Acta Universitatis Apulensis No. 43/2015
ISSN: 1582-5329 pp. 79-92
http://www.uab.ro/auajournal/ doi: 10.17114/j.aua.2015.43.07

# ON SEMI-INVARIANT SUBMANIFOLDS OF A GENERALIZED KENMOTSU MANIFOLD ADMITTING A SEMI-SYMMETRIC METRIC CONNECTION

A. Turgut Vanli, R. Sari

Abstract. In this paper, semi-invariant submanifolds of a generalized Kenmotsu manifold endowed with a semi-symmetric metric connection are studied. Necessary and sufficient conditions are given on a submanifold of a generalized Kenmotsu manifold to be semi-invarinat submanifold with the semi-symmetric metric connection. Moreover, the integrability conditions of the distribution on semi-invariant submanifolds of a generalized Kenmotsu manifold with the semi-symmetric metric connection are studied.

2010 *Mathematics Subject Classification*: 53C17, 53C25, 53C40.

*Keywords:* generalized Kenmotsu manifold, semi-invariant submanifolds, semi-symmetric metric connection.

## 1. Introduction

In [3] , K. Kenmotsu has introduced a Kenmotsu manifold. A. Turgut Vanlı and R. Sarı [6], introduced the notion of a generalized Kenmotsu manifold. Semi-invariant submanifolds are studied by some authors (for examples, M. Kobayashi [4], B.B. Sinha, A.K. Srivastava [5]). In [9], K. Yano have introduced a semi-symmetric metric connection on a Riemannian manifold. He studied some properties of the curvature tensor with respect to the semi-symmetric metric connection. In this paper, semi-invariant submanifolds of a generalized Kenmotsu manifold with a semi-symmetric metric connection are studied.

Let $\nabla$ be a linear connection in an $n$-dimensional differentiable manifold $M$. The torsion tensor $T$ of $\nabla$ is given by

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y].$$

The connection $\nabla$ is symmetric if torsion tensor $T$ vanishes, otherwise it is non-symmetric. A lineer connection $\nabla$ is said to be a semi-symmetric connection if it torsion tensor $T$ is of the form $T(X,Y) = \eta(Y)X - \eta(X)Y$ where $\eta$ is a $1-$form. The connection $\nabla$ is metric connection if there is a Riemannian metric $g$ in $M$ such that $\nabla g = 0$, otherwise it is non-metric. It is well known that a linear connection is symmetric and metric if it is the Levi-Civita connection.

The paper is organized as follows : In section 2, a brief introduction of a generalized Kenmotsu manifolds is given. A semi-symmetric metric connection on a generalized Kenmotsu manifold is defined . In section 3, some basic results for semi-invariant submanifolds of a generalized Kenmotsu manifold with the semi-symmetric metric connection are given. In last section, some necessary and sufficient conditions for integrability of certain distributions on semi-invariant submanifolds of a generalized Kenmotsu manifold with the semi-symmetric metric connection are obtained.

## 2. Semi-invariant submanifolds of generalized Kenmotsu manifold

In [8], K.Yano has introduced the notion of a $f$-structure on a differentianable manifold $M$, i.e., a tensor fields $\varphi$ of type $(1,1)$ and $rank$ $2n$ satisfying $\varphi^3 + \varphi = 0$. The existence of which is equivalent to a reduction of the structural group of the tangent bundle to $U(n) \times O(s)$ [1]. Let $M$ be $(2n+s)$ dimensional and a differentiable manifold with a $f$-structure of rank $2n$. If there exists on $M$ vector fields $\xi_i$, $i \in \{1, 2, ..., s\}$ and $\eta^i$ are dual $1-$forms such that

$$\varphi^2 = -I + \sum_{i=1}^{s} \eta^i \otimes \xi_i, \qquad \eta^i \circ \xi_j = \delta_{ij} \tag{1}$$

then $M$ is called a $f$-manifold. Moreover, we have $\varphi \circ \xi_i = 0$, $\eta^i \circ \varphi = 0$, $i \in \{1, 2, ..., s\}$ [2].

