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P-VALENT MEROMORPHIC FUNCTIONS WITH TWO FIXED POINTS ASSOCIATED WITH INTEGRAL OPERATOR

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Abstract. In this paper we used integral operator defined on the space of p-valent meromorphic functions with two fixed points. By making use of this integral operator, we introduce and investigate some new subclasses of p-valent meromorphic starlike functions, coefficient estimate, distortion bounds,the radii of meromorphically starlikeness, some convolution properties and closure properties of p-valent meromorphic functions.

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1. Introduction

Let Σ_p denote the class of functions of the form

$$f(z)=\frac{a_p}{z^p}+\sum_{n=p}^{\infty}a_n z^n \quad (a_n\geq 0; p\in\mathbb{N}=\{1,2,...\}), \tag{1}$$

which are analytic and p-valent in the punctured unit disc $U^*=\{z : z\in\mathbb{C}$ and $0<|z|<1\}=U\backslash\{0\}$.
For function $f(z)$ given by (1) and $g(z)$ is given by

$$g(z)=\frac{b_p}{z^p}+\sum_{n=p}^{\infty}b_n z^n$$

Hadamard product (or convolution) of $f(z)$ and $g(z)$ in Σ_p is given by

$$f(z)*g(z)=(f*g)(z)=\frac{a_pb_p}{z^p}+\sum_{n=p}^{\infty}a_nb_nz^n \quad (a_n,c_n\geq 0; p\in\mathbb{N}).$$

For p-valently meromorphic functions $f(z) \in \Sigma_p$, the normalization

$$z^{1+p} f(z)\,|_{z=0} = 0 \text{ and } z^p f(z)\,|_{z=0} = 1 \tag{2}$$

is classical. We can obtain interesting results by applying Montel's normalization [14] of the form

$$z^{1+p} f(z)\,|_{z=0} = 0 \text{ and } z^p f(z)\,|_{z=\rho} = 1, \tag{3}$$

where ρ is a fixed point from the unit disc U^*.
Meromorphic p-valent functions have been extensively studied by Joshi and Aouf [6], Raina [15], Aouf et al. [3], Uralegaddi and Somanatha [19] and Mogra [13] (see also [2, 7, 8, 9, 11, 12, 16, 18, 20, 21]).
El-Ashwah and Hassan [5] defined the integral operator as follows:

$$\mathcal{J}_{p,\mu}^{a,c} f(z) = a_p z^{-p} + \frac{\Gamma(c-\mu p)}{\Gamma(a-\mu p)} \sum_{n=p}^{\infty} \frac{\Gamma(a+\mu n)}{\Gamma(c+\mu n)} a_n z^n, \tag{4}$$

$$(\mu > 0, a, c \in \mathbb{C}, Re(c-a) \geq 0; Re(a) > p\mu; p \in \mathbb{N}; k > -p)$$

where $f \in \Sigma_p$ is in the form (1) and $(\nu)_n$ denotes the Pochhammer symbol given by

$$(\nu)_n = \frac{\Gamma(\nu+n)}{\Gamma(\nu)} = \begin{cases} 1 & (n=0) \\ \nu(\nu+1)...(\nu+n-1) & (n \in \mathbb{N}). \end{cases}$$

By specializing the parameters in (4), we note that:
(i) $\mathcal{J}_{p,1}^{a+p,c+p} f(z) = \ell_p(a,c) f(z)$ $(a \in \mathbb{R}; c \in \mathbb{R}\backslash\mathbb{Z}_0^-, \mathbb{Z}_0^- = \{1,2,...\}; p \in \mathbb{N})$ (see Liu and Srivastava [10]);
(ii) $\mathcal{J}_{p,1}^{k+2p,p+1} f(z) = D^{k+p-1} f(z)$ (k is an integer,$k > -p, p \in \mathbb{N}$) (see Uralegaddi and Somanatha [19], Aouf [1] and Aouf and Srivastava [4]).

By using the integral operator defined by (4), we define a new class $\mathcal{M}_{p,\mu}^{a,c}(\alpha,\beta)$ of Σ_p as follows:

$$\left| \frac{z\left(\mathcal{J}_{p,\mu}^{a,c} f(z)\right)'}{p\left(\mathcal{J}_{p,\mu}^{a,c} f(z)\right)} + \alpha + \alpha\beta \right| \leq Re\left\{ \frac{-z\left(\mathcal{J}_{p,\mu}^{a,c} f(z)\right)'}{p\left(\mathcal{J}_{p,\mu}^{a,c} f(z)\right)} + \alpha - \alpha\beta \right\}, \tag{5}$$

$$\left(\alpha \geq \frac{1}{2+\beta}; 0 \leq \beta < 1.\right)$$

Also, we note that:

(i) Putting $a = c$, we have

$$\mathcal{M}^{c,c}_{p,\mu}(\alpha,\beta) = \left|\frac{zf'(z)}{pf(z)} + \alpha + \alpha\beta\right| \leq Re\left\{\frac{-zf'(z)}{pf(z)} + \alpha - \alpha\beta\right\}, \left(\alpha \geq \frac{1}{2+\beta}; 0 \leq \beta < 1.\right);$$

