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ON A NEW CLASS OF INTEGRALS INVOLVING PRODUCT OF GENERALIZED BESSEL FUNCTION OF THE FIRST KIND AND GENERAL CLASS OF POLYNOMIALS

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Abstract. In this paper, we aim at establishing two generalized integral formulae involving product of generalized Bessel function of the first kind $w_v(z)$ and General class of polynomials $S_n^m[x]$ which are expressed in terms of the generalized Wright hypergeometric function. Some interesting special cases of our main results are also considered. The results are derived with the help of an interesting integral due to Lavoie and Trottier.

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1. Introduction and Preliminaries

In recent years, many integral formulae involving a variety of special functions have been developed by many authors [2],[5] ,[6] for a very recent work, see also [1].Those integrals involving generalized Bessel functions are of great importance since they are used in applied physics and in many branches of engineering.

In present paper, we established two generalized integral formulae involving product of generalized Bessel function of the first kind $w_v(z)$ and General class of polynomials $S_n^m[x]$ which are expressed in terms of the generalized Wright hypergeometric function. For this purpose we begin by recalling some known functions and earlier results.

The general class of polynomials $S_n^m[x]$ introduced by Srivastava [9]

$$S_n^m[x] = \sum_{k=0}^{\left[\frac{m}{n}\right]} \frac{(-n)_{mk}}{k!} A_{n,k} x^k \quad (n = 0, 1, 2, ...) \tag{1}$$

where m is an arbitrary positive integer and the coefficient $A_{n,k}$ $(n, k \geq 0)$ are arbitrary constants, real or complexion suitably specializing the coefficients $A_{n,k}$,the polynomial family $S_n^m[x]$ yields a number of known polynomials as its special cases.

The generalized Bessel function of the first kind, $w_v(z)$ [5] is defined for $z \in \mathbb{C} \setminus \{0\}$and $b, c, v \in \mathbb{C}$ with $\Re(v) > -1$ by the following series

$$w_v(z) = \sum_{l=0}^{\infty} \frac{(-1)^l c^l \left(\frac{z}{2}\right)^{v+2l}}{l!\Gamma\left(v + l + \frac{l+b}{2}\right)} \tag{2}$$

where $\mathbb{C}$ denotes set of complex numbers and $\Gamma(z)$ is the familiar gamma function [7]

An interesting further generalization of the generalized hypergeometric series ${}_pF_q$ is due to Fox [3] and Wright ([10] ,[11] ,[12])who studied the asymptotic expansion of the generalized (Wright) hypergeometric function defined by [8]

$${}_p\Psi_q\left[\begin{array}{c}(\alpha_1, A_1), ..., (\alpha_p, A_p); z\\(\beta_1, B_1), ..., (\beta_p, B_p)\end{array}\right] = \sum_{k=0}^{\infty} \frac{\Pi_{j=1}^{q}\Gamma(\alpha_j + A_j k)}{\Pi_{j=1}^{q}\Gamma(\beta_j + B_j k)} \frac{z^k}{k!} \tag{3}$$

Where the coefficients $A_1, \ldots, A_p$ and $B_1, \ldots, B_q$ are real positive numbers such that

$$1 + \sum_{j=1}^{q} B_j - \sum_{j=1}^{q} A_j \geq 0 \tag{4}$$

A special case of (3) is

$$\begin{aligned}&{}_p\Psi_q\left[\begin{array}{c}(\alpha_1, 1), ..., (\alpha_p, 1); z\\(\beta_1, 1), ..., (\beta_q, 1)\end{array}\right]\\ =\ &\sum_{l=0}^{\infty} \frac{\Pi_{j=1}^{q}\Gamma(\alpha_j)}{\Pi_{j=1}^{q}\Gamma(\beta_j)}\ {}_pF_q\left[\begin{array}{c}\alpha_1, ..., \alpha_p; z\\\beta_1, ..., \beta_q\end{array}\right]\end{aligned} \tag{5}$$

where ${}_pF_q$ is the generalized hypergeometric series defined by [7]