Let $M$ be a $(2n+s)$ dimensional $f$-manifold. $M$ is called a metric $f$-manifold if there exists on $M$ a Riemannian metric $g$ such that

$$g(\varphi X, \varphi Y) = g(X,Y) - \sum_{i=1}^{s} \eta^i(X)\eta^i(Y). \tag{2}$$

In addition, we have

$$\eta^i(X) = g(X, \xi_i), \qquad g(X, \varphi Y) = -g(\varphi X, Y). \tag{3}$$

Then, a 2-form $\Phi$ is defined by $\Phi(X,Y) = g(X, \varphi Y)$, for any $X, Y \in \Gamma(TM)$, called the *fundamental 2-form*. Moreover, a metric $f$-manifold is *normal* if

$$[\varphi, \varphi] + 2\sum_{i=1}^{s} d\eta^i \otimes \xi_i = 0$$

where $[\varphi, \varphi]$ is denoting the Nijenhuis tensor field associated to $\varphi$.

In [7], let $M$, $(2n+s)-$dimensional a metric $f-$manifold. If there exists $2-$form $\Phi$ such that

$$\eta^1 \wedge ... \wedge \eta^s \wedge \Phi^n \neq 0$$

on $M$ then $M$ is called an almost $s-$contact metric structure.

**Definition 1.** *Let $M$ be an almost $s-$contact metric manifold of dimension $(2n+s)$, $s \geq 1$, with $(\varphi, \xi_i, \eta^i, g)$. $M$ is said to be a generalized almost Kenmotsu manifold if for all $1 \leq i \leq s$, $1-$forms $\eta^i$ are closed and $d\Phi = 2\sum\limits_{i=1}^{s} \eta^i \wedge \Phi$. A normal generalized almost Kenmotsu manifold $M$ is called a generalized Kenmotsu manifold [6].*

In [6], A $(2n+s), s \geqslant 1$, dimensional almost $s-$contact metric manifold $\tilde{M}$ is a generalized Kenmotsu manifold if it satisfies the condition

$$(\widetilde{\nabla}_X \varphi)Y = \sum_{i=1}^{s} \{g(\varphi X, Y)\xi_i - \eta^i(Y)\varphi X\} \tag{4}$$

where $\widetilde{\nabla}$ denotes the Riemannian connection with respect to $g$.In [6], from the formula (4) we have

$$\widetilde{\nabla}_X \xi_j = -\varphi^2 X. \tag{5}$$

**Definition 2.** *Let $M$ be a submanifold of the $(2n+s)$-dimensional a generalized Kenmotsu manifold $\tilde{M}$. $M$ is called a semi-invariant submanifold if vector fields $\xi_i$ , $i \in \{1, 2, ..., s\}$ are tangent to $M$ and there exists on $M$ a pair of orthogonal distribution $\{D, D^{\perp}\}$ such that*

*$(i)$ $TM = D \oplus D^{\perp} \oplus sp\{\xi_1, ..., \xi_s\}$*

*$(ii)$ The distribution $D$ is invariant under $\varphi$, that is $\varphi D_x = D_x$, for all $x \in M$*

*$(iii)$ The distribution $D^{\perp}$ is anti-invariant under $\varphi$, that is $\varphi D_x^{\perp} \subset T_x^{\perp} M$, for all $x \in M$,*

*where $T_x M$ is the tangent space of $M$ at $x$.*

A semi-invariant submanifold $M$ is said to be an *invariant* (resp. a*nti-invariant*) submanifold if we have $D_x^{\perp} = \{0\}$ (resp. $D_x = \{0\}$) for each $x \in M$. We say that $M$ is a proper semi-invariant submanifold, which is neither an invariant nor an anti-invariant submanifold.

Let $\widetilde{\nabla}$ be the Riemannian connection of $\tilde{M}$ with respect to the induced metric $g$. Then the Gauss and Weingarten formulas are given by

$$\widetilde{\nabla}_X Y = \nabla^*_X Y + h(X, Y) \tag{6}$$

$$\widetilde{\nabla}_X N = \nabla^{*\perp}_X N - A_N X \tag{7}$$

for any $X, Y \in \Gamma(TM)$ and $N \in \Gamma(TM)^\perp$. $\nabla^*$ is the induced connection on $M$, $\nabla^{*\perp}$ is the connection in the normal bundle, $h$ is the second fundamental from of $M$ and $A_N$ is the Weingarten endomorphism associated with $N$. The second fundamental form $h$ and the shape operator $A$ related by

$$g(h(X,Y),N) = g(A_N X, Y). \tag{8}$$

Now, a linear connection $\overline{\nabla}$ is defined as

$$\overline{\nabla}_X Y = \widetilde{\nabla}_X Y + \sum_{i=1}^{s}\{\eta^i(Y)X - g(X,Y)\xi_i\}.$$