(ii) Putting $a = c$ and $p = 1$, we have

$$\mathcal{M}^{c,c}_{1,\mu}(\alpha,\beta) = \left|\frac{zf'(z)}{f(z)} + \alpha + \alpha\beta\right| \leq Re\left\{\frac{-zf'(z)}{f(z)} + \alpha - \alpha\beta\right\}, \left(\alpha \geq \frac{1}{2+\beta}; 0 \leq \beta < 1.\right).$$

Further more, we introduce the subclass $\mathcal{M}^{a,c}_{p,\mu}(\alpha,\beta,\rho)$ satisfying the condition (5) with Montel's normalization (3).

2. Mean results

Now, we state and prove the following theorem.

Theorem 1. *Let $f(z) \in \Sigma_p$, then $f(z)$ is in the class $\mathcal{M}^{a,c}_{p,\mu}(\alpha,\beta)$ if and only if*

$$\sum_{n=p}^{\infty} B_n d_n |a_n| \leq p(1-\alpha\beta)a_p, \tag{6}$$

where

$$B_n = \frac{\Gamma(c-\mu p)}{\Gamma(a-\mu p)} \frac{\Gamma(a+\mu n)}{\Gamma(c+\mu n)} \tag{7}$$

and

$$d_n = n + p\alpha\beta \tag{8}$$

Proof. We assume $f(z) \in \mathcal{M}^{a,c}_{p,\mu}(\alpha,\beta)$, by using (1.5), we have

$$\left|\frac{z\left(\mathcal{J}^{a,c}_{p,\mu}f(z)\right)'}{p\left(\mathcal{J}^{a,c}_{p,\mu}f(z)\right)} + \alpha + \alpha\beta\right| \leq Re\left\{\frac{-z\left(\mathcal{J}^{a,c}_{p,\mu}f(z)\right)'}{p\left(\mathcal{J}^{a,c}_{p,\mu}f(z)\right)} + \alpha - \alpha\beta\right\}$$

that is,

$$\begin{aligned} Re\left\{\frac{z\left(\mathcal{J}^{a,c}_{p,\mu}f(z)\right)'}{p\left(\mathcal{J}^{a,c}_{p,\mu}f(z)\right)} + \alpha + \alpha\beta\right\} &\leq \left|\frac{z\left(\mathcal{J}^{a,c}_{p,\mu}f(z)\right)'}{p\left(\mathcal{J}^{a,c}_{p,\mu}f(z)\right)} + \alpha + \alpha\beta\right| \\ &\leq Re\left\{\frac{-z\left(\mathcal{J}^{a,c}_{p,\mu}f(z)\right)'}{p\left(\mathcal{J}^{a,c}_{p,\mu}f(z)\right)} + \alpha - \alpha\beta\right\}, \end{aligned}$$

then, we have

$$Re\left\{\frac{z\left(\mathcal{J}_{p,\mu}^{a,c}f(z)\right)'}{p\left(\mathcal{J}_{p,\mu}^{a,c}f(z)\right)}+\alpha\beta\right\}\leq 0.$$

Hence,

$$Re\left\{\frac{\frac{-pa_p}{z^p}+\sum_{n=p}^{\infty}nB_na_nz^n}{\frac{pa_p}{z^p}+\sum_{n=p}^{\infty}pB_na_nz^n}+\alpha\beta\right\}\leq 0.$$

Since $Re(z)\leq|z|$, we obtain

$$\left|-pa_p+\sum_{n=p}^{\infty}nB_na_nz^{n+p}+\alpha\beta pa_p+p\alpha\beta\sum_{n=p}^{\infty}B_na_nz^{n+p}\right|\leq 0$$

and by letting $|z|\longrightarrow 1^-$, we have

$$\sum_{n=p}^{\infty}B_n(n+p\alpha\beta)\left|a_n\right|\leq p(1-\alpha\beta)a_p.$$

Conversely, we suppose that the inequality (6) holds true. Then, if we let $z\in\partial U$, we get from (1) and (6), that

$$Re\left\{\frac{z\left(\mathcal{J}_{p,\mu}^{a,c}f(z)\right)'}{p\left(\mathcal{J}_{p,\mu}^{a,c}f(z)\right)}+\alpha\beta\right\}\leq 0.$$

Then, we have

$$Re\left\{\frac{\frac{-pa_p}{z^p}+\sum_{n=p}^{\infty}nB_na_nz^n}{\frac{pa_p}{z^p}+\sum_{n=p}^{\infty}pB_na_nz^n}+\alpha\beta\right\}\leq 0.$$

Since $Re(z)\leq|z|$, we get

$$\sum_{n=p}^{\infty}\frac{B_n(n+p\alpha\beta)\left|a_n\right|}{p(1-\alpha\beta)a_p}\leq 1,$$

which completes the proof of Theorem 2.1.

