2,\ \rho > 1,\ \beta > -1,\ \tau > 0,\ a \in D^n$

$$M(a) = \int_{D^n} \frac{|f_{\varepsilon_1}(z_1, \dots, z_n)| \prod_{k=1}^{n} (1-|z_k|)^{\beta}}{\prod_{k=1}^{n} |(1-\bar{a}_k z_k)|^{\tau}} dm_{2n}(z) \prod_{k=1}^{n} (1-|a_k|)^{\rho}$$

$$\geq \prod_{k=1}^{n}(1-|a_k|)^{\rho}\int_{D^n}\frac{|f_1(z_1,\dots,z_n)|\prod_{k=1}^{n}(1-|z_k|)^{\beta}}{|\prod_{k=1}^{n}(1-\bar{a}_k z_k)|^{\tau}}dm_{2n}(z)$$

$$-\left(\sup_{a\in D^n}\right)\int_{D^n}\frac{|f_{\varepsilon_1}(z)-f(z)|\prod_{k=1}^{n}(1-|z_k|)^{\beta}}{|\prod_{k=1}^{n}(1-\bar{a}_k z_k)|^{\tau}}dm_{2n}(z)\prod_{k=1}^{n}(1-|a_k|)^{\rho}.$$

Hence $M(a)\geq\left((\varepsilon-\varepsilon_1)\left(\chi_{N_f}(a)\right)\right)$ and by (1) after choosing appropriate $\rho>1$ and $\beta>-1$, we will have

$$\int_{D^n}|f_{\varepsilon_1}(z)|(1-|z_1|)^{r-2}(1-|z_n|)^{r-2}dm_{2n}(z)\geq(\varepsilon-\varepsilon_1)K.$$

So we have an obvious contradiction. Our theorem is proved.

The careful analysis of the proof of this theorem in polydisk shows that only four tools were used in it. First Forelly-Rudin type estimate, then Bergman representation formula, then two simple estimates of Bergman kernel. The first estimate of Bergman kernel allows to pose a dist problem. The second estimate provides integral representation for all functions from our BMOA type classes. These tools are available in various other domains in higher dimension which were under study in recent decades and this fact will be heavily used in this paper. We will discuss in detail below these conditions for various domains. In following theorem we repeat the same proof and formulate first sharp theorem for *dist* function in BMOA type spaces in bounded pseudoconvex domains with smooth boundary. First we formulate the same result in most typical case of such type domains namely in the unit ball (the model case of such type domains). We need some standard definitions from the theory in the unit ball (see, for example, [11] and [26]). Let dv volume measure on the unit ball B, $d\sigma$ be the standard Lebesque measure on $\mathbb{S}$ sphere. Let $r>n$, $\alpha>r$, $\alpha>n$. We define BMOA type spaces and Bergman spaces in the ball $Q_{r,\alpha}$ and A^1_{r-n-1} and note that the following estimate is valid for norms of these spaces.

$$\|f\|_{Q_{r,\alpha}(B)}=(\sup_{a\in B})\int_B\frac{|f(\omega)|(1-|\omega|)^{\alpha-n-1}}{(|1-<\omega,a>|)^{\alpha}}dv(z)\times(1-|a|)^{r}$$

$$\leq C\int_B|f(\omega)|(1-|\omega|)^{r-n-1}dv(\omega)=\|f\|_{A^1_{r-n-1}(B)}.$$

Let

$$\widetilde{N}_f=\left\{a\in B:\left(\int_B\frac{|f(\omega)|(1-|\omega|)^{\alpha-n-1}dv(\omega)}{(|1-<a,\omega>|)^{\alpha}}(1-|a|)^{r}\geq\varepsilon\right)\right\}.$$

Theorem 2. *Let $r > n$, $\alpha > n$, $r < \alpha - n$, $f \in Q_{r,\alpha}(B)$, then the following are equivalent:*

(1) $dist_{Q_{r,\alpha}(B)}\left(f,\ A^1_{r-n-1}(B)\right)$;

(2) $\inf\left\{\varepsilon > 0,\ \int_B \left(\chi_{\widetilde{N}_f}\right)(a)\,(1-|a|)^{-n-1}\,dv\,(a) < \infty\right\}$.

The proof is simply repetition of the proof of our previous theorem, but in case of unit ball.

