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ON STARLIKE FUNCTIONS INVOLVING A CERTAIN DIFFERENTIAL OPERATOR

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Abstract. Using the technique of differential subordination, we here, obtain certain sufficient conditions involving the differential operator $1+\dfrac{\alpha z f''(z)}{f'(z)}-\dfrac{zf'(z)}{f(z)}$ for $f \in \mathcal{A}_p$ to be parabolic starlike and starlike.

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1. Introduction

Let $\mathcal{A}_p$ denote the class of functions of the form

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k, \qquad (p \in \mathbb{N}; z \in \mathbb{E})$$

which are analytic and p-valent in the open unit disk $\mathbb{E} = \{z : |z| < 1\}$. Obviously, $\mathcal{A}_1 = \mathcal{A}$, the class of all analytic functions f, normalized by the conditions $f(0) = f'(0)-1 = 0$. Let the functions f and g be analytic in $\mathbb{E}$. We say that f is subordinate to g in $\mathbb{E}$ (written as $f \prec g$), if there exists a Schwarz function ϕ in $\mathbb{E}$ (i.e. ϕ is regular in $|z| < 1, \phi(0) = 0$ and $|\phi(z)| \leq |z| < 1$) such that

$$f(z) = g(\phi(z)),\ |z| < 1.$$

Let $\Phi : \mathbb{C}^2 \times \mathbb{E} \to \mathbb{C}$ be an analytic function, p an analytic function in $\mathbb{E}$ with $(p(z), zp'(z); z) \in \mathbb{C}^2 \times \mathbb{E}$ for all $z \in \mathbb{E}$ and h be univalent in $\mathbb{E}$. Then the function p is said to satisfy first order differential subordination if

$$\Phi(p(z), zp'(z); z) \prec h(z), \Phi(p(0), 0; 0) = h(0). \tag{1}$$

A univalent function q is called a dominant of the differential subordination (1) if $p(0) = q(0)$ and $p \prec q$ for all p satisfying (1). A dominant $\tilde{q}$ that satisfies $\tilde{q} \prec q$ for all dominants q of (1), is said to be the best dominant of (1). The best dominant is unique up to a rotation of $\mathbb{E}$.
A function $f \in \mathcal{A}_p$ is said to be p-valent starlike of order $\alpha(0 \leq \alpha < p)$ in $\mathbb{E}$, if

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha, z \in \mathbb{E}. \tag{2}$$

Let $\mathcal{S}_p^*(\alpha)$ denote the class of p-valent starlike functions of order α. Note that $\mathcal{S}_p^*(0) = \mathcal{S}_p^*$, which is the class of p-valent starlike functions.
A function $f \in \mathcal{A}_p$ is said to be p-valent parabolic starlike in $\mathbb{E}$, if

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \left|\frac{zf'(z)}{f(z)} - p\right|, z \in \mathbb{E}. \tag{3}$$

We denote by $\mathcal{S}_{\mathcal{P}}^p$, the class of p-valent parabolic starlike functions. Note that $\mathcal{S}_{\mathcal{P}}^1 = \mathcal{S}_{\mathcal{P}}$, the class of parabolic starlike functions. Define the parabolic domain Ω as under

$$\Omega = \{u + iv : u > \sqrt{(u-p)^2 + v^2}\}.$$

Clearly the function

$$q(z) = p + \frac{2p}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2$$

maps the unit disk $\mathbb{E}$ onto the domain Ω. Hence the condition (3) is equivalent to

$$\frac{zf'(z)}{f(z)} \prec q(z), z \in \mathbb{E},$$

where q(z) is given above.
Ronning [2] and Ma and Minda [1] studied the domain Ω and the above function $q(z)$ in a special case where $p = 1$.
In univalent function theory, the operators $\frac{zf'(z)}{f(z)}$ and $1 + \frac{zf''(z)}{f'(z)}$ have played an important role in obtaining the sufficient conditions for starlikeness and convexity of normalised analytic functions. Many authors have used various combinations of the above operators to define different classes of analytic functions and have obtained the different criteria for univalence, starlikeness and convexity of analytic functions. One such combination comprising the differential operator $1+\frac{\alpha zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}$ has been studied by different authors to obtain the sufficient conditions for starlikeness of normalised analytic functions. In fact, in 2006, Obradovič et al. [4] studied the

differential operator $1+\dfrac{\alpha zf''(z)}{f'(z)}-\dfrac{zf'(z)}{f(z)}$ and obtain certain sufficient conditions for starlikeness involving the above operator. They proved the following result:

Theorem 1. *Let $f\in\mathcal{A}$ and $\alpha\in(-\infty,0)\cup[2/3,\infty)$. Then we have*

$$1+\frac{\alpha zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)} \prec \frac{(\alpha-1)|\alpha|}{\alpha}\frac{2z}{1-z}+|\alpha|\frac{2z}{1-z^2}=G(z)\Rightarrow f\in\mathcal{S}^*,$$

where G is the conformal mapping of the unit disk $\mathbb{E}$ with $G(0)=1$ and

$$G(\mathbb{E})=\mathbb{C}\setminus\left\{w\in\mathbb{C}:\Re(w)=\frac{(1-\alpha)|\alpha|}{\alpha},|\Im(w)|\geq|\alpha|\sqrt{3-2/\alpha}\right\}.$$

Recently, Singh et al.[5] further investigated the work of obradovič and obtained certain sufficient conditions for starlikeness which in particular, represent the correct version of the above result of Obradovič et al. In fact they proved the following result:

Theorem 2. *Let $\alpha\in(-\infty,0)\cup[2/3,\infty)$ be a real number. If $f\in\mathcal{A}$, $\dfrac{zf'(z)}{f(z)}\neq 0$ in $\mathbb{E}$, satisfies the differential subordination*

$$1+\frac{\alpha zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)} \prec 2(\alpha-1)\frac{z}{1-z}+2\alpha\frac{z}{1-z^2}=F(z),\ \textit{then}\ f\in\mathcal{S}^*,$$

where $F(\mathbb{E})$ is given by

$$F(\mathbb{E})=\mathbb{C}\setminus\left\{w\in\mathbb{C}:\Re(w)=1-\alpha,|\Im(w)|\geq\sqrt{3\alpha^2-2\alpha}\right\}.$$

The main objective of the present paper is to obtain the sufficient conditions for parabolic starlikeness of $f\in\mathcal{A}_p$, in terms of the differential operator $1+\dfrac{\alpha zf''(z)}{f'(z)}-\dfrac{zf'(z)}{f(z)}$. As a consequence of our main result, we also obtain the starlikeness of $f\in\mathcal{A}_p$ and in particular $f\in\mathcal{A}$, expressed in terms of above operator.

2. Preliminaries

To prove our main results, we shall use the following lemma of Miller and Mocanu [[3], p.132].

Lemma 3. *Let q be a univalent in $\mathbb{E}$ and let θ and ϕ be analytic in a domain $\mathbb{D}$ containing $q(\mathbb{E})$, with $\phi(w) \neq 0$, when $w \in q(\mathbb{E})$. Set $Q(z) = zq'(z)\phi[q(z)], h(z) = \theta[q(z)] + Q(z)$ and suppose that either*
(i) h is convex, or
(ii) Q is starlike.
In addition, assume that
(iii) $\Re\left(\frac{zh'(z)}{Q(z)}\right) > 0$ for all z in $\mathbb{E}$.
If p is analytic in $\mathbb{E}$, with $p(0) = q(0)$, $p(\mathbb{E}) \subset \mathbb{D}$ and

$$\theta[p(z)] + zp'(z)\phi[p(z)] \prec \theta[q(z)] + zq'(z)\phi[q(z)], z \in \mathbb{E},$$

then $p(z) \prec q(z)$ and q is the best dominant.

3. Main Results

Theorem 4. *Let $\alpha \neq 0$, be a complex number. Let $q(z) \neq 0$, be a univalent function in $\mathbb{E}$ such that*

$$\Re\left[1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)}\right] > max\left\{0, p\, \Re\left(\frac{1-\alpha}{\alpha}\right) q(z)\right\}. \tag{4}$$

If $f \in \mathcal{A}_p$ satisfies

$$1 + \frac{\alpha z f''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} \prec (1-\alpha)(1 - pq(z)) + \frac{\alpha z q'(z)}{q(z)}, z \in \mathbb{E}, \tag{5}$$

then

$$\frac{zf'(z)}{pf(z)} \prec q(z), z \in \mathbb{E}.$$

Proof. Write $\dfrac{zf'(z)}{pf(z)} = u(z)$, in (5), we obtain:

$$(1-\alpha)(1 - pu(z)) + \frac{\alpha z u'(z)}{u(z)} \prec (1-\alpha)(1 - pq(z)) + \frac{\alpha z q'(z)}{q(z)}.$$