$$\begin{aligned}{}_pF_q\left[\begin{array}{c}\alpha_1, ..., \alpha_p; z\\\beta_1, ..., \beta_q\end{array}\right] &= \sum_{n=0}^{\infty} \frac{(\alpha_1)_n, ..., (\alpha_p)_n}{(\beta_1)_n, ..., (\beta_q)_n} \frac{z^n}{n!}\\ &= {}_pF_q(\alpha_1, ..., \alpha_p; \beta_1, ..., \beta_q; z)\end{aligned} \tag{6}$$

where$(\lambda)_n$ is the Pochhammer symbol defined (for $\lambda \in \mathbb{C}$) by [7] :

$$\begin{aligned}(\lambda)_n &= \begin{cases} 1, n=0 \\ \lambda(\lambda+1)\dots(\lambda+n-1) \quad , n\in N \end{cases} \\ &= \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} \quad \left(\lambda\in C\backslash Z_0^-\right)\end{aligned} \tag{7}$$

and $\mathbb{Z}_0^-$- denotes the set of non positive integers.

We also recall Lavoie-Trottier integral formula [4] for our present study

$$\begin{aligned}&\int_0^1 x^{\alpha-1}(1-x)^{2\beta-1}\left(1-\frac{x}{3}\right)^{2\alpha-1}\left(1-\frac{x}{4}\right)^{\beta-1}dx \\ =& \left(\frac{2}{3}\right)^{2\alpha}\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}, (\Re(\alpha)>0 \text{ and } \Re(\beta)>0).\end{aligned} \tag{8}$$

2. Main Results

In this section, we established two generalized integral formulae involving product of generalized Bessel function of the first kind $w_v(z)$ and general class of polynomials $S_n^m[x]$ which are expressed in terms of the generalized Wright hypergeometric function.

Theorem 1. *The following integral formula holds true: for $\rho, j, v, b, c\in\mathbb{C}$ and $z\in\mathbb{C}$ with $\Re(v)>-1, \Re(\rho)>0, \Re(\rho+j)>0, \Re(\rho+k+v)>0, x>0$*

$$\begin{aligned}&\int_0^1 x^{\rho+j-1}(1-x)^{2\rho-1}\left(1-\frac{x}{3}\right)^{2(\rho+j)-1}\left(1-\frac{x}{4}\right)^{\rho-1} \\ &\times S_n^m\left[y\left(1-\frac{x}{4}\right)(1-x)^2\right]w_v\left(y\left(1-\frac{x}{4}\right)\left(1-x^2\right)\right)dx \\ =& \sum_{k=0}^{\left[\frac{m}{n}\right]}\frac{(-n)_{mk}}{k!}A_{n,k}y^k\left(\frac{y}{2}\right)^v\left(\frac{2}{3}\right)^{2(\rho+j)}\Gamma(\rho+j) \\ &\times {}_1\Psi_2\left[\begin{matrix}(\rho+k+v,2) \\ \left(v+\frac{1+b}{2},1\right),(2\rho+k+v+j,2)\end{matrix};-\left(\frac{y}{3}\right)^2 c\right]\end{aligned} \tag{9}$$

Proof. By applying product of (1) and (2) in the integrand of (9) and interchanging the order of integral sign and summation which is verified by uniform convergence of the involved series under the given condition, we get

$$\begin{aligned}
&\int_0^1 x^{\rho+j-1}(1-x)^{2\rho-1}\left(1-\frac{x}{3}\right)^{2(\rho+j)-1}\left(1-\frac{x}{4}\right)^{\rho-1} \\
&\times S_n^m\left[y\left(1-\frac{x}{4}\right)(1-x^2)\right] w_v\left(y\left(1-\frac{x}{4}\right)(1-x^2)\right)dx \\
=\ &\sum_{k=0}^{\left[\frac{m}{n}\right]}\frac{(-n)_{mk}}{k!}A_{n,k}y^k\sum_{l=0}^{\infty}\frac{(-1)^l c^l}{l!\Gamma\left(v+l+\frac{1+b}{2}\right)}\left(\frac{y}{2}\right)^{v+2l} \\
&\times\int_0^1 x^{\rho+j-1}(1-x)^{2(\rho+k+v+2l)-1}\left(1-\frac{x}{3}\right)^{2(\rho+j)-1}\left(1-\frac{x}{4}\right)^{(\rho+k+v+2l)-1}
\end{aligned} \tag{10}$$