Theorem 2. *(Coefficient estimate). Let $f(z)\in\mathcal{M}_{p,\mu}^{a,c}(\alpha,\beta)$, then*

$$\sum_{n=p}^{\infty}\left|a_n\right|\leq\frac{p(1-\alpha\beta)a_p}{B_nd_n},\tag{9}$$

where B_n and d_n given by (7) and (8), respectively.

Theorem 3. *Let* $f(z) \in \mathcal{M}_{p,\mu}^{a,c}(\alpha,\beta,\rho)$, *then*

$$\sum_{n=p}^{\infty} |a_n| \leq \frac{p(1-\alpha\beta)}{B_n d_n + p(1-\alpha\beta)\rho^{n+p}}, \tag{10}$$

where B_n and d_n given by (7) and (8), respectively.

Proof. Let $f(z) \in \mathcal{M}_{p,\mu}^{a,c}(\alpha,\beta,\rho)$. Since $f(z) \in \mathcal{M}_{p,\mu}^{a,c}(\alpha,\beta)$ by Theorem 2.1, we have

$$\sum_{n=p}^{\infty} B_n d_n |a_n| \leq p(1-\alpha\beta)a_p. \tag{11}$$

For $f(z) \in \Sigma_p$, by Montel's normalization (3), we obtain

$$\begin{aligned} z^p \left(a_p z^{-p} + \sum_{n=p}^{\infty} a_n z^n \right) \Big|_{z=\rho} = 1, \\ a_p \quad = \quad 1 - \sum_{n=p}^{\infty} a_n \rho^{n+p}. \end{aligned}$$

By using (6), we have

$$\begin{aligned} \sum_{n=p}^{\infty} B_n d_n |a_n| &\leq p(1-\alpha\beta)(1 - \sum_{n=p}^{\infty} a_n \rho^{n+p}), \\ \sum_{n=p}^{\infty} \left[B_n d_n + p(1-\alpha\beta)\rho^{n+p} \right] |a_n| &\leq p(1-\alpha\beta). \end{aligned}$$

Hence

$$\sum_{n=p}^{\infty} |a_n| \leq \frac{p(1-\alpha\beta)}{B_n d_n + p(1-\alpha\beta)\rho^{n+p}}.$$

Theorem 4. *(Distortion bounds). If* $f(z) \in \mathcal{M}_{p,\mu}^{a,c}(\alpha,\beta,\rho)$, *then*

$$\left(\frac{B_p d_p - p(1-\alpha\beta)r^{2p}}{B_p d_p + p(1-\alpha\beta)\rho^{2p}} \right) r^{-p} \leq |f(z)| \leq \left(\frac{B_p d_p + p(1-\alpha\beta)r^{2p}}{B_p d_p + p(1-\alpha\beta)\rho^{2p}} \right) r^{-p} \quad (0 < |z| = r < 1).$$

Proof. Let $f(z) \in \mathcal{M}_{p,\mu}^{a,c}(\alpha,\beta)$. From Theorem 2.3, we have

$$\sum_{n=p}^{\infty}\left[B_n d_n + p(1-\alpha\beta)\rho^{n+p}\right]|a_n| \le p(1-\alpha\beta),$$

which yields,

$$\sum_{n=p}^{\infty}|a_n| \le \frac{p(1-\alpha\beta)}{B_p d_p + p(1-\alpha\beta)\rho^{2p}}.$$

$$\begin{aligned}
|f(z)| &= \left|a_p z^{-p} + \sum_{n=p}^{\infty} a_n z^n\right| \\
&\le \left(1 - \sum_{n=p}^{\infty}|a_n|\rho^{n+p} + \sum_{n=p}^{\infty}|a_n| r^{n+p}\right) r^{-p} \\
&\le \left(1 - (\rho^{2p} - r^{2p})\sum_{n=p}^{\infty}|a_n|\right) r^{-p} \\
&\le \left(\frac{B_p d_p + p(1-\alpha\beta)r^{2p}}{B_p d_p + p(1-\alpha\beta)\rho^{2p}}\right) r^{-p}.
\end{aligned}$$

To prove the other hand side, by using the same arguments above we derive

$$|f(z)| \ge \left(\frac{B_p d_p - p(1-\alpha\beta)r^{2p}}{B_p d_p + p(1-\alpha\beta)\rho^{2p}}\right) r^{-p}.$$

The proof of Theorem 2.4 is completed.

Taking $\rho = 0$ in Theorem 4, we state the following theorem without proof.

Theorem 5. *If* $f(z) \in \mathcal{M}_{p,\mu}^{a,c}(\alpha,\beta)$, *then*

$$\left(\frac{B_p d_p - p(1-\alpha\beta)r^{2p}}{B_p d_p}\right) r^{-p} \le |f(z)| \le \left(\frac{B_p d_p + p(1-\alpha\beta)r^{2p}}{B_p d_p}\right) r^{-p} \quad (0 < |z| = r < 1).$$

3. The radii of meromorphically starlikeness

Now we determine the radii of meromorphically starlikeness of order μ for function in the class $\mathcal{M}_{p,\mu}^{a,c}(\alpha,\beta)$.