The same proof allows to formulate even more general version for bounded pseudoconvex domains. We need some standard definitions in bounded strongly pseudoconvex domains (see, for example, [1] and [23]). Let $\Omega \subset\subset \mathbb{C}^n$ be a bounded strongly pseudoconvex domain in $\mathbb{C}^n$. We shall use the following notations:

- $\delta : \Omega \to \mathbb{R}^+$ will denote as usual the Euclidean distance from the boundary, that is $\delta(z) = d(z, \partial\Omega)$;
- We denote as usual by $K_\alpha(z,w)$ the weighted Bergman kernel with index α in our domain Ω.
- We denote by dv the Lebegues measure in Ω.

Let further,

$$\widetilde{\widetilde{N}}_f = \left\{a \in \Omega : \left(\int_\Omega |f(\omega)|\,(\delta(w))^{\alpha-n-1}|K_\alpha(a,w)|dv(\delta(a))^r \geq \varepsilon\right)\right\}.$$

The proof of the following theorem in bounded strictly pseudoconvex domains with smooth boundary is the same as the proof of previous theorem in the unit ball based on Forelly-Rudin estimate for these domains and standard estimates for the Bergman kernel (see, for example, [1] and references there). We define BMOA and Bergman spaces in this domain and note that the following estimate is valid for norms of these spaces.

For $r > n$, $\alpha > r$, $\alpha > n$ we have the following estimate.

$$\|f\|_{\tilde{Q}_{r,\alpha}(\Omega)} = (\sup_{a\in\Omega}) \int_\Omega |f(\omega)|\,(\delta(w))^{\alpha-n-1}|K_\alpha(a,w)|dv\,(z) \times (\delta(a))^r$$
$$\leq C \int_\Omega |f\,(\omega)|\,(\delta(w))^{r-n-1}dv\,(\omega) = \|f\|_{A^1_{r-n-1}(\Omega)}.$$

Indeed we have for weighted Bergman kernel (see, for example, [1], [6], [7]) $|K_\alpha(z,w)| \leq |K_\alpha(z,z)|$, $z, w \in \Omega$. And this immediately as in other cases above leads to related embeddings between BMOA-type and Bergman type spaces in bounded strictly pseudoconvex domains with smooth boundary in $\mathbb{C}^n$.

Hence then we have that $dist_X(f, Y)$ problem can be posed. The remaining is the Bergman representation formula and Forelly-Rudin type estimate which are also available in these type of domains (see [1]). For some other similar results on such type extremal problems in pseudoconvex domain we refer the reader to our papers [5], [23].

We now need to justify the Bergman integral representation, since it was used in our proof in simpler cases. Note when we deal with the unit disk (or the unit ball) we have that if $f \in H(D)$ (or $f \in H(B)$) then

$$f(z) = c(\alpha) \int_D \frac{f(w)(1-|w|)^\alpha dm_2(w)}{(1-\bar{w}z)^{\alpha+2}},\ \alpha > -1$$

or

$$f(z) = c(\beta) \int_B \frac{f(w)(1-|w|)^\beta dv(w)}{(1-\bar{w}z)^{\beta+n+1}},\ \beta > -1.$$

This is a classical fact.

In case of bounded strictly pseudoconvex domains with smooth boundary Ω in $\mathbb{C}^n$ we must use the theorem of Beatros first (see [7]) which says that for any f function $f \in A^p_\alpha$, $\alpha > -1$, $0 < p < \infty$ admits the same Bergman integral representation, $\beta > \beta_0$

$$f(z) = c(\beta) \int_\Omega f(w)\delta(w)^\beta K_{\beta+n+1}(z,w)dv(w),\ z \in \Omega$$

for sufficiently large β_0. Note since domain is bounded we have that the same is valid for any f, $f \in A^\infty_\tau = \{f \in H(\Omega) : |f(z)|\delta(z)^\tau\}$, $\tau \geq 0$ since in this case $A^\infty_\tau \subset A^p_\alpha$, $\alpha > \alpha_0$ with estimates of norms (quazinorms) (see for this also [7]).

Now we must show that for these bounded strongly pseudoconvex domains any functions f, $f \in Q_{r,\alpha}(\Omega)$ admits similar representation of Bergman (at least for large indexes).

