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SOME RESULTS CONSISTING OF CERTAIN INEQUALITIES SPECIFIED BY NORMALIZED ANALYTIC FUNCTIONS AND THEIR IMPLICATIONS

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Abstract. The aim of this work is to determine some useful results consisting of certain inequalities and normalized functions analytic in the unit open disk and then to present certain geometric and analytic implications of them, as our conclusion.

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1. Introduction and Definitions

First of all, let us denote by $\mathbb{N}$, $\mathbb{C}$, $\mathbb{U}$ and $\mathcal{H}$ the set of natural numbers, the set of complex numbers, the unit open disk, i.e., the set $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ and the class of all analytic functions in $\mathbb{U}$, respectively. Also let $\mathcal{A}_n$ be the subclass of all functions $f(z)$ in the class $\mathcal{H}$ satisfying the conditions:

$$f(0) = 0 \quad \text{and} \quad f'(0) = 1.$$

In other words, the functions $f(z)$ belonging to the class $\mathcal{A}_n$ have the complex power series representation:

$$f(z) = z + a_{n+1}z^{n+1} + a_{n+2}z^{n+2} + \cdots \quad \big(a_{n+1} \in \mathbb{C}; n \in \mathbb{N}; z \in \mathbb{U}\big).$$

In the light of the literature, as is known, certain geometrical and analytical properties of complex valued functions are quite important in the univalent function theory. Therefore, the well-known function classes $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$ are very important and they are also called the classes of all starlike functions and convex functions of order α $(0 \leq \alpha < 1)$ in $\mathbb{U}$. In general, the subclass consisting of univalent functions is denoted by $\mathcal{S}$ and the special subclasses $\mathcal{K}$ and $\mathcal{S}^*$ of the class $\mathcal{A}_n$ are known as the classes of convex functions and starlike functions, respectively. For the details of the definitions of those functions classes (and also certain results about them), one may see [1], [2], [3], [4], [5], [11] and also [12].

Furthermore, in the literature, there are several papers including important or interesting results relating to certain inequalities and also certain classes of the functions which are analytic and univalent in the domain $\mathbb{U}$. For those, one may look over the results in the papers in nearly all references. But, especially, the results concerning the problem of finding λ satisfying the condition $|f''(z)| \leq \lambda \ \big(f(z) \in \mathcal{A}_n; z \in \mathbb{U}\big)$ implies $f(z) \in \mathcal{S}^*$ which was first considered by Mocanu [8] for the value of $\lambda = 2/3$. Later, Ponnusamy and Singh [10] obtained a better value of the parameter $\lambda = 2/\sqrt{3}$. Afterwards, Obradovic [9] centered on same problem with the value of $\lambda = 1$ by proving that his result is sharp. Tuneski [13] also obtained certain results consisting of the same problems, by using techniques used by Obradovic [9].

In this investigation, two general results in relation with certain conditions concerning $|f''(z)| \leq \lambda \ \big(f(z) \in \mathcal{A}_n; z \in \mathbb{U}\big)$ and also the classes indicated above are first stated and some of their consequences which will be important for analytic or geometric function theory are then presented. In particular, for the proofs of some consequences of our results, both the novel result generated by the assertion obtained by Miller and Mocanu [7] (see p. 33-35), i.e. Lemma 3, together with the results obtained by Tuneski [13] (Lemmas 1 and 2 below) and the well-known result obtained by Jack [6] (Lemma 4) are there used. In addition, as example and, particularly, for the methods used in the proofs of the earlier results given by [3], [4], [5] and [12], one may check the assertions obtained by [6] and the novel form of the result given by [7]. (See also [11].)

The following important assertions that we indicated above, i.e., Lemma 1, Lemma 2, Lemma 3 and Lemma 4 below, will be required in our main results.

Lemma 1. ([13]) *Let $f(z) \in \mathcal{A}_n$, $z \in \mathbb{U}$ and $0 \leq \alpha < 1$. Then,*

$$|f''(z)| \leq \frac{2(1-\alpha)}{2-\alpha} \implies f(z) \in \mathcal{S}^*(\alpha).$$

Lemma 2. ([13]) *Let $f(z) \in \mathcal{A}_n$, $z \in \mathbb{U}$ and $0 \leq \alpha < 1$. Then,*

$$|f''(z)| \leq \frac{1-\alpha}{2-\alpha} \implies f(z) \in \mathcal{K}(\alpha).$$