Theorem 6. *Let the function $f(z)$ defined by (1) be in the class $\mathcal{M}_{p,\mu}^{a,c}(\alpha,\beta)$, then $f(z)$ is meromorphically p-valent starlike of order $\delta\,(0 \leq \sigma < p)$ in $0<|z| < r_1$, that is,*

$$Re\left\{-\frac{zf'(z)}{f(z)}\right\} > \sigma \ (0 \leq \sigma < p, 0 < |z| < r_1),$$

where

$$|z| \leq \left(\frac{B_n d_n (p-\sigma)}{(n+\sigma)p(1-\alpha\beta)}\right)^{\frac{1}{n+p}}. \tag{12}$$

Proof. Suppose $f(z)$ is given by (1). Then we have

$$\left|\frac{\frac{zf'(z)}{f(z)}+p}{\frac{zf'(z)}{f(z)}-p+2\sigma}\right| \leq \frac{\sum_{n=p}^{\infty}(n+p)\,a_n\,|z|^{n+p}}{2\,(p-\sigma)\,a_p - \sum_{n=p}^{\infty}(n-p+2\sigma)a_n\,|z|^{n+p}},$$
$$(0 \leq \delta < 1, 0 < |z| < r_1(\lambda,\alpha,\beta,k,c,\delta)).$$

Thus, we have the desired inequality

$$\left|\frac{\frac{zf'(z)}{f(z)}+p}{\frac{zf'(z)}{f(z)}-p+2\sigma}\right| \leq 1, \text{ if, } \sum_{n=p}^{\infty}\frac{(n+\sigma)}{(p-\sigma)\,a_p}\,|a_n|\,|z|^{n+p} \leq 1. \tag{13}$$

Since $f(z) \in \mathcal{M}_{p,\mu}^{a,c}(\alpha,\beta)$, from Theorem 2.1, we have

$$\sum_{n=p}^{\infty}\frac{B_n d_n\,|a_n|}{p(1-\alpha\beta)a_p} \leq 1. \tag{14}$$

By using (13) and (14), we get

$$\frac{(n+\sigma)}{(p-\sigma)}\,|z|^{n+p} \leq \frac{B_n d_n}{p(1-\alpha\beta)},$$

hence,

$$|z| = r \leq \left(\frac{B_n d_n (p-\sigma)}{(n+\sigma)p(1-\alpha\beta)}\right)^{\frac{1}{n+p}}.$$

This completes the proof of Theorem 3.1.

4. Convolution properties

For two functions $f_j(z) \in \Sigma_p (j = 1, 2)$ are given by

$$f_j(z) = a_{p,j} z^{-p} + \sum_{n=p}^{\infty} |a_{n,j}| z^n \quad (j = 1, 2; p \in \mathbb{N}). \tag{15}$$

Hadamard product (or convolution) of $f_1(z)$ and $f_2(z)$ in Σ_p is given by

$$(f_1 * f_2)(z) = a_{p,1} a_{p,2} z^{-p} + \sum_{n=p}^{\infty} |a_{n,1}| |a_{n,2}| z^n = (f_2 * f_1)(z). \tag{16}$$

Theorem 7. *Let the functions $f_j(z)$ $(j = 1, 2)$ defined by* (15) *be in the class $\mathcal{M}_{p,\mu}^{a,c}(\alpha, \beta)$. Then $(f_1 * f_2)(z) \in \mathcal{M}_{p,\mu}^{a,c}(\alpha, \delta)$, where*

$$\delta \le \tfrac{1}{\alpha} \left(1 - \frac{p(1 - \alpha\beta)^2 B_p}{p^2 (1 - \alpha\beta)^2 B_p + B_p^2 d_p^2} \right)$$

where $d_p = p + p\alpha\beta$ and $B_p = \frac{\Gamma(c-\mu p)}{\Gamma(a-\mu p)} \frac{\Gamma(a+\mu p)}{\Gamma(c+\mu p)}$.

Proof. Let $f_1(z) = a_{p,1} z^{-p} + \sum_{n=p}^{\infty} |a_{n,1}| z^n$ and $f_2(z) = a_{p,2} z^{-p} + \sum_{n=p}^{\infty} |a_{n,2}| z^n$ are in the class $\mathcal{M}_{p,\mu}^{a,c}(\alpha, \beta)$.