**Theorem 1.** *Let $\widetilde{\nabla}$ be the Riemannian connection on a generalized Kenmotsu manifold $\tilde{M}$. Then the linear connection which is defined as*

$$\overline{\nabla}_X Y = \widetilde{\nabla}_X Y + \sum_{i=1}^{s}\{\eta^i(Y)X - g(X,Y)\xi_i\} \qquad X, Y \in \Gamma(TM)$$

*is a semi-symmetric metric connection on $\tilde{M}$.*

*Proof.* Let $\bar{T}$ be the torsion tensor of $\overline{\nabla}$. Then,

$$\begin{aligned}
\bar{T}(X,Y) &= \overline{\nabla}_X Y - \overline{\nabla}_Y X - [X,Y] \\
&= \widetilde{\nabla}_X Y + \sum_{i=1}^{s}\{\eta^i(Y)X - g(X,Y)\xi_i\} \\
&\quad -\widetilde{\nabla}_Y X - \sum_{i=1}^{s}\{\eta^i(X)Y - g(Y,X)\xi_i\} - [X,Y] \\
&= \sum_{i=1}^{s}\{\eta^i(Y)X - \eta^i(X)Y\}.
\end{aligned}$$

Moreover we get,

$$\begin{aligned}
(\overline{\nabla}_X g)(Y,Z) &= X[g(Y,Z)] - g(\overline{\nabla}_X Y, Z) - g(Y, \overline{\nabla}_X Z) \\
&= X[g(Y,Z)] - g(\widetilde{\nabla}_X Y + \sum_{i=1}^{s}\{\eta^i(Y)X - g(X,Y)\xi_i\}, Z) \\
&\quad -g(Y, \widetilde{\nabla}_X Z + \sum_{i=1}^{s}\{\eta^i(Z)X - g(X,Z)\xi_i\}) \\
&= 0.
\end{aligned}$$

**Corollary 2.** *Let $\widetilde{\nabla}$ be the Riemannian connection on a generalized Kenmotsu manifold $\tilde{M}$. Then the linear connection which is defined as*

$$\overline{\nabla}_X Y = \widetilde{\nabla}_X Y + \sum_{i=1}^{s} \{\eta^i(Y)X - g(X,Y)\xi_i\} \qquad X, Y \in \Gamma(TM) \tag{9}$$

*is a semi-symmetric metric connection on $\tilde{M}$.*

**Theorem 3.** *Let $M$ be a semi-invariant submanifold of a generalized Kenmotsu manifold $\tilde{M}$.Then we have*

$$(\overline{\nabla}_X \varphi)Y = 2\sum_{i=1}^{s} \{g(\varphi X, Y)\xi_i - \eta^i(Y)\varphi X\} \tag{10}$$

*for all $X, Y \in \Gamma(TM)$.*

*Proof.* From (4) and (9), we have

$$(\widetilde{\nabla}_X \varphi)Y = \sum_{i=1}^{s} \{g(\varphi X, Y)\xi_i - \eta^i(Y)\varphi X\}$$

$$\overline{\nabla}_X \varphi Y - \sum_{i=1}^{s} \{\eta^i(\varphi Y)X - g(X, \varphi Y)\xi_i\} - \varphi\{\overline{\nabla}_X Y - \sum_{i=1}^{s} \{\eta^i(Y)X - g(X,Y)\xi_i\}\}$$

$$= \sum_{i=1}^{s} \{g(\varphi X, Y)\xi_i - \eta^i(Y)\varphi X\}$$

$$(\overline{\nabla}_X \varphi)Y = \sum_{i=1}^{s} \{g(\varphi X, Y)\xi_i - \eta^i(Y)\varphi X - g(X, \varphi Y)\xi_i - \eta^i(Y)\varphi X\}.$$

**Theorem 4.** *Let $M$ be a semi-invariant submanifold of a generalized Kenmotsu manifold $\tilde{M}$ with the semi-symmetric metric connection $\overline{\nabla}$. Then*

$$\overline{\nabla}_X \xi_j = 2X - \sum_{i=1}^{s} \{\eta^i(X) + \eta^j(X)\}\xi_i \tag{11}$$

*for all $X, Y \in \Gamma(TM)$.*

*Proof.* Using (9) then we have

$$\overline{\nabla}_X \xi_j = \widetilde{\nabla}_X \xi_j + \sum_{i=1}^{s} \{\eta^i(\xi_j)X - g(X, \xi_j)\xi_i\}$$

from (5),

$$\overline{\nabla}_X \xi_j = X - \sum_{i=1}^{s} \eta^i(X)\xi_i + \sum_{i=1}^{s} \{X - \eta^j(X)\xi_i\}.$$