Lemma 3. ([13], p. 33-34) *Let $\Omega \subset \mathbb{C}$ and suppose that the function $\psi : \mathbb{C}^2 \times \mathbb{U} \to \mathbb{C}$* satisfies *$\psi(Me^{i\theta}, Ke^{i\theta}; z) \notin \Omega$ for all $K \geq mM\frac{M-|a|}{M+|a|}$, $\theta \in \mathbb{R}$, and $z \in \mathbb{U}$. If the function $p(z)$ is in the class $\mathcal{H}[a, m] \equiv \{p(z) \in \mathcal{H}(\mathbb{U}) \ : \ p(z) = a + a_m z^m + \dots\}$ and $\psi(p(z), zp'(z); z) \in \Omega$ for all $z \in \mathbb{U}$, then $|p(z)| < M$, where $M > |a| \geq 0$ and $m \in \mathbb{N} = \{1, 2, 3, \cdots\}$.*

Lemma 4. ([6]) *Let $w(z)$ be non-constant and analytic in $\mathbb{U}$ with $w(0) = 0$. If $|w(z)|$ attains its maximum value on the circle $|z| = r < 1$ at a point z_0, then $z_0 w'(z_0) = cw(z_0)$, where c is a real number and $c \geq 1$.*

2. The Main Results

By making use of the Lemma 3, firstly, we shall state and then prove the following result, which is given by the following theorem (Theorem 1 below).

Theorem 1. *Let $z \in \mathbb{U}$, $f(z) = z + a_2 z^2 + a_3 z^3 + ... \in \mathcal{A}_n$* and $0 \leq 2|a_2| < M$. *Then,*

$$\left|\frac{zf'''(z)}{1 \pm f''(z)}\right| < \frac{M(M - 2|a_2|)}{(M+1)(M+2|a_2|)} \implies |f''(z)| < M. \tag{1}$$

Proof. Define a function $p(z)$ as in the form

$$p(z) = f''(z), \tag{2}$$

where $f(z) = z + a_2 z^2 + a_3 z^3 + ... \in \mathcal{A}_n$ and $z \in \mathbb{U}$. Then, obviously, the function $p(z)$ is in the class $\mathcal{H}[2a_2, 1]$ with, of course, $a_3 \neq 0$. From (2), it follows that

$$\frac{zp'(z)}{1 \pm p(z)} = \frac{zf'''(z)}{1 \pm f''(z)} \quad \big(z \in \mathbb{U};\ f''(z) \neq \pm 1\big).$$

Now, let $\psi(r, s; z)$ and Ω denote by

$$\psi(r, s; z) := \frac{s}{1 \pm r} \quad \big(r \neq \pm 1\big)$$

and

$$\Omega := \left\{ w \in \mathbb{C} \ : \ |w| < \frac{M(M - 2|a_2|)}{(M+1)(M+2|a_2|)} \quad (0 \leq 2|a_2| < M) \right\}, \tag{3}$$

respectively. Then, we receive that

$$\psi\big(p(z), zp'(z); z\big) := \frac{zf'''(z)}{1 \pm f''(z)} \quad \big(z \in \mathbb{U};\ f''(z) \neq \pm 1\big)$$

belongs to the domain Ω, defined in (3), for all $z \in \mathbb{U}$. Further, in view of Lemma 3, for any

$$\theta \in \mathbb{R}\ ,\ K\left(\geq mM\frac{M - 2|a_2|}{M + 2|a_2|}\right) \geq M\frac{M - 2|a_2|}{M + 2|a_2|} \ \text{ and } \ z \in \mathbb{U},$$

we then get

$$\left|\psi(Me^{i\theta},Ke^{i\theta};z)\right| = \left|\frac{Ke^{i\theta}}{1\pm Me^{i\theta}}\right| \geq \frac{M(M-2|a_2|)}{(M+1)(M+2|a_2|)} \quad (since\ m\geq 1),$$

that is, that

$$\psi\left(Me^{i\theta},Ke^{i\theta};z\right)\notin\Omega.$$

Therefore, according to the Lemma 3, the definition of the function $p(z)$ in (2) yields that

$$|p(z)| = |f''(z)| < M \quad \left(z\in\mathbb{U}; M > 2|a_2|\geq 0\right),$$

which completes the proof of Theorem 1.

By making use of the Lemma 4, secondly, we shall state and then prove the following result, which is given by the following theorem (Theorem 2 below).

Theorem 2. *Let $f(z)=z+a_2z^2+a_3z^3+...\in\mathcal{A}_n$, $\delta>0$, $0\leq\alpha<1$, and $z\in\mathbb{U}$. Then,*

$$\Re e\left(\frac{zf''(z)}{f'(z)}\right) > \frac{\alpha-1}{2\delta(1+\alpha)} \implies \Re e\left(\left[f'(z)\right]^{\delta}\right) > \frac{1+\alpha}{2}, \tag{4}$$

where the value of each one of the above complex power and its applications is taken as its principal value.