The key ingredient is the fact that

$$\|K_\alpha(z,w)\| = |K_{n+1}(z,w)|^{\frac{\alpha}{n+1}} \geq |K_{n+1}(z,z)|^{\frac{\alpha}{n+1}},\ z,w \in B_\Omega(z,r) \tag{2}$$

for $\alpha = m(n+1)$, $m \in \mathbb{N}$, where $B_\Omega(z,r)$ is a Kobayashi ball.

This can be seen, for example, in [1].

Now we have (as model case we look at the unit disk case) by subharmonicity of f for $r \geq 0$, $\alpha > 1$, the following estimate

$$(1-|z|)^r|f(z)| \leq \frac{c}{(1-|z|)^\alpha} \int_{D(z,r)} \frac{|f(w)|(1-|w|)^{\alpha-2}dm_2(w)}{|1-z\bar{w}|^\alpha}(1-|z|)^\alpha(1-|z|)^r;$$

$z \in D = \{|\tilde{z}| < 1\}$, where $D(z,r)$ is a Bergman ball in the unit disk. Hence we have that

$$\sup_{z\in D}(1-|z|)^r|f(z)| \leq c \sup_{a\in D}\int_D \frac{|f(w)|(1-|w|)^{\alpha-2}}{|1-\bar{a}w|^{\alpha}} dm_2(w)(1-|a|)^r,$$

$\alpha > 1$, $r > 0$, $f \in H(D)$. The same arguments allows to show the same estimate in Ω pseudoconvex domains based on (2) and on analogue of the estimate which follows (2) based on lemmas from [1]. Hence based on discussion above if $f \in Q_{r,\alpha}(\Omega)$ then Bergman representation formula for large enough index β is valid. This is important ingredient of the proof in this Ω domain and in all domains we consider in this paper. We have the following theorem.

Theorem 3. *Let $r > n$, $\alpha > n$, $r < \alpha - n$, $f \in \tilde{Q}_{r,\alpha}(\Omega)$, then the following are equivalent:*

(1) $dist_{\tilde{Q}_{r,\alpha}(\Omega)}\left(f,\ A^1_{r-n-1}(\Omega)\right)$;

(2) $\inf\left\{\varepsilon > 0,\ \int_\Omega \left(\chi_{\tilde{\tilde{N}}_f}\right)(a)\,(\delta(a))^{-n-1}d\sigma(a) < \infty\right\}$.

Now we provide complete analogue of this result in tube. The proof is the repetition of same arguments. This is the first result (sharp) of this type for tube. We first need some standard definitions of the theory of analytic functions in tubular domains over symmetric cones. Let $T_\Omega = V + i\Omega$ be the tube domain over an irreducible symmetric cone Ω in the complexification $V^{\mathbb{C}}$ of an n-dimensional Euclidean space V. $\mathcal{H}(T_\Omega)$ denotes the space of all holomorphic functions on T_Ω. Following the notation of [8] we denote the rank of the cone Ω by r and by Δ the determinant function on V.

Below we denote by Δ_s the generalized power function, (see [8]), dv is the Lebegues measure on tube.

Let

$$Q_{\tilde{r},\alpha}(T_\Omega) = \left\{f \in H(T_\Omega) : \sup_{a\in T_\Omega}\int_{T_\Omega} \frac{|f(w)|\Delta^{\alpha-\frac{2n}{r}}(Im\ w)dv(w)}{\left|\Delta(\frac{a-\bar{z}}{i})\right|^{\alpha}}\Delta^{\tilde{r}}(Im\ a) < \infty\right\};$$

$\tilde{r} > 0$; $\alpha > \frac{2n}{r} - 1$.

$$A^1_{\tilde{r}-\frac{2n}{r}}(T_\Omega) = A^1_{\tilde{r}-\frac{2n}{r}} = \left\{f \in H(T_\Omega) : \int_{T_\Omega} |f(w)|\Delta^{\tilde{r}-\frac{2n}{r}}(Im\ w)dv(w) < \infty\right\};$$

$\tilde{r} > \frac{2n}{r} - 1$.

These are Banach spaces. The proof of the following sharp theorem as it was indicated already is the repetition of the proof of previous theorem but based on some known and standard facts of the theory of functions in tubular domains from [9], [10]. Recently a series of similar sharp results concerning to distances in this direction in analytic spaces in tubular domains over symmetric cones was obtained in an interesting paper [25].