**Example 1.** *Now , we construct an example of generalized Kenmotsu manifold for 4-dimensional.*

*Let,* $n = 1$ *and* $s = 2$*. The vector fields*

$$e_1 = f_1(z_1, z_2)\frac{\partial}{\partial x} + f_2(z_1, z_2)\frac{\partial}{\partial y},$$

$$e_2 = -f_2(z_1, z_2)\frac{\partial}{\partial x} + f_1(z_1, z_2)\frac{\partial}{\partial y},$$

$$e_3 = \frac{\partial}{\partial z_1},$$

$$e_4 = \frac{\partial}{\partial z_2}$$

*where* $f_1$ *and* $f_2$ *are given by*

$$\begin{aligned} f_1(z_1, z_2) &= c_2 e^{-(z_1+z_2)} Cos(z_1 + z_2) - c_1 e^{-(z_1+z_2)} Sin(z_1 + z_2), \\ f_2(z_1, z_2) &= c_1 e^{-(z_1+z_2)} Cos(z_1 + z_2) + c_2 e^{-(z_1+z_2)} Sin(z_1 + z_2) \end{aligned}$$

*for nonzero constant* $c_1, c_2$*. It is obvious that* $\{e_1, e_2, e_3, e_4\}$ *are linearly independent at each point of* $M$*. Let* $g$ *be the Riemannian metric defined by*

$$g(e_i, e_j) = \begin{cases} 1, & \text{for } i = j \\ 0, & \text{for } i \neq j \end{cases}$$

*for all* $i, j \in \{1, 2, 3, 4\}$ *and given by the tensor product*

$$g = \frac{1}{f_1^2 + f_2^2}(dx \otimes dx + dy \otimes dy) + dz_1 \otimes dz_1 + dz_2 \otimes dz_2,$$

*where* $\{x, y, z_1, z_2\}$ *are standart coordinates in* $\mathbb{R}^4$*. Let* $\eta^1$ *and* $\eta^2$ *be the 1-form defined by*

$$\eta^1(X) = g(X, e_3) \text{ and } \eta^2(X) = g(X, e_4),$$

*respectively, for any vector field $X$ on $M$ and $\varphi$ be the $(1,1)$ tensor field defined by*

$$\varphi(e_1) = e_2, \qquad \varphi(e_2) = -e_1, \qquad \varphi(e_3 = \xi_1) = 0, \qquad \varphi(e_4 = \xi_2) = 0.$$

*We have $\Phi(e_{1,}, e_2) = -1$ and otherwise $\Phi(e_{i,}, e_j) = 0$ for $i < j$. Therefore, the essential non-zero component of $\Phi$ is*

$$\Phi(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}) = \frac{1}{f_1^2 + f_2^2} = \frac{e^{2(z_1+z_2)}}{c_1^2 + c_2^2}$$

*and hence*

$$\Phi = \frac{e^{2(z_1+z_2)}}{c_1^2 + c_2^2} dx \wedge dy.$$

*Consequently, the exterior derivative $d\Phi$ is given by*

$$d\Phi = \frac{2e^{2(z_1+z_2)}}{c_1^2 + c_2^2} dx \wedge dy \wedge (dz_1 + dz_2).$$

*Since $\eta^1 = dz_1$ and $\eta^2 = dz_2$, we find*

$$d\Phi = 2(\eta^1 + \eta^2) \wedge \Phi.$$

*So, we have 4-dimensional a generalized Kenmotsu manifold [6]. Let $\nabla$ be the Riemannian connection (the Levi-Civita connection) of $g$. Then, we have*

$$[e_1, e_4] = [e_1, e_3] = e_1 + e_2, \qquad [e_2, e_4] = [e_2, e_3] = e_1 + e_2,$$

$$[e_1, e_2] = 0, \qquad [e_3, e_4] = 0.$$

*By Koszul's formula, we get*

$$\nabla_{e_1} e_1 = \nabla_{e_1} e_2 = \nabla_{e_2} e_1 = \nabla_{e_2} e_2 = -e_3 - e_4,$$

$$\nabla_{e_1} e_3 = \nabla_{e_1} e_4 = \nabla_{e_2} e_3 = \nabla_{e_2} e_4 = e_1 + e_2$$