In view of the conditions given in Theorem 1, since $\Re(v)>-1, \Re(\rho+j)>0, \Re(\rho+k+v+2l)>0 (n,k)\geq 0 (k,l\in N_0\in N\cup\{0\})$, we can apply the integral formula (8) to the integral in (2) and obtain the following expression

$$\begin{aligned}
=\ &\sum_{k=0}^{\left[\frac{m}{n}\right]}\frac{(-n)_{mk}}{k!}A_{n,k}y^k\sum_{l=0}^{\infty}\frac{(-1)^l c^l}{l!\Gamma\left(v+l+\frac{1+b}{2}\right)}\left(\frac{y}{2}\right)^{v+2l} \\
&\times\left(\frac{2}{3}\right)^{2(\rho+j)}\frac{\Gamma(\rho+j)\Gamma(\rho+k+v+2l)}{\Gamma(2\rho+k+v+j+2l)}
\end{aligned}$$

$$\begin{aligned}
=\ &\sum_{k=0}^{\left[\frac{m}{n}\right]}\frac{(-n)_{mk}}{k!}A_{n,k}y^k\sum_{l=0}^{\infty}\frac{(-1)^l c^l}{l!\Gamma\left(v+l+\frac{1+b}{2}\right)}\left(\frac{y}{2}\right)^{v}\left(\frac{2}{3}\right)^{2(\rho+j)}\Gamma(\rho+j) \\
&\times\sum_{l=0}^{\infty}\frac{(-1)^l c^l}{l!\Gamma\left(v+l+\frac{1+b}{2}\right)}\left(\frac{y}{2}\right)^{v+2l}\frac{\Gamma(\rho+k+v+2l)}{\Gamma(2\rho+k+v+j+2l)}\left(\frac{y}{2}\right)^{2l}
\end{aligned}$$

which, upon using (3), yields (9).This completes the proof of Theorem 1.

Theorem 2. *The following integral formula holds true: For $\rho, j, v\in\mathbb{C}$, $z\in\mathbb{C}$ and $b, c, v\in\mathbb{C}$ with $\Re(v)>-1, \Re(\rho)>0, \Re(\rho+j)>0, \Re(\rho+k+v)>0, x>0$.*

$$\begin{aligned}&\int_0^1 x^{\rho-1}(1-x)^{2(\rho+j)-1}\left(1-\frac{x}{3}\right)^{2\rho-1}\left(1-\frac{x}{4}\right)^{(\rho+j)-1}\\&\times S_n^m\left[yx\left(1-\frac{x}{3}\right)^2\left(1-x^2\right)\right]w_v\left(yx\left(1-\frac{x}{3}\right)^2\right)dx\\=\;&\sum_{k=0}^{\left[\frac{m}{n}\right]}\frac{(-n)_{mk}}{k!}A_{n,k}y^k\left(\frac{y}{2}\right)^v\left(\frac{2}{3}\right)^{2(\rho+k+v)}\Gamma(\rho+j)\\&\times{}_1\Psi_2\left[\begin{array}{c}(\rho+k+v,2);\\\left(v+\frac{1+b}{2},1\right),(2\rho+k+v+j,2)\end{array}\;-\left(\frac{2y}{9}\right)^2c\right]\end{aligned}\tag{11}$$

Proof. By applying product of (1) and (2) in the integrand of (11) and interchanging the order of integral sign and summation which is verified by uniform convergence of the involved series under the given condition, we get

$$\begin{aligned}&\int_0^1 x^{\rho-1}(1-x)^{2(\rho+j)-1}\left(1-\frac{x}{3}\right)^{2\rho-1}\left(1-\frac{x}{4}\right)^{(\rho+j)-1}\\&\times S_n^m\left[yx\left(1-\frac{x}{3}\right)^2\left(1-x^2\right)\right]w_v\left(yx\left(1-\frac{x}{3}\right)^2\right)dx\\=\;&\sum_{k=0}^{\left[\frac{m}{n}\right]}\frac{(-n)_{mk}}{k!}A_{n,k}y^k\sum_{l=0}^{\infty}\frac{(-1)^l c^l}{l!\Gamma\left(v+l+\frac{1+b}{2}\right)}\left(\frac{y}{2}\right)^{v+2l}\\&\times\int_0^1 x^{\rho+k+v+2l-1}(1-x)^{2(\rho+j)-1}\left(1-\frac{x}{3}\right)^{2(\rho+k+v+2l)-1}\left(1-\frac{x}{4}\right)^{(\rho+j)-1}dx\end{aligned}\tag{12}$$