Then by Theorem 2.1, we have

$$\sum_{n=p}^{\infty} \frac{B_n d_n}{p(1 - \alpha\beta) a_{p,1}} |a_{n,1}| \le 1,$$
$$\sum_{n=p}^{\infty} \frac{B_n d_n}{p(1 - \alpha\beta) a_{p,2}} |a_{n,2}| \le 1.$$

Employing the techniques used earlier by Schild and Silverman [17], we need to find the smallest δ such that

$$\sum_{n=p}^{\infty} \frac{(n + p\alpha\delta) B_n}{p(1 - \alpha\delta) a_{p,1} a_{p,2}} |a_{n,1}| |a_{n,2}| \le 1. \tag{17}$$

Since $f_j(z) \in \mathcal{M}_{p,\mu}^{a,c}(\alpha, \beta)$ $(j = 1, 2)$, we have

$$\sum_{n=p}^{\infty} \frac{[n + p\alpha\beta)] B_n}{p(1 - \alpha\beta) a_{p,j}} |a_{n,j}| \le 1 \ (j = 1, 2). \tag{18}$$

Therefore, by the Cauchy-Schwarz inequality, we obtain

$$\sum_{n=p}^{\infty} \frac{d_n B_n}{p(1-\alpha\beta)\sqrt{a_{p,1}a_{p,2}}} \sqrt{|a_{n,1}|\,|a_{n,2}|} \leq 1. \tag{19}$$

Thus it is sufficient to show that

$$\begin{aligned} & \frac{(n+p\alpha\delta)B_n\,|a_{n,1}|\,|a_{n,2}|}{p(1-\alpha\delta)a_{p,1}a_{p,2}} \\ \leq\ & \frac{d_n B_n}{p(1-\alpha\beta)\sqrt{a_{p,1}a_{p,2}}} \sqrt{|a_{n,1}|\,|a_{n,2}|}, \end{aligned} \tag{20}$$

or, equivalently that

$$\sqrt{|a_{n,1}|\,|a_{n,2}|} \leq \frac{d_n\,(1-\alpha\delta)\sqrt{a_{p,1}a_{p,2}}}{(n+p\alpha\delta)\,(1-\alpha\beta)}. \tag{21}$$

Hence, by inequality (7), we obtain

$$\sqrt{|a_{n,1}|\,|a_{n,2}|} \leq \frac{p(1-\alpha\beta)\sqrt{a_{p,1}a_{p,1}}}{B_n d_n} \tag{22}$$

It follows from (21) and (22) that

$$\frac{p(1-\alpha\beta)}{B_n d_n} \leq \frac{d_n\,(1-\alpha\delta)}{(n+p\alpha\delta)\,(1-\alpha\beta)}. \tag{23}$$

It follows that

$$\delta \leq \tfrac{1}{\alpha}\left(1 - \frac{(n+p)\,p(1-\alpha\beta)^2}{p^2(1-\alpha\beta)^2 + B_n d_n^2}\right)$$

Now, defining the function $\varphi(n)$ by

$$\varphi(n) = \tfrac{1}{\alpha}\left(1 - \frac{(n+p)\,p(1-\alpha\beta)^2}{p^2(1-\alpha\beta)^2 + B_n d_n^2}\right)\ (n \geq p).$$

We see that $\varphi(n)$ is an increasing function of n $(n \geq p)$. Therefore we conclude that

$$\delta \leq \varphi(p) = \tfrac{1}{\alpha}\left(1 - \frac{2(1-\alpha\beta)^2}{(1-\alpha\beta)^2 + (1+\alpha\beta)^2\,B_p}\right)$$

which completes the proof of Theorem 4.1.

Theorem 8. *Let the function $f_1(z)$ be in the class $\mathcal{M}_{p,\mu}^{a,c}(\alpha,\beta)$ and the function $f_2(z)$ be in the class $\mathcal{M}_{p,\mu}^{a,c}(\alpha,\gamma)$, then $(f_1 * f_2)(z) \in \mathcal{M}_{p,\mu}^{a,c}(\alpha,\zeta)$, where*

$$\zeta \leq \frac{1}{\alpha}\left[\frac{B_p(1+\alpha\beta)(1+\alpha\gamma)-(1-\alpha\beta)(1-\alpha\gamma)}{(1-\alpha\beta)(1-\alpha\gamma)+B_p(1+\alpha\beta)(1+\alpha\gamma)}\right]. \tag{24}$$

Theorem 9. *Suppose that $f_1(z) \in \mathcal{M}_{p,\mu}^{a,c}(\alpha,\beta)$ and $f_2(z) \in \mathcal{M}_{p,\mu}^{a,c}(\alpha,\gamma)$. By using Theorem 2.1, we get*

$$\sum_{n=p}^{\infty} \frac{[n+p\alpha\beta]\, B_n}{p(1-\alpha\beta)a_{p,1}} |a_{n,1}| \leq 1 \tag{25}$$

and

$$\sum_{n=p}^{\infty} \frac{[n+p\alpha\gamma)]\, B_n}{p(1-\alpha\gamma)a_{p,2}} |a_{n,2}| \leq 1. \tag{26}$$

Proof. Since $(f_1 * f_2)(z) \in \mathcal{M}_{p,\mu}^{a,c}(\alpha,\zeta)$, then by Theorem 2.1, we have

$$\sum_{n=p}^{\infty} \frac{[n+p\alpha\zeta)]\, B_n}{p(1-\alpha\zeta)a_{p,1}a_{p,2}} |a_{n,1}|\,|a_{n,2}| \leq 1. \tag{27}$$