*and anothers are zero.*

$$\overline{\nabla}_X Y = \nabla_X Y + \eta^1(Y)X - g(X,Y)\xi_1 + \eta^2(Y)X - g(X,Y)\xi_2$$

*is a semi-symmetric metric connection . Therefore, we have*

$$\bar{\nabla}_{e_1} e_1 = \bar{\nabla}_{e_2} e_2 = -2(e_3 + e_4), \qquad \bar{\nabla}_{e_2} e_1 = \bar{\nabla}_{e_1} e_2 = -e_3 - e_4$$

$$\bar{\nabla}_{e_1} e_3 = \bar{\nabla}_{e_1} e_4 = 2e_1 + e_2, \qquad \bar{\nabla}_{e_2} e_3 = \bar{\nabla}_{e_2} e_4 = e_1 + 2e_2,$$

$$-\bar{\nabla}_{e_3} e_3 = \bar{\nabla}_{e_4} e_3 = e_4, \qquad \bar{\nabla}_{e_3} e_4 = -\bar{\nabla}_{e_4} e_4 = e_3$$

*and anothers are zero.*

We denote by same symbol $g$ both metrices on $\tilde{M}$ and $M$. Let $\overline{\nabla}$ be the semi-symmetric metric connection on $\tilde{M}$ and $\nabla$ be the induced connection on $M$. Then,

$$\overline{\nabla}_X Y = \nabla_X Y + m(X, Y) \tag{12}$$

where $m$ is a tensor field of type $(0, 2)$ on a semi-invariant submanifold $M$. Using (6) and (9) we have,

$$\nabla_X Y + m(X, Y) = \nabla^*_X Y + h(X, Y) + \sum_{i=1}^{s}\{\eta^i(Y)X - g(X, Y)\xi_i\}.$$

So equation tangential and normal components from both the sides, we get

$$m(X, Y) = h(X, Y)$$

$$\nabla_X Y = \nabla^*_X Y + \sum_{i=1}^{s}\{\eta^i(Y)X - g(X, Y)\xi_i\}. \tag{13}$$

From (13) and (7)

$$\begin{aligned}
\nabla_X N &= \nabla^*_X N + \sum_{i=1}^{s}\{\eta^i(N)X - g(X, N)\xi_i\} \\
&= -A_N X + \sum_{i=1}^{s}\eta^i(N)X \\
&= (-A_N + a)X
\end{aligned}$$

where $a = \sum\limits_{i=1}^{s}\eta^i(N)$ is a function on $M$ and $N \in \Gamma(TM)^\perp$.

Now, the Gauss and Weingarten formulas for semi-invariant submanifolds of a generalized Kenmotsu manifold with the semi-symmetric metric connection is

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y) \tag{14}$$

$$\overline{\nabla}_X N = (-A_N + a)X + \nabla^\perp_X N \tag{15}$$

for all $X, Y \in \Gamma(TM)$, $N \in \Gamma(TM)^\perp$, $h$ second fundamental form of $M$ and $A_N$ is the Weingarten endomorphism associated with $N$. The second fundamental form $h$ and the shape operator $A$ related by

$$g(h(X, Y), N) = g((-A_N + a)X, Y). \tag{16}$$

**Theorem 5.** *The connection induced on a semi-invariant submanifold of a generalized Kenmotsu manifold with the semi-symmetric metric connection is also a semi-symmetric metric connection.*

*Proof.* From (14) we have

$$\bar{T}(X,Y) = T(X,Y) \text{ and } (\overline{\nabla}_X g)(Y,Z) = (\nabla_X g)(Y,Z)$$

for any $X, Y \in \Gamma(TM)$, where $T$ is the torsion tensor of $\nabla$.

The projection morphisms of $TM$ to $D$ and $D^{\perp}$ are denoted by $P$ and $Q$ respectively. For any $X, Y \in \Gamma(TM)$ and $N \in \Gamma(TM)^{\perp}$, we have

$$X = PX + QX + \sum_{i=1}^{s} \eta^i(X)\xi_i \tag{17}$$

$$\varphi N = BN + CN \tag{18}$$

where $BN$ (resp. $CN$) denotes the tangential (resp. normal) component of $\varphi N$.