Now,we apply the integral formula (8) to the integral in (11) and obtain the following expression

$$\begin{aligned}=\;&\sum_{k=0}^{\left[\frac{m}{n}\right]}\frac{(-n)_{mk}}{k!}A_{n,k}y^k\sum_{l=0}^{\infty}\frac{(-1)^l c^l}{l!\Gamma\left(v+l+\frac{1+b}{2}\right)}\\&\times\left(\frac{y}{2}\right)^{v+2l}\left[\left(\frac{2}{3}\right)^{2(\rho+k+v)}\right]\frac{\Gamma(\rho+j)\Gamma(2\rho+k+v+2l)}{\Gamma(2\rho+k+v+j+2l)}\left(\frac{2}{3}\right)^{4l}\left(\frac{y}{2}\right)^{2l}\end{aligned}$$

which, upon using (3), yields (11).This completes the proof of Theorem 2

Next we consider other variations of Theorem 1 and Theorem 2. We express result of Theorem 1 and Theorem 2 in terms of hypergeometric function ${}_pF_q$.To do this, we recall the well-known Legendre duplication formula for the gamma function Γ:

$$\sqrt{\pi}\Gamma\left(2z\right)=2^{2z-1}\Gamma\left(z\right)\Gamma\left(z+\frac{1}{2}\right)\ ,\quad \left(z\neq 0,-\frac{1}{2},-1,-\frac{3}{2},...\right) \tag{13}$$

which is equivalently written in terms of the Pochhammer symbol (7) as follows

$$(\lambda)_{2n}=2^{2n}\left(\frac{1}{2}\lambda\right)_{n}\left(\frac{1}{2}\lambda+\frac{1}{2}\right)_{n}\qquad (n\in\mathbb{N}_0) \tag{14}$$

Now we have two corollaries.

Corollary 3. *Let the condition of Theorem 1 be satisfied and* $+j,\rho+v+k\in\mathbb{C}\backslash Z_0^-$ *. Then the following integral formula holds true:*

$$\int_0^1 x^{\rho+j-1}\left(1-x\right)^{2\rho-1}\left(1-\frac{x}{3}\right)^{2(\rho+j)-1}\left(1-\frac{x}{4}\right)^{\rho-1}$$
$$\times S_n^m\left[y\left(1-\frac{x}{4}\right)^2\left(1-x\right)^2\right]w_v\left(y\left(1-\frac{x}{4}\right)\left(1-x\right)^2\right)dx$$
$$=\sum_{k=0}^{\left[\frac{m}{n}\right]}\frac{(-n)_{mk}}{k!}A_{n,k}y^k\left(\frac{y}{2}\right)^v\left(\frac{2}{3}\right)^{2(\rho+j)}\frac{\Gamma\left(\rho+j\right)\Gamma\left(\rho+v+k\right)}{\Gamma\left(v+\frac{1+b}{2}\right)\Gamma\left(2\rho+v+j+k\right)}$$
$$\times {}_2F_3\left[\begin{array}{c}\frac{(\rho+v+k)}{2},\frac{(\rho+v+k+)}{2}\\ \left(v+\frac{1+b}{2}\right),\left(\frac{2\rho+v+j+k+1}{2}\right),\frac{(2\rho+v+k+j)}{2}\end{array};\frac{-y^2c}{4}\right] \tag{15}$$