Proof. Let $f(z)=z+a_2z^2+a_3z^3+...$ be in the class $\mathcal{A}_n$ and define an implicit function $w(z)$ by

$$\left[f'(z)\right]^{\delta} = \frac{1+\alpha w(z)}{1+w(z)} \quad \left(f(z)\in\mathcal{A}; z\in\mathbb{U}; 0\leq\alpha<1; \delta>0\right). \tag{5}$$

Then, clearly, $w(z)$ is an analytic function in $\mathbb{U}$ with $w(0)=0$. By differantiating of the both sides of (5) and then by making use of (5) once again there, we easy derive that

$$\frac{zf''(z)}{f'(z)} = \frac{1}{\delta}\cdot\left(\frac{\alpha zw'(z)}{1+\alpha w(z)} - \frac{zw'(z)}{1+w(z)}\right), \tag{6}$$

where $w(z)\neq -1$, $z\in\mathbb{U}$, $0\leq\alpha<1$ and $\delta>0$.

Assume now that there exists a point $z_0\in\mathbb{U}$ such that

$$\max|w(z)| = |w(z_0)| = 1 \ \ when\ \ |z|\leq|z_0|, \tag{7}$$

where $z\in\mathbb{U}$. Then, applying Lemma 4, we have

$$z_0w'(z_0) = cw(z_0) \quad \left(c\geq 1; w(z_0)=e^{i\theta}\neq -1\right) \tag{8}$$

Therefore, we obtain from (6), (7) and (8) that

$$\begin{aligned} \Re e\left(\frac{\delta z_0 f''(z_0)}{f'(z_0)}\right) &= \Re e\left(\frac{\alpha z_0 w'(z_0)}{1+\alpha w(z_0)} - \frac{z_0 w'(z_0)}{1+w(z_0)}\right) \\ &= \Re e\left(\frac{c\alpha e^{i\theta}}{1+\alpha e^{i\theta}}\right) - \Re e\left(\frac{c e^{i\theta}}{1+e^{i\theta}}\right) \\ &= \frac{c\alpha(\alpha + Cos(\theta))}{1+\alpha^2+2\alpha Cos(\theta)} - \frac{c}{2} \\ &\leq \frac{\alpha-1}{2(1+\alpha)}. \end{aligned} \tag{9}$$

The inequality in (9) obviously contradicts the hypothesis of the proposition in (4). Hence, $|w(z)| < 1$ for all z in the domain $\mathbb{U}$. Consequently, we conclude from (5) that

$$\left|\frac{1-[f'(z)]^\delta}{[f'(z)]^\delta - \alpha}\right| = |w(z)| < 1 \quad \big(z \in \mathbb{U}; 0 \leq \alpha < 1; \delta > 0\big), \tag{10}$$

that is, that the inequality (10) immediately yields the conclusion of the proposition given by (4). Thus, it completes the proof of the Theorem 2.

3. Certain Implications of the Main Results

In this section, as certain implications of our main results, by selecting the suitable values of the parameters in the both theorems (Theorems 1 and 2 above), with the help of certain new definitions or by putting extra conditions to the both theorems, there can be revealed several important or interesting results concerning analytic or univalent functions in the functions class $\mathcal{A}_n$. It is not possible to expose all of them. But, it can present only some of them, which deal with geometric properties of the related complex functions. For both those and their applications, one may refer to the works in [1], [2] and (see also) [5] and [12].

In view of the Theorem 1 and the Lemma 1, the following result can be first stated, which is Proposition 1 below.

Proposition 1. *Let* $f(z) = z + a_2 z^2 + a_3 z^3 + ... \in \mathcal{A}_n$, $f''(z) \neq \pm 1$, $0 \leq |a_2| < \frac{1-\alpha}{2-\alpha}$ *and* $z \in \mathbb{U}$. *Then,*

$$\left|\frac{z f'''(z)}{1 \pm f''(z)}\right| < \frac{2(1-\alpha)[1-\alpha-(2-\alpha)|a_2|]}{(4-3\alpha)[1-\alpha+(2-\alpha)|a_2|]} \implies f(z) \in \mathcal{S}^*(\alpha).$$

Proof. By taking

$$M := \frac{2(1-\alpha)}{2-\alpha} \quad \big(0 \leq \alpha < 1\big)$$

in the Theorem 1 and then using of the Lemma 1, it can be easily obtained the proof of the proposition 1.

By letting $\alpha := 0$ in the Proposition 1, the following result (Corollary 1 below) can be then given.

Corollary 1. *Let* $f(z) = z + a_2 z^2 + a_3 z^3 + ... \in \mathcal{A}_n$, $f''(z) \neq \pm 1$, $0 \leq |a_2| < \frac{1}{2}$ *and* $z \in \mathbb{U}$. *Then,*

$$\left| \frac{z f'''(z)}{1 \pm f''(z)} \right| < \frac{1 - 2|a_2|}{2(1 + 2|a_2|)} \implies f(z) \in \mathcal{S}^*.$$

In the light of the Theorem 1 and the Lemma 1, the following result can be second stated, which is Proposition 2 below.