Let us define the function θ and ϕ as follows:

$$\theta(w) = (1-\alpha)(1 - pw)$$

and

$$\phi(w) = \frac{\alpha}{w}.$$

Obviously, the function θ and ϕ are analytic in domain $\mathbb{D} = \mathbb{C} \setminus \{0\}$ and $\phi(w) \neq 0$ in $\mathbb{D}$. Therefore,

$$Q(z) = \phi(q(z))zq'(z) = \frac{\alpha z q'(z)}{q(z)}$$

and

$$h(z) = \theta(q(z)) + Q(z) = (1-\alpha)(1-pq(z)) + \frac{\alpha z q'(z)}{q(z)}.$$

On differentiating, we obtain $\dfrac{zQ'(z)}{Q(z)} = 1 + \dfrac{zq''(z)}{q'(z)} - \dfrac{zq'(z)}{q(z)}$ and

$$\frac{zh'(z)}{Q(z)} = 1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)} + \frac{p(\alpha-1)}{\alpha} q(z).$$

In view of the given conditions, we see that Q is starlike and $\Re\left(\dfrac{zh'(z)}{Q(z)}\right) > 0$. Therefore, the proof, now follows from Lemma 1.1.

4. Parabolic Starlikeness:

Remark 1. *When we select the dominant* $q(z) = 1 + \dfrac{2}{\pi^2}\left(\log\left(\dfrac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2$ *in Theorem 4, a little calculation yields that*

$$1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)} = \frac{1+z}{2(1-z)} + \frac{\sqrt{z}}{(1-z)log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)} - \frac{\frac{4\sqrt{z}}{\pi^2(1-z)}\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)}{1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2}$$

$$1+\frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)} + \frac{p(\alpha-1)}{\alpha}q(z) = \frac{1+z}{2(1-z)} + \frac{\sqrt{z}}{(1-z)log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)} - \frac{\frac{4\sqrt{z}}{\pi^2(1-z)}\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)}{1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2}$$
$$+\frac{p(\alpha-1)}{\alpha}\left[1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2\right].$$

Thus for $\alpha \in \mathbb{R} \setminus [0,1)$, we notice that q(z) satisfies the condition (4). Therefore, we immediately arrive at the following result.

Theorem 5. *Suppose $\alpha \in \mathbb{R} \setminus [0,1)$ and if $f \in \mathcal{A}_p$ satisfies*

$$1 + \frac{\alpha z f''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} \prec (1-\alpha)\left\{1 - p - \frac{2p}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2\right)\right\}$$

$$+\frac{\frac{4\alpha\sqrt{z}}{\pi^2(1-z)}\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)}{1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2}, z\in\mathbb{E},$$

then

$$\frac{zf'(z)}{f(z)}\prec p+\frac{2p}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2 \ \ i.e.\ f\in\mathcal{S}^p_{\mathcal{P}}.$$

Setting $\alpha=1$ in Theorem 5, we get:

Corollary 6. *If $f\in\mathcal{A}_p$ satisfies*

$$1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\prec\frac{\frac{4}{\pi^2}\frac{\sqrt{z}}{1-z}\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)}{1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2}, z\in\mathbb{E},$$

then

$$\frac{zf'(z)}{f(z)}\prec p+\frac{2p}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2 \ \ i.e.\ f\in\mathcal{S}^p_{\mathcal{P}}.$$

Setting $p=1$ in Theorem 5, we get:

Corollary 7. *Suppose $\alpha\in\mathbb{R}\setminus[0,1)$ and if $f\in\mathcal{A}$ satisfies*

$$1+\frac{\alpha zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\prec \ \frac{2(\alpha-1)}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2\right)$$

$$+\frac{\frac{4\alpha\sqrt{z}}{\pi^2(1-z)}\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)}{1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2}, z\in\mathbb{E},$$

then

$$\frac{zf'(z)}{f(z)}\prec 1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2 \ \ i.e.\ f\in\mathcal{S}_{\mathcal{P}}.$$