Corollary 4. *Let the condition of Theorem 2 be satisfied and* $+j,\rho+v+k\in\mathbb{C}\backslash\mathbb{Z}_0^-$ *. Then the following integral formula holds true:*

$$\int_0^1 x^{\rho-1}\left(1-x\right)^{2(\rho+j)-1}\left(1-\frac{x}{3}\right)^{2\rho-1}\left(1-\frac{x}{4}\right)^{(\rho+j)-1}$$
$$\times S_n^m\left[yx\left(1-\frac{x}{3}\right)^2\right]w_v\left(yx\left(1-\frac{x}{3}\right)^2\right)dx$$
$$=\sum_{k=0}^{\left[\frac{m}{n}\right]}\frac{(-n)_{mk}}{k!}A_{n,k}y^k\left(\frac{y}{2}\right)^v\left(\frac{2}{3}\right)^{2(\rho+k+v)}\frac{\Gamma\left(\rho+j\right)\Gamma\left(\rho+v+k\right)}{\Gamma\left(v+\frac{1+b}{2}\right)\Gamma\left(2\rho+v+j+k\right)}$$
$$\times {}_2F_3\left[\begin{array}{c}\frac{(\rho+v+k)}{2},\frac{(\rho+v+k+1)}{2}\\ \left(v+\frac{1+b}{2}\right),\left(\frac{2\rho+v+j+k+1}{2}\right),\frac{(2\rho+v+k+j)}{2}\end{array};\frac{-4cy^2}{81}\right] \tag{16}$$

Proof. By writing the right-hand side of equation (8) in the original summation and applying (14) to the resulting summation, after a little simplification, we find that, when the last resulting summation is expressed in terms of pFq in (6), this completes the proof of corollary 1. Similarly, it is easy to see that a similar argument as in proof of corollary 1 will establish the integral formula (16). Therefore, we omit the details of the proof of corollary 2.

3. Special Cases

In this section we derive some new integral formulae by using Bessel function of the first kind and general class of polynomials.

Corollary 5. *Let the condition of Theorem 1 be satisfied and for $b = c = 1$ Theorem 1 reduces in following form*

$$\begin{aligned} & \int_0^1 x^{\rho+j-1}(1-x)^{2\rho-1}\left(1-\frac{x}{3}\right)^{2(\rho+j)-1}\left(1-\frac{x}{4}\right)^{\rho-1} \\ & \times S_n^m\left[y\left(1-\frac{x}{4}\right)(1-x)^2\right] J_v\left(y\left(1-\frac{x}{4}\right)(1-x)^2\right) dx \\ = & \sum_{k=0}^{\left[\frac{m}{n}\right]} \frac{(-n)_{mk}}{k!} A_{n,k} y^k \left(\frac{y}{2}\right)^v \left(\frac{2}{3}\right)^{2(\rho+j)} \Gamma(\rho+j) \\ & \times {}_1\Psi_2\left[\begin{matrix} (\rho+k+v,2); \\ (v+1,1),(2\rho+k+v+j,2) \end{matrix}; \frac{-y^2}{4}\right] \end{aligned} \tag{17}$$

Corollary 6. *Let the condition of Theorem 2 be satisfied for $b = c = 1$, Theorem 2 reduces in following form*

$$\begin{aligned} & \int_0^1 x^{\rho-1}(1-x)^{2(\rho+j)-1}\left(1-\frac{x}{3}\right)^{2\rho-1}\left(1-\frac{x}{4}\right)^{(\rho+j)-1} \\ & \times S_n^m\left[yx\left(1-\frac{x}{3}\right)^2\right] J_v\left(yx\left(1-\frac{x}{3}\right)^2\right) dx \\ = & \sum_{k=0}^{\left[\frac{m}{n}\right]} \frac{(-n)_{mk}}{k!} A_{n,k} y^k \left(\frac{y}{2}\right)^v \left(\frac{2}{3}\right)^{2(\rho+k+v)} \Gamma(\rho+j) \\ & \times {}_1\Psi_2\left[\begin{matrix} (\rho+k+v,2); \\ (v+1,1),(2\rho+k+v+j,2) \end{matrix}; -\left(\frac{2y}{9}\right)^2\right] \end{aligned} \tag{18}$$

Where $J_v(z)$ is given by (19) which is a well known Bessel function of the first kind [5] defined for $z \in \mathbb{C}\backslash\{0\}$ and $v \in \mathbb{C}$ with $\Re(l) > -1$.

$$J_v\left(z\right)=\sum_{l=0}^{\infty}\frac{\left(-1\right)^l\left(\frac{z}{2}\right)^{v+2l}}{l!\Gamma\left(v+l+1\right)} \tag{19}$$

By applying product of (1) and (19) in the integrand of (17) and (18) respectively and then using same method as we used earlier we can obtain corollary 3 and 4.

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