Proposition 2. *Let* $f(z) = z + a_2 z^2 + a_3 z^3 + ... \in \mathcal{A}_n$, $f''(z) \neq -1$, $0 \leq |a_2| < \frac{1-\alpha}{2(2-\alpha)}$ *and* $z \in \mathbb{U}$. *Then,*

$$\left| \frac{z f'''(z)}{1 \pm f''(z)} \right| < \frac{(1-\alpha)[1 - \alpha - (2-\alpha)|a_2|]}{(4 - 3\alpha)[1 - \alpha + (2-\alpha)|a_2|]} \implies f(z) \in \mathcal{K}^*(\alpha).$$

Proof. By taking

$$M := \frac{1-\alpha}{2-\alpha} \quad \big(0 \leq \alpha < 1\big)$$

in the Theorem 1 and then making use of the Lemma 2, the proof of Proposition 2 can be easily obtained.

By putting $\alpha := 0$ in the Proposition 2 above, the following special result is also obtained, which is Corollary 2 below.

Corollary 2. *Let* $f(z) = z + a_2 z^2 + a_3 z^3 + ... \in \mathcal{A}_n$, $f''(z) \neq -1$, $0 < |a_2| < \frac{1}{4}$ and $z \in \mathbb{U}$. Then,

$$\left| \frac{z f'''(z)}{1 \pm f''(z)} \right| < \frac{1 - 2|a_2|}{4(1 + 2|a_2|)} \implies f(z) \in \mathcal{K}.$$

In consideration of the Theorem 2, the following result can be also determined, which is Proposition 3 below.

Proposition 3. *Let* $f(z) = z + a_2 z^2 + a_3 z^3 + ... \in \mathcal{A}_n$, $\delta > 0$, $0 \leq \alpha < 1$, *and* $z \in \mathbb{U}$. *Then,*

$$\Re e\left(\frac{z f'(z)}{f(z)}\right) > 1 - \frac{1-\alpha}{2\delta(1+\alpha)} \implies \Re e\left(\left[\frac{f(z)}{z}\right]^{\delta}\right) > \frac{1+\alpha}{2},$$

where the value of each one of the above complex power and its applications is considered a s its principal value.

Proof. By using of the definition:

$$\left[\frac{f(z)}{z}\right]^{\delta} = \frac{1+\alpha w(z)}{1+w(z)} \quad \big(f(z) \in \mathcal{A}_n; z \in \mathbb{U}; 0 \leq \alpha < 1; \delta > 0\big),$$

and also by following the same steps used in the proof of Theorem 2, the desired proof is easily obtained.

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References

[1] P. L. Duren, *Univalent Functions,* in: A Series of Comprehensive Studies in Mathematics, vol. 259, Springer-Verlag, New York, Berlin, Heidelberg, Tokyo, 1983.

[2] A. W. Goodman, *Univalent Functions,* Vols. I and II, Polygonal Publishing House, Washington, New Jersey, 1983.

[3] H. Irmak and M. Şan, *Some relations between certain inequalities concerning analytic and univalent functions,* Appl. Math. Lett. 23, 8 (2010), 897-901.

[4] H. Irmak, T. Bulbuoca and N. Tuneski, *Some relations between certain classes consisting of α-convex type and Bazilevic type functions,* Appl. Math. Lett. 23, 8 (2010), 897-901.

[5] H. Irmak, *The ordinary differential operator and some of its applications to p-valently analytic functions,* Electron. J. Math. Anal. Appl. 4, 1 (2016), 205-210.

[6] I. S. Jack, *Functions starlike and convex of order α,* J. London Math. Soc. 3 (1971), 469-474 .

[7] S.S. Miller and P. T. Mocanu, *Differential Subordinations, Theory and Applications*, Marcel Dekker, New York-Basel (2000).

[8] P. T. Mocanu, *Two simple sufficient conditions for starlikenes,* Mathematica (Cluj) 34, 57 (1992), 175-181.

[9] M. Obradovic, *Simple sufficient conditions for univalence,* Mat. Vesnik 49, 3-4 (1997), 241-244.

[10] S. Ponnusamy and V. Singh, *Criteria for univalent, starlike and convex functions,* Bull. Belg. Math. Soc. Simon Stevin 9, 4 (2002), 511-531.

[11] H. M. Srivastava and S. Owa, *Close-to-convexity, starlikeness and convexity of certain analytic functions,* Appl. Math. Lett. 15 (2002), 63-69.

[12] M. Şan and H. Irmak, *Some novel applications of certain higher order ordinary complex differential equations to normalized analytic functions,* J. Appl. Anal. Comput. 5, 3 (2015), 479-484.

[13] N. Tuneski, *On some simple sufficient conditions for univalence,* Math. Bohem. 126, 1 (2001), 229-236.

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