5. Starlikeness with different dominants:

Remark 2. *When we select the dominant $q(z)=\dfrac{1+(1-2\beta)z}{1-z}, 0\leq\beta<1$ in Theorem 4, a little calculation yields that*

$$1+\frac{zq''(z)}{q'(z)}-\frac{zq'(z)}{q(z)}=\frac{1+(1-2\beta)z^2}{(1-z)(1+(1-2\beta)z)},$$

$$1+\frac{zq''(z)}{q'(z)}-\frac{zq'(z)}{q(z)}+\frac{p(\alpha-1)}{\alpha}q(z)=\frac{1+(1-2\beta)z^2}{(1-z)(1+(1-2\beta)z)}+\frac{p(\alpha-1)}{\alpha}\left[\frac{1+(1-2\beta)z}{1-z}\right].$$

Thus for $\alpha \in \mathbb{R} \setminus [0,1)$, we notice that q(z) satisfies the condition (4). Therefore, we, immediately arrive at the following result.

Theorem 8. *Suppose $\alpha \in \mathbb{R} \setminus [0,1)$ and if $f \in \mathcal{A}_p$ satisfies*

$$1+\frac{\alpha z f''(z)}{f'(z)}-\frac{zf'(z)}{f(z)} \prec (1-\alpha)\left[1-\frac{p(1+(1-2\beta)z)}{1-z}\right]+\frac{2\alpha(1-\beta)z}{(1-z)(1+(1-2\beta)z)},$$
$$0 \le \beta < 1, z \in \mathbb{E},$$

then

$$\frac{zf'(z)}{f(z)} \prec \frac{p[1+(1-2\beta)z]}{1-z} \textit{ i.e. } f \in \mathcal{S}_p^*(\beta).$$

Setting $\alpha = 1$ in Theorem 8, we get:

Corollary 9. *If $f \in \mathcal{A}_p$ satisfies*

$$1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)} \prec \frac{2(1-\beta)z}{(1-z)(1+(1-2\beta)z)}, 0 \le \beta < 1, z \in \mathbb{E},$$

then

$$\frac{zf'(z)}{f(z)} \prec \frac{p[1+(1-2\beta)z]}{1-z}.$$

Setting $p = 1$ in Theorem 8, we get:

Corollary 10. *Suppose $\alpha \in \mathbb{R} \setminus [0,1)$ and if $f \in \mathcal{A}$ satisfies*

$$1+\frac{\alpha z f''(z)}{f'(z)}-\frac{zf'(z)}{f(z)} \prec (1-\alpha)\left[1-\left(\frac{1+(1-2\beta)z}{1-z}\right)\right]+\frac{2\alpha(1-\beta)z}{(1-z)(1+(1-2\beta)z)},$$
$$0 \le \beta < 1, z \in \mathbb{E},$$

then

$$\frac{zf'(z)}{f(z)} \prec \frac{1+(1-2\beta)z}{1-z}.$$

Remark 3. *When we select the dominant $q(z) = \dfrac{1+z}{1-z}$ in Theorem 4, a little calculation yields that*

$$1+\frac{zq''(z)}{q'(z)}-\frac{zq'(z)}{q(z)}=\frac{1+z^2}{1-z^2}$$

$$1+\frac{zq''(z)}{q'(z)}-\frac{zq'(z)}{q(z)}+\frac{p(\alpha-1)}{\alpha}q(z)=\frac{1+z^2}{1-z^2}+\frac{p(\alpha-1)}{\alpha}\left(\frac{1+z}{1-z}\right).$$

Thus for $\alpha \in \mathbb{R}\setminus[0,1)$, we notice that q(z) satisfies the condition (4). Therefore, we, immediately arrive at the following result.

Theorem 11. *Suppose $\alpha \in \mathbb{R}\setminus[0,1)$ and if $f \in \mathcal{A}_p$ satisfies*

$$1+\frac{\alpha zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)} \prec (1-\alpha)\left[1-\frac{p(1+z)}{1-z}\right]+\frac{2\alpha z}{1-z^2}, z \in \mathbb{E},$$

then

$$\frac{zf'(z)}{f(z)} \prec \frac{p(1+z)}{1-z} \text{ i.e. } f \in \mathcal{S}_p^*.$$

Setting $\alpha = 1$ in Theorem 11, we get:

Corollary 12. *If $f \in \mathcal{A}_p$ satisfies*

$$1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)} \prec \frac{2z}{1-z^2}, z \in \mathbb{E},$$

then

$$\frac{zf'(z)}{f(z)} \prec \frac{p(1+z)}{1-z}.$$

Note: For $p = 1$ in Theorem 11, we obtain the result of Singh et al. [5] stated in Theorem 2.