## 3. Basic Results

**Lemma 6.** *Let $M$ be a semi-invariant submanifold of a generalized Kenmotsu manifold $\tilde{M}$ with the semi-symmetric metric connection, then we have*

$$\begin{aligned}(\overline{\nabla}_X \varphi)Y &= (\nabla_X P)Y + (-A_{QY} + a)X - Bh(X,Y) \\ &\quad +(\nabla_X Q)Y + h(X,PY) - Ch(X,Y)\end{aligned} \tag{19}$$

$$\begin{aligned}(\overline{\nabla}_X \varphi)N &= (\nabla_X B)N + (-A_{CN} + a)X + P(-A_N + a)X \\ &\quad +(\nabla_X C)N + h(X,BN) + Q(-A_N + a)X\end{aligned} \tag{20}$$

*for all $X, Y \in \Gamma(TM)$, $N \in \Gamma(TM)^{\perp}$ where $a = \sum_{i=1}^{s} \eta^i(CN) = 0$.*

*Proof.* Using (17) , (18), the Gauss and Weingarten formulas, necessary arrangements are made to obtain the desired.

**Lemma 7.** *Let $M$ be a semi-invariant submanifold of a generalized Kenmotsu manifold $\tilde{M}$ with the semi-symmetric metric connection, we have*

$$(\nabla_X P)Y + (-A_{QY} + a)X - Bh(X,Y) = -2\sum_{i=1}^{s} \eta^i(Y)PX \tag{21}$$

$$(\nabla_X Q)Y + h(X,PY) - Ch(X,Y) = -2\sum_{i=1}^{s} \eta^i(Y)QX \tag{22}$$

$$(\nabla_X B)N + (-A_{CN} + a)X + P(-A_N + a)X = 0 \tag{23}$$

$$(\nabla_X C)N + h(X, BN) + Q(-A_N + a)X = 0 \tag{24}$$

$$g(PX, Y) = 0 \tag{25}$$

$$g(QX, Y) = 0 \tag{26}$$

*for all* $X, Y \in \Gamma(TM)$, $N \in \Gamma(TM)^{\perp}$.

*Proof.* Using (10) in (19) and (20) we get (21)-,(26).

**Corollary 8.** *Let* $M$ *be a semi-invariant submanifold of a generalized Kenmotsu manifold* $\tilde{M}$ *with semi-symmetric metric connection such that* $\xi_i \in \Gamma(TM)$, *we have*

$$(\nabla_X P)\xi_j = -2PX \tag{27}$$

$$(\nabla_X Q)\xi_j = -2QX \tag{28}$$

$$(\nabla_{\xi_j} B)N = 0, \quad \nabla_{\xi_j} B = 0 \tag{29}$$

$$(\nabla_{\xi_j} C)N = 0, \quad \nabla_{\xi_j} C = 0. \tag{30}$$

**Lemma 9.** *Let* $M$ *be a semi-invariant submanifold of a generalized Kenmotsu manifold* $\tilde{M}$ *with the semi-symmetric metric connection such that* $\xi_i \in \Gamma(TM)$, *we have*

$$\nabla_X \xi_j = 2X - \sum_{i=1}^{s} \{\eta^i(X) + \eta^j(X)\}\xi_i, \qquad h(X, \xi_j) = 0 \tag{31}$$

$$\nabla_{\xi_i} \xi_j = 0, \qquad h(\xi_i, \xi_j) = 0, \qquad A_N \xi_j = 0. \tag{32}$$

*Proof.* Using (9)and (11) we have (31).In addition, we get

$$0 = g(h(X, \xi_j), N) = g(h(\xi_j, X), N) = g(A_N \xi_j, X).$$

## 4. Integrability of distribution on a semi-invariant submanifold generalized Kenmotsu manifold

**Theorem 10.** *Let* $M$ *be a semi-invariant submanifold of a generalized Kenmotsu manifold* $\tilde{M}$ *with the semi-symmetric metric connection. Then the distribution* $D$ *is integrable.*

*Proof.* We have

$$\begin{aligned} g([X,Y],\xi_j) &= g(\widetilde{\nabla}_X Y,\xi_j) - g(\widetilde{\nabla}_Y X,\xi_j) \\ &= -g(Y,\widetilde{\nabla}_X \xi_j) + g(X,\widetilde{\nabla}_Y \xi_j) \end{aligned}$$

for all $X,Y \in \Gamma(D)$. Using (9) and (11), we get

$$\begin{aligned} g([X,Y],\xi_j) &= -g(Y,\overline{\nabla}_X\xi_j - X + \sum_{i=1}^{s} g(X,\xi_j)\xi_i) + g(X,\overline{\nabla}_Y\xi_j - Y + \sum_{i=1}^{s} g(Y,\xi_j)\xi_i) \\ &= -g(Y,2X - \sum_{i=1}^{s}\{\eta^i(X)+\eta^j(X)\}\xi_i - X + \sum_{i=1}^{s} g(X,\xi_j)\xi_i) \\ &\quad + g(X,2Y - \sum_{i=1}^{s}\{\eta^i(Y)+\eta^j(Y)\}\xi_i - Y + \sum_{i=1}^{s} g(Y,\xi_j)\xi_i) \\ &= 0. \end{aligned}$$

So $\eta^j([X,Y]) = 0$ for $j = 1,2,...,s$. Then, we have $[X,Y] \in \Gamma(D)$.