Applying Cauchy-Schwartz inequality, we have

$$\frac{B_n\sqrt{[n+p\alpha\beta]\,[n+p\alpha\gamma)]}}{p\sqrt{(1-\alpha\beta)(1-\alpha\gamma)a_{p,1}a_{p,2}}}\sqrt{|a_{n,1}|\,|a_{n,2}|} \leq 1. \tag{28}$$

From (27) and (28), we have

$$\begin{aligned} \frac{[n+p\alpha\zeta]\, B_n}{p(1-\alpha\zeta)a_{p,1}a_{p,2}} |a_{n,1}|\,|a_{n,2}| &\leq \frac{B_n\sqrt{[n+p\alpha\beta][n+p\alpha\gamma)]}}{p\sqrt{(1-\alpha\beta)(1-\alpha\gamma)a_{p,1}a_{p,2}}}\sqrt{|a_{n,1}|\,|a_{n,2}|} \\ \sqrt{|a_{n,1}|\,|a_{n,2}|} &\leq \frac{\sqrt{[n+p\alpha\beta][n+p\alpha\gamma)]}(1-\alpha\zeta)\sqrt{a_{p,1}a_{p,2}}}{\sqrt{(1-\alpha\beta)(1-\alpha\gamma)}(n+p\alpha\zeta)}. \end{aligned} \tag{29}$$

We know that

$$\sqrt{|a_{n,1}|\,|a_{n,2}|} \leq \frac{p\sqrt{(1-\alpha\beta)(1-\alpha\gamma)a_{p,1}a_{p,2}}}{B_n\sqrt{[n+p\alpha\beta]\,[n+p\alpha\gamma)]}}, \tag{30}$$

It follows from (29) and (30) that

$$\frac{p\sqrt{(1-\alpha\beta)(1-\alpha\gamma)}}{B_n\sqrt{[n+p\alpha\beta]\,[n+p\alpha\gamma)]}} \leq \frac{\sqrt{[n+p\alpha\beta][n+p\alpha\gamma)]}(1-\alpha\zeta)}{\sqrt{(1-\alpha\beta)(1-\alpha\gamma)}(n+p\alpha\zeta)} \tag{31}$$

$$\zeta \leq \frac{1}{\alpha}\left[\frac{B_n(n+p\alpha\beta)(n+p\alpha\gamma)-np(1-\alpha\beta)(1-\alpha\gamma)}{p^2(1-\alpha\beta)(1-\alpha\gamma)+B_n(n+p\alpha\beta)(n+p\alpha\gamma)}\right] \tag{32}$$

Now,we define the function $\phi(n)$ by

$$\phi(n) = \frac{1}{\alpha}\left[\frac{B_n(n+p\alpha\beta)(n+p\alpha\gamma)-np(1-\alpha\beta)(1-\alpha\gamma)}{p^2(1-\alpha\beta)(1-\alpha\gamma)+B_n(n+p\alpha\beta)(n+p\alpha\gamma)}\right] \ (n \geq p),$$

We see that $\phi(n)$ is an increasing function of n $(n \geq p)$. Therefore we conclude that

$$\zeta \leq \frac{1}{\alpha}\left[\frac{B_p(1+\alpha\beta)(1+\alpha\gamma)-(1-\alpha\beta)(1-\alpha\gamma)}{(1-\alpha\beta)(1-\alpha\gamma)+B_p(1+\alpha\beta)(1+\alpha\gamma)}\right]$$

which completes the proof of Theorem 4.3.

Theorem 10. *Let the functions* $f_j(z)$ $(j = 1, 2)$ *defined by*

$$f_j(z) = a_{p,j}z^{-p} + \sum_{n=p}^{\infty} |a_{n,j}|\, z^n \ (j = 1, 2)$$

be in the class $\mathcal{M}_{p,\mu}^{a,c}(\alpha, \beta, \rho)$, then the function $h(z)$ defined by

$$h(z) = (a_{p,1} + a_{p,2})\, z^{-p} + \sum_{n=p}^{\infty} \left(|a_{n,1}|^2 + |a_{n,2}|^2\right) z^n \tag{33}$$

belongs to the class $\mathcal{M}_{p,\mu}^{a,c}(\alpha, \gamma, \rho)$, where

$$\gamma \leq \frac{1}{\alpha}\left[\frac{c_p^2-2p^2(1-\alpha\beta)^2(B_p+\rho^{2p})}{c_p^2+2p^2(1-\alpha\beta)^2(B_p-\rho^{2p})}\right], \tag{34}$$

where

$$c_n = (n + p\alpha\beta)\, B_n + p(1 - \alpha\beta)\rho^{n+p}.$$

Proof. Noting that

$$\sum_{n=p}^{\infty}\left[\frac{c_n}{p(1-\alpha\beta)}\right]^2 |a_{n,j}|^2 \leq \left[\sum_{n=p}^{\infty}\frac{c_n}{p(1-\alpha\beta)}\,|a_{n,j}|\right]^2 \leq 1 \ (j = 1, 2).$$