Remark 4. *When we select the dominant $q(z) = e^z$ in Theorem 4, a little calculation yields that*

$$1+\frac{zq''(z)}{q'(z)}-\frac{zq'(z)}{q(z)}=1$$

$$1+\frac{zq''(z)}{q'(z)}-\frac{zq'(z)}{q(z)}+\frac{p(\alpha-1)}{\alpha}q(z)=1+\frac{p(\alpha-1)}{\alpha}e^z.$$

Thus for $\alpha \in \mathbb{R}\setminus[0,1)$, we notice that q(z) satisfies the condition (4). Therefore, we, immediately arrive at the following result.

Theorem 13. *Suppose $\alpha \in \mathbb{R}\setminus[0,1)$ and if $f \in \mathcal{A}_p$ satisfies*

$$1+\frac{\alpha zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)} \prec (1-\alpha)(1-pe^z)+\alpha z, z \in \mathbb{E},$$

then

$$\frac{zf'(z)}{f(z)} \prec pe^z \text{ i.e. } f \in \mathcal{S}_p^*.$$

Setting $\alpha = 1$ in Theorem 13, we get:

Corollary 14. *If $f \in \mathcal{A}_p$ satisfies*

$$1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} \prec z, z \in \mathbb{E},$$

then

$$\frac{zf'(z)}{f(z)} \prec pe^z.$$

Setting $p = 1$ in Theorem 13, we get:

Corollary 15. *Suppose $\alpha \in \mathbb{R} \setminus [0, 1)$ and if $f \in \mathcal{A}$ satisfies*

$$1 + \frac{\alpha zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} \prec \ (1 - \alpha)(1 - e^z) + \alpha z, z \in \mathbb{E},$$

then

$$\frac{zf'(z)}{f(z)} \prec e^z.$$

Remark 5. *When we select the dominant $q(z) = 1 + az$; $0 \le a < 1$ in Theorem 4, a little calculation yields that*

$$1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)} = \frac{1}{1 + az}$$

$$1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)} + \frac{p(\alpha - 1)}{\alpha}q(z) = \frac{1}{1 + az} + \frac{p(\alpha - 1)}{\alpha}(1 + az).$$

Thus for $\alpha \in \mathbb{R} \setminus [0, 1)$, we notice that q(z) satisfies the condition (4). Therefore, we, immediately arrive at the following result.

Theorem 16. *Suppose $\alpha \in \mathbb{R} \setminus [0, 1)$ and if $f \in \mathcal{A}_p$ satisfies*

$$1 + \frac{\alpha zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} \prec \ (1 - \alpha)[1 - p(1 + az)] + \frac{\alpha az}{1 + az}, z \in \mathbb{E}, 0 \le a < 1,$$

then

$$\frac{zf'(z)}{f(z)} \prec p(1 + az) \ i.e. \ f \in \mathcal{S}_p^*.$$

Setting $\alpha = 1$ in Theorem 16, we get:

Corollary 17. *If $f \in \mathcal{A}_p$ satisfies*

$$1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)} \prec \frac{az}{1+az}, z \in \mathbb{E}, 0 \leq a < 1,$$

then

$$\frac{zf'(z)}{f(z)} \prec p(1+az).$$

Setting $p=1$ in Theorem 16, we get:

Corollary 18. *Suppose $\alpha \in \mathbb{R} \setminus [0,1)$ and if $f \in \mathcal{A}$ satisfies*

$$1+\frac{\alpha zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)} \prec \; a(\alpha-1)z+\frac{\alpha az}{1+az}, z \in \mathbb{E}, 0 \leq a < 1,$$

then

$$\frac{zf'(z)}{f(z)} \prec 1+az.$$

Remark 6. *When we select the dominant $q(z)=\left(\frac{1+z}{1-z}\right)^{\gamma}$, $0<\gamma \leq 1$ in Theorem 4, a little calculation yields that*

$$1+\frac{zq''(z)}{q'(z)}-\frac{zq'(z)}{q(z)}=\frac{1+z^2}{1-z^2}$$

$$1+\frac{zq''(z)}{q'(z)}-\frac{zq'(z)}{q(z)}+\frac{p(\alpha-1)}{\alpha}q(z)=\frac{1+z^2}{1-z^2}+\frac{p(\alpha-1)}{\alpha}\left(\frac{1+z}{1-z}\right)^{\gamma}.$$

Thus for $\alpha \in \mathbb{R} \setminus [0,1)$, we notice that q(z) satisfies the condition (4). Therefore, we, immediately arrive at the following result.