**Theorem 11.** *Let $M$ be a semi-invariant submanifold of a generalized Kenmotsu manifold $\tilde{M}$ with the semi-symmetric metric connection. The distribution $D \oplus sp\{\xi_1,...,\xi_s\}$ is integrable if and only if*

$$h(X,\varphi Y) = h(\varphi X,Y)$$

*for all $X,Y \in \Gamma(D \oplus sp\{\xi_1,...,\xi_s\})$ is satisfied.*

*Proof.* Using (6) and (9), then

$$\begin{aligned} \varphi([X,Y]) &= \varphi(\nabla^*_X Y - \nabla^*_Y X) \\ &= \varphi(\widetilde{\nabla}_X Y - h(X,Y) - \widetilde{\nabla}_Y X + h(Y,X)) \\ &= \varphi(\overline{\nabla}_X Y - \sum_{i=1}^{s}\{\eta^i(Y)X - g(X,Y)\xi_i\} - \overline{\nabla}_Y X + \sum_{i=1}^{s}\{\eta^i(X)Y - g(Y,X)\xi_i\}) \\ &= \overline{\nabla}_X\varphi Y - (\overline{\nabla}_X\varphi)Y - \sum_{i=1}^{s}\eta^i(Y)\varphi X - \overline{\nabla}_Y\varphi X + (\overline{\nabla}_Y\varphi)X + \sum_{i=1}^{s}\eta^i(X)\varphi Y. \end{aligned}$$

for all $X, Y \in \Gamma(D)$. For (10) and (14), we have

$$\varphi([X,Y]) = \nabla_X \varphi Y - \nabla_Y \varphi X + \sum_{i=1}^{s} \{4g(X,\varphi Y)\xi_i + \eta^i(Y)\varphi X - \eta^i(X)\varphi Y\} + h(X,\varphi Y) - h(\varphi X, Y).$$

Then, we have $[X,Y] \in \Gamma(D \oplus sp\{\xi_1, ..., \xi_s\})$ if and only if $h(X,\varphi Y) = h(\varphi X, Y)$, where $\varphi([X,Y])$ shows the component of $\nabla_X Y$ from the ortohogonal complementary distribution of $D \oplus Sp\{\xi_{1,...,}\xi_s\}$ in $M$. Then, we have $[X,Y] \in \Gamma(D \oplus sp\{\xi_1, ..., \xi_s\})$ if and only if $h(X,\varphi Y) = h(Y, \varphi X)$.

**Theorem 12.** *Let $M$ be a semi-invariant submanifold of a generalized Kenmotsu manifold $\tilde{M}$ with the semi-symmetric metric connection.*

*The distribution $D^\perp \oplus sp\{\xi_1, ..., \xi_s\}$ is integrable if and only if*

$$A_{\varphi X} Y = A_{\varphi Y} X$$

*for all $X, Y \in \Gamma(D^\perp \oplus sp\{\xi_1, ..., \xi_s\})$ is satisfied.*

*Proof.* We have for all $X, Y \in \Gamma(D^\perp)$

$$\begin{aligned} g([X,Y],\xi_j) &= g(\widetilde{\nabla}_X Y, \xi_j) - g(\widetilde{\nabla}_Y X, \xi_j) \\ &= -g(Y, \widetilde{\nabla}_X \xi_j) + g(X, \widetilde{\nabla}_Y \xi_j). \end{aligned}$$

Using (9) and (11), we have

$$\begin{aligned} g([X,Y],\xi_i) &= -g(Y, 2X - \sum_{i=1}^{s}\{\eta^i(X) + \eta^j(X)\}\xi_i - X + \sum_{i=1}^{s} g(X,\xi_j)\xi_i) \\ &\quad + g(X, 2Y - \sum_{i=1}^{s}\{\eta^i(Y) + \eta^j(Y)\}\xi_i - Y + \sum_{i=1}^{s} g(Y,\xi_j)\xi_i) \\ &= 0. \end{aligned}$$