For $f_j(z) \in \mathcal{M}_{p,\mu}^{a,c}(\alpha, \beta, \rho)$ $(j = 1, 2)$, we obtain

$$\sum_{n=p}^{\infty}\frac{1}{2}\left[\frac{c_n}{p(1-\alpha\beta)}\right]^2 (|a_{n,1}|^2 + |a_{n,2}|^2) \leq 1. \tag{35}$$

Therefore we have to find the largest γ such that

$$\sum_{n=p}^{\infty}\left[\frac{B_n(n+p\alpha\gamma)+p(1-\alpha\gamma)\rho^{n+p}}{p(1-\alpha\gamma)}\right](|a_{n,1}|^2+|a_{n,2}|^2)\leq 1\ (n\geq p). \tag{36}$$

It follows from (35) and (36) that

$$\begin{aligned}\left[\frac{B_n[n+p\alpha\gamma)]+p(1-\alpha\gamma)\rho^{n+p}}{p(1-\alpha\gamma)}\right] &\leq \frac{1}{2}\left[\frac{c_n}{p(1-\alpha\beta)}\right]^2, \\ \gamma &\leq \frac{1}{\alpha}\left[\frac{c_n^2-2p(1-\alpha\beta)^2(nB_n+p\rho^{n+p})}{c_n^2+2p^2(1-\alpha\beta)^2(B_n-\rho^{n+p})}\right]\ (n\geq p). \end{aligned}\tag{37}$$

Now, defining a function $\psi(n)$ by

$$\psi(n)=\frac{1}{\alpha}\left[\frac{c_n^2-2p(1-\alpha\beta)^2(nB_n+p\rho^{n+p})}{c_n^2+2p^2(1-\alpha\beta)^2(B_n-\rho^{n+p})}\right]\ (n\geq p).$$

We see that $\psi(n)$ is an increasing function of n $(n\geq p)$. Therefore we conclude that

$$\gamma\leq\frac{1}{\alpha}\left[\frac{c_p^2-2p^2(1-\alpha\beta)^2(B_p+\rho^{2p})}{c_p^2+2p^2(1-\alpha\beta)^2(B_p-\rho^{2p})}\right].$$

which completes the proof of Theorem 4.4.

5. Closure properties

Theorem 11. *Let the function $f(z)$ given by (1) be in the class $\mathcal{M}_{p,\mu}^{a,c}(\alpha,\delta)$. Then the integral operator*

$$F(z)=c_0^1u^{c+p-1}f(uz)du,(0<u\leq 1,\ 0<c<\infty), \tag{38}$$

is in $\mathcal{M}_{p,\mu}^{a,c}(\alpha,\delta)$, where

$$\delta=\frac{1}{\alpha}\left(\frac{(n+c+p)\,(n+p\alpha\beta)-cn\,(1-\alpha\beta)}{(n+c+p)\,(n+p\alpha\beta)+cp(1-\alpha\beta)}\right). \tag{39}$$

Proof. Supoose that $f(z)\in\mathcal{M}_{p,\mu}^{a,c}(\alpha,\beta)$. Then

$$\begin{aligned}F(z) &= c_0^1u^{c+p-1}f(uz)du \\ &= a_pz^{-p}+\sum_{n=p}^{\infty}\left(\frac{c}{n+c+p}\right)a_nz^n.\end{aligned}$$

It is sufficient to show that

$$\sum_{n=p}^{\infty} \frac{(n+p\alpha\delta)}{p\,(1-\alpha\delta)\,a_p}\left(\frac{c}{n+c+p}\right) a_n \leq 1. \tag{40}$$

Since $f(z) \in \mathcal{M}_{p,\mu}^{a,c}(\alpha,\beta)$, we get

$$\sum_{n=p}^{\infty} \frac{(n+p\alpha\beta)}{p\,(1-\alpha\beta)\,a_p} a_n \leq 1. \tag{41}$$

From (40) and (41) we have

$$\frac{(n+p\alpha\delta)}{(1-\alpha\delta)}\left(\frac{c}{n+c+p}\right) \leq \frac{(n+p\alpha\beta)}{(1-\alpha\beta)},$$

hence

$$c\,(n+p\alpha\delta)\,(1-\alpha\beta) \leq (n+c+p)\,(1-\alpha\delta)\,(n+p\alpha\beta)\,,$$

so,

$$\delta \leq \frac{1}{\alpha}\left(\frac{(n+c+p)\,(n+p\alpha\beta) - cn\,(1-\alpha\beta)}{(n+c+p)\,(n+p\alpha\beta) + cp(1-\alpha\beta)}\right) = F(n).$$

We see that $F(n)$ is an increasing function of n $(n \geq p)$, which completes the proof of Theorem 5.1.