A well known estimate of the Bergman kernel from above allows to provide an embedding of Bergman space A^1 into BMOA type spaces In tube similarly as we did in other domains and hence a distance problem can be posed again for this pair of spaces.(see for this estimate for example [1])

Theorem 4. *Let $\tilde{r} > \frac{2n}{r} - 1$, $\alpha > \frac{2n}{r} - 1$, $\tilde{r} < \alpha - \frac{2n}{r} + 1$. Let $f \in Q_{\tilde{r},\alpha}(T_\Omega)$. Then the following are equivalent:*

(1) $dist_{Q_{\tilde{r},\alpha}}(f, A^1_{\tilde{r}-\frac{2n}{r}})$

(2) $\inf\left\{\varepsilon : \int_{T_\Omega} \chi_{\tilde{N}_{f,\varepsilon}}(a)[\Delta(Im\ a)]^{-\frac{2n}{r}} dv(a) < \infty\right\}$, *where*

$$\tilde{N}_f = \left\{a \in T_\Omega : \int_{T_\Omega} |f(w)|(\Delta^{\alpha-\frac{2n}{r}}(Im\ w))dv(w)\frac{1}{\Delta(\frac{a-\bar{w}}{i})^\alpha}\Delta^{\tilde{r}}(Im\ a) \geq \varepsilon\right\}.$$

The proof repeats previous cases based on basics of theory of tubular domains (see, for example, [8], [9], [10]). We provide the full scheme of the proof leaving details to readers. The distance problem can be posed since the Bergman kernel can be estimated from above by Δ function. This fact is well-known (see, for example, [8], [9], [10]). Let

$$B_\nu(z, w) = C_\nu \Delta^{-(\nu+\frac{n}{r})}((z - \overline{w})/i)$$

be the Bergman reproducing kernel for Bergman space

$$A^2_\nu(T_\Omega)$$

(see [8]). The following vital Forelly-Rudin type estimate for Bergman kernel (A_1) which we use in proof of our main result (see [22]).

Lemma 5.

$$\int_{T_\Omega} \Delta^\beta(y)|B_{\alpha+\beta+\frac{n}{r}}(z, w)|dV(z) \leq C\Delta^{-\alpha}(v), \tag{3}$$

$\beta > -1$, $\alpha > \frac{n}{r} - 1$, $z = x + iy$, $w = u + iv$ (see [22]).

Further the Bergman representation formula for all f, $f \in Q_{\tilde{r},\alpha}$ is valid since we note that for $\tilde{r} > \frac{2n}{r} - 1$ we have estimate (G)

$$\sup_{z\in T_\Omega} |f(z)|\Delta^{\tilde{r}}(Im\ z) \leq c \sup_{a\in T_\Omega} \int_{T_\Omega} \frac{|f(w)|\Delta(Im\ w)^{\alpha-\frac{2n}{r}}}{\left|\Delta(\frac{a-\bar{w}}{i})^\alpha\right|}(\Delta(Im\ a))^{\tilde{r}} dv(w). \tag{4}$$

Based on important estimate from below of the Bergman kernel on Bergman balls this estimate can be shown immediately as above in the unit disk where the same estimate from below of the Bergman kernel for proof was used(see recent paper [22] for this estimate from below of Bergman kernel on Bergman ball in tubular domains over symmetric cone). From this estimate (4) we have that for each analytic f function from BMOA type $Q_{\tilde{r},\alpha}$ space the Bergman representation formula is valid for large Bergman kernel ν index, since this type integral representation is valid for all functions from A^{∞}_{α} spaces for all positive α indexes (see for this known fact, for example, [22], [9], [10]). This is an important part of our proof in tubular domains as well as in analytic spaces defined in polydisk and pseudoconevx domains which studied earlier in this paper.

We finally remark that some results of similar type with the same proof can be shown in bounded symmetric domains, in minimal homogeneous domains and even in Siegel domains of second type. This will be done in our next papers. Indeed a careful analysis of proof of our results shows some (not sharp) analogues of our results under some restrictions on behavior and some additional conditions on the Bergman kernel can be provided even in general weakly pseudoconvex domains, in Siegel domains, in bounded symmetric domains and minimal bounded homogeneous domains where some analogues of our main lemmas (in particular Forelly-Rudin type estimates and Bergman type integral representation) needed for proof can be seen (see [6] and [14], for example, for minimal homogeneous domains and for Siegel domains of second type).

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