Using (6) and (9), then

$$\begin{aligned} \varphi([X,Y]) &= \varphi(\nabla^*_X Y - \nabla^*_Y X) \\ &= \overline{\nabla}_X \varphi Y - (\overline{\nabla}_X \varphi) Y - \sum_{i=1}^{s} \eta^i(Y)\varphi X - \overline{\nabla}_Y \varphi X + (\overline{\nabla}_Y \varphi) X + \sum_{i=1}^{s} \eta^i(X)\varphi Y. \end{aligned}$$

For (10) and (15), we have

$$\begin{aligned} \varphi([X,Y]) &= (-A_{\varphi Y} + a)X + \nabla^\perp_X \varphi Y - 2\sum_{i=1}^{s}\{g(\varphi X, Y)\xi_i - \eta^i(Y)\varphi X\} - \sum_{i=1}^{s} \eta^i(Y)\varphi X \\ &\quad -(-A_{\varphi X} + a)Y - \nabla^\perp_Y \varphi X + 2\sum_{i=1}^{s}\{g(\varphi Y, X)\xi_i - \eta^i(X)\varphi Y\} + \sum_{i=1}^{s} \eta^i(X)\varphi Y \\ &= A_{\varphi X} Y - A_{\varphi Y} X + \nabla^\perp_X \varphi Y - \nabla^\perp_Y \varphi X + \sum_{i=1}^{s}\{4g(X,\varphi Y)\xi_i + \eta^i(Y)\varphi X - \eta^i(X)\varphi Y\}. \end{aligned}$$

Then we obtain,

$$[X,Y] \in \Gamma(D^{\perp} \oplus Sp\{\xi_1, ..., \xi_s\}) \Rightarrow A_{\varphi X}Y = A_{\varphi Y}X.$$

Conversely

$$\varphi^2([X,Y]) = \varphi(A_{\varphi X}Y - A_{\varphi Y}X + \nabla^{\perp}_X \varphi Y - \nabla^{\perp}_Y \varphi X + \sum_{i=1}^{s}\{4g(X,\varphi Y)\xi_i + \eta^i(Y)\varphi X - \eta^i(X)\varphi Y\})$$

$$[X,Y] = \sum_{i=1}^{s}\{-\eta^i(Y)X + \eta^i(X)Y + \sum_{k=1}^{s}(\eta^i(Y)\eta^k(X)\xi_k - \eta^i(X)\eta^k(X)\xi_k)\} + \varphi(\nabla^{\perp}_X \varphi Y) - \varphi(\nabla^{\perp}_Y \varphi X)$$

then, we have $[X,Y] \in \Gamma(D^{\perp} \oplus Sp\{\xi_1, ..., \xi_s\})$.

## References

[1] D.E. Blair, *Geometry of manifolds with structural group U(n)X O(s)*, J. Differ. Geom.,4,(1970),155-167.

[2] S. Goldberg and K. Yano, Globally framed f-Manifolds, Illinois J. Math.,15,(1971),456-474.

[3] K. Kenmotsu, *A class of almost contact Riemannian manifolds*, Tohoku Math. J., 24, (1972), 93-103.

[4] M. Kobayashi, *Semi-invariant submanifolds of a certain class of almost contact manifolds*, Tensor N. S.,43,(1986)28-36.

[5] B. B. Sinha, A. K. Srivastava, *Semi-invariant submanifolds of a Kenmotsu manifold with constant $\phi$-holomorphic sectional curvature*, Indian J. pure appl. Math., 23(1992),783-789.

[6] A. Turgut Vanli, R. Sari, *Generalized Kenmotsu manifolds*, arXiv:1406.1032v1.

[7] J. Vanzura, *Almost r-contact structures*, Ann. Scuola Norm. Sup. Pisa Sci. Fis. Mat. 26 (1972), 97–115.

[8] K. Yano, *On a structure f satisfying $f^3 + f = 0$*, Technical report No.12, Univ. of Washington, 1961.

[9] K. Yano, *On semi-symmetric metric connections*, Rev. Roumaine Math. Pures Appl. 15(1970),1579-1586.

Aysel Turgut Vanli
Department of Mathematics, Faculty of Arts and Sciences,
Gazi University, 06500 Ankara, Turkey
email: *avanli@gazi.edu.tr*

Ramazan Sari
Department of Mathematics, Faculty of Arts and Sciences,
Gazi University, 06500 Ankara, Turkey
email: *ramazansr@gmail.com*