Theorem 19. *Suppose $\alpha \in \mathbb{R} \setminus [0,1)$ and if $f \in \mathcal{A}_p$ satisfies*

$$1+\frac{\alpha zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)} \prec \; (1-\alpha)\left[1-\frac{p(1+z)^{\gamma}}{(1-z)^{\gamma}}\right]+\frac{2\alpha\gamma z}{1-z^2}, 0<\gamma \leq 1, z \in \mathbb{E},$$

then

$$\frac{zf'(z)}{f(z)} \prec \frac{p(1+z)^{\gamma}}{(1-z)^{\gamma}}, \; i.e. \; f \in \mathcal{S}_p^*.$$

Setting $\alpha=1$ in Theorem 19, we get:

Corollary 20. *If $f \in \mathcal{A}_p$ satisfies*

$$1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} \prec \frac{2\gamma z}{1-z^2}, 0 < \gamma \leq 1, z \in \mathbb{E},$$

then

$$\frac{zf'(z)}{f(z)} \prec \frac{p(1+z)^\gamma}{(1-z)^\gamma}.$$

Setting $p = 1$ in Theorem 19, we get:

Corollary 21. *Suppose $\alpha \in \mathbb{R} \setminus [0, 1)$ and if $f \in \mathcal{A}$ satisfies*

$$1 + \frac{\alpha zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} \prec (1-\alpha)\left[1 - \left(\frac{1+z}{1-z}\right)^\gamma\right] + \frac{2\alpha\gamma z}{1-z^2}, 0 < \gamma \leq 1, z \in \mathbb{E},$$

then

$$\frac{zf'(z)}{f(z)} \prec \left(\frac{1+z}{1-z}\right)^\gamma.$$

Remark 7. *When we select the dominant $q(z) = \dfrac{\alpha'(1-z)}{\alpha' - z}$, $\alpha' > 1$ in Theorem 4, a little calculation yields that*

$$1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)} = \frac{1}{1-z} + \frac{z}{\alpha' - z}$$

$$1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)} + \frac{p(\alpha-1)}{\alpha}q(z) = \frac{1}{1-z} + \frac{z}{\alpha'-z} + \frac{p(\alpha-1)}{\alpha}\left(\frac{\alpha'(1-z)}{\alpha'-z}\right).$$

Thus for $\alpha \in \mathbb{R} \setminus [0, 1)$, we notice that q(z) satisfies the condition (4). Therefore, we, immediately arrive at the following result.

Theorem 22. *Suppose $\alpha \in \mathbb{R} \setminus [0, 1)$ and if $f \in \mathcal{A}_p$ satisfies*

$$1 + \frac{\alpha zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} \prec (1-\alpha)\left[1 - \frac{p\alpha'(1-z)}{\alpha'-z}\right] + \frac{\alpha z(1-\alpha')}{(1-z)(\alpha'-z)}, \ \alpha' > 1, z \in \mathbb{E},$$

then

$$\frac{zf'(z)}{f(z)} \prec \frac{p\alpha'(1-z)}{\alpha'-z}, \ i.e.\ f \in \mathcal{S}_p^*.$$

Setting $\alpha = 1$ in Theorem 22, we get:

Corollary 23. *If $f \in \mathcal{A}_p$ satisfies*

$$1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)} \prec \frac{z(1-\alpha')}{(1-z)(\alpha'-z)},\ \alpha'>1, z\in\mathbb{E},$$

then

$$\frac{zf'(z)}{f(z)} \prec \frac{p\alpha'(1-z)}{\alpha'-z}.$$

Setting $p=1$ in Theorem 22, we get:

Corollary 24. *Suppose $\alpha \in \mathbb{R}\setminus[0,1)$ and if $f \in \mathcal{A}$ satisfies*

$$1+\frac{\alpha zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)} \prec (1-\alpha)\left[1-\frac{\alpha'(1-z)}{\alpha'-z}\right]+\frac{\alpha z(1-\alpha')}{(1-z)(\alpha'-z)},\ \alpha'>1, z\in\mathbb{E},$$

then

$$\frac{zf'(z)}{f(z)} \prec \frac{\alpha'(1-z)}{\alpha'-z}.$$

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