Theorem 12. *Let the function $f(z)$ given by (1) be in the class $\mathcal{M}_{p,\mu}^{a,c}(\alpha,\beta)$. Then*

$$F(z) = \frac{1}{c}\left[(c+p)f(z) + zf'(z)\right] = a_p z^{-p} +_{n=p}^{\infty} \left(\frac{n+c+p}{c}\right) a_n z^n,\ (c>0), \tag{42}$$

is in $\mathcal{M}_{p,\mu}^{a,c}(\alpha,\beta)$ for $|z| < r(\alpha,\beta,\delta)$,
where

$$r(\alpha,\beta,\delta) = \inf_n \left(\frac{c\,(1-\alpha\delta)\,(n+p\alpha\beta)}{(1-\alpha\delta)\,(n+c+p)\,(n+p\alpha\beta)}\right)^{\frac{1}{n+p}} \quad (n = p, p+1, ...).$$

Proof. Let $f(z)$ be given by (42), then we get

$$\left|\frac{\frac{-z\left(\mathcal{J}_{p,\mu}^{a,c}F(z)\right)'}{p\alpha\left(\mathcal{J}_{p,\mu}^{a,c}F(z)\right)} + p}{\frac{-z\left(\mathcal{J}_{p,\mu}^{a,c}F(z)\right)'}{p\alpha\left(\mathcal{J}_{p,\mu}^{a,c}F(z)\right)} - p + 2\delta}\right| \leq \frac{\left(p^2\alpha-p\right)a_p + \sum_{n=p}^{\infty}(p^2\alpha-n)\left(\frac{n+c+p}{c}\right)B_n a_n |z|^{n+p}}{(2p\alpha\delta-p^2\alpha-p)a_p - \sum_{n=p}^{\infty}(2p\alpha\delta-p^2\alpha-n)\left(\frac{n+c+p}{c}\right)B_n a_n |z|^{n+p}}.$$

Thus, we have the desired inequality

$$\left| \frac{\frac{-z\left(\mathcal{J}_{p,\mu}^{a,c}F(z)\right)'}{p\alpha\left(\mathcal{J}_{p,\mu}^{a,c}F(z)\right)} + p}{\frac{-z\left(\mathcal{J}_{p,\mu}^{a,c}F(z)\right)'}{p\alpha\left(\mathcal{J}_{p,\mu}^{a,c}F(z)\right)} - p + 2\delta} \right| \leq 1, \text{ if,} \sum_{n=p}^{\infty} . \frac{(n + p\alpha\delta)}{p\,(1-\alpha\delta)\,a_p} \left(\frac{n+c+p}{c}\right) B_n\,|a_n|\,|z|^{n+p} \leq 1. \tag{43}$$

Since $f(z) \in \mathcal{M}_{p,\mu}^{a,c}(\alpha,\beta)$, from Theorem 2.1 we have

$$\sum_{n=p}^{\infty} \frac{(n+p\alpha\beta)\,B_n\,|a_n|}{p(1-\alpha\beta)a_p} \leq 1. \tag{44}$$

By using (43) and (44), we get

$$\frac{(n+p\alpha\delta)}{(1-\alpha\delta)} \left(\frac{n+c+p}{c}\right) |z|^{n+p} \leq \frac{(n+p\alpha\beta)}{(1-\alpha\beta)},$$

hence,

$$|z| = r(\alpha,\beta,\delta) \leq \left(\frac{c\,(1-\alpha\delta)\,(n+p\alpha\beta)}{(n+c+p)\,(n+p\alpha\delta)\,(1-\alpha\beta)}\right)^{\frac{1}{n+p}}.$$

This completes the proof of Theorem 5.2.

Theorem 13. *(Arithmetic Mean). Let the function $f_i(z)$ $(i = 1,2,...,\tau)$ defined by*

$$f_i(z) = a_{p,i}z^{-p} + \sum_{n=p}^{\infty} a_{n,i}z^n \;\; (a_{n,i} \geq 0, i = 1,2,...,\tau, n \geq p)$$

be in the class $\mathcal{M}_{p,\mu}^{a,c}(\alpha,\beta,\rho)$. Then the arithmetic mean of $f_i(z)$ $(i = 1,2,...,\tau)$ defined by

$$h(z) = \frac{1}{\tau}\sum_{i=1}^{\tau} f_i(z)$$

is also in the class $\mathcal{M}_{p,\mu}^{a,c}(\alpha,\beta,\rho)$.

Proof. Since $f_i(z) \in \mathcal{M}_{p,\mu}^{a,c}(\alpha,\beta,\rho) \quad (i=1,2,...,\tau)$, then from Theorem 2.2, we obtain

$$\begin{aligned}
&\sum_{n=p}^{\infty}[(n+p\alpha\beta)+p(1-\alpha\beta)\rho^{n+p}]\left(\frac{1}{\tau}\sum_{i=1}^{\tau}a_{n,i}\right)\\
=\ & \frac{1}{\tau}\sum_{i=1}^{\tau}\left(\sum_{n=p}^{\infty}[(n+p\alpha\beta)+p(1-\alpha\beta)\rho^{n+p}]\right)a_{n,i}\\
\leq\ & \frac{1}{\tau}\sum_{i=1}^{\tau}p(1-\alpha\beta)\\
\leq\ & p(1-\alpha\beta),
\end{aligned}$$

this implies that $h(z) \in \mathcal{M}_{p,\mu}^{a,c}(\alpha,\beta,\rho)$, which completes the proof of Theorem 5.3.

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