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SOME RESULTS ON LIMITED OPERATORS

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Abstract. We investigate conditions under which each limited operator is L-weakly compact (resp. M-weakly compact) and the converse.

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1. Introduction

Throughout this paper X, Y will denote Banach spaces, and E, F will denote Banach lattices. B_X is the closed ball of X.

In [3] the authors studied the L-weak (resp. M-weak) compactness of semi-compact operators. They proved that if E and F are nonzero Banach lattices, then each semi-compact operator $T : E \longrightarrow F$ is L-weakly compact if, and only if, the norm of F is order continuous [[3], Theorem 1]. Also, if F is Dedekind $\sigma-$complete, then each positive semi-compact operator $T : E \longrightarrow F$ is M-weakly compact if, and only if, the norms of E' and F are order continuous or E is finite dimensional [[3], Theorem 2]. Our objective in this paper is to continue the investigation of Banach lattices on which each limited operator is L-weakly compact (resp. M-weakly compact) and the converse.

The article is organized as follows, after the introduction, we give notations, definitions and what we will need from the Banach lattice theory in a preliminary section. In section 3, we start with a characterization for limited (compact) operators being L-weakly compact (Theorem 1). Also, we give necessary conditions under which each L-weakly compact operator is limited (Theorem 3). Finally, in section 4 we characterize Banach lattices on which each positive limited operators is M-weakly compact (Theorem 6), and we give some sufficient conditions for which the class of regular limited operators coincides with that of L-weakly compact operators and M-weakly compact (Corollary 8).

2. Preliminaries

A bounded subset A of X is called limited if, for every weak* null sequence (x'_n) in the dual space X', we have $x'_n(x) \longrightarrow 0$ uniformly for x in A. Based on this concept, the class of limited operators, first appeared in 1984 in connection with studying problems of the strictly cosingular operators (see [2]). We recall that an operator $T : X \longrightarrow Y$ is said to be limited if $T(B_X)$ is a limited subset of Y. Alternatively, the operator T is limited if, and only if, $\|T'(f_n)\| \longrightarrow 0$ for every weak* null sequence $(f_n) \subset Y'$.

It is well known that each compact operator is limited but there exists a limited operator which is not compact. Indeed, the canonical injection $i : c_0 \hookrightarrow \ell^\infty$ is limited (see [1, Theorem 4.67]) but fails to be compact. On the other hand, an operator T from E into X is M-weakly compact if, $\|T(x_n)\| \longrightarrow 0$ holds for every norm bounded disjoint sequence (x_n) in E. An operator T from X into E is called L-weakly compact if, $\|y_n\| \longrightarrow 0$ holds for every disjoint sequence (y_n) in the solid hull of $T(B_X)$. That L-weakly compact and M-weakly compact operators are weakly compact operators was shown by P. Meyer-Nieberg [5, Proposition 3.6.12].

To state our results, we need to fix some notations and recall some definitions. A Banach lattice is a Banach space $(E, \|.\|)$ such that E is a vector lattice and its norm satisfies the following property: for each $x, y \in E$ such that $|x| \leq |y|$, we have $\|x\| \leq \|y\|$. A norm $\|\cdot\|$ of a Banach lattice E is order continuous if for each generalized sequence (x_α) such that $x_\alpha \downarrow 0$ in E, (x_α) converges to 0 for the norm $\|\cdot\|$ where the notation $x_\alpha \downarrow 0$ means that (x_α) is decreasing, its infimum exists and $\inf(x_\alpha) = 0$. For the element x in a Riesz space E, if the order ideal generated by x coincides with the vector space generated by x then x is called a discrete element of E. The Riesz space E is called discrete if all discrete elements of E are order dense. For instance, the spaces c, c_0 and ℓ^p $(1 \leq p \leq \infty)$ are discrete Riesz spaces but the spaces $(\ell^\infty)'$ and $L^2[0, 1]$ are not discrete. If all limited sets in Banach space X are relatively compact, then X is said to be a Gelfand-Phillips space (has GP-property). For example, the classical Banach spaces c_0 and ℓ^1 have the GP-property and every separable Banach space, every Schur space (i.e., weak and norm convergence of sequences in X are coincide), and spaces containing no copy of ℓ^1, such as reflexive spaces, have the same property [2].

Note that if E is a Banach lattice, its topological dual E', endowed with the dual norm and the dual order, is also a Banach lattice. Also, a vector lattice E is Dedekind σ-complete if every majorized countable nonempty subset of E has a supremum. The lattice operations in E (resp. in E') are called weak (resp. weak*) sequentially continuous if the sequence $(|x_n|)$ (resp. $(|f_n|)$) converges to 0 in the weak (resp. weak*) topology, whenever the sequence (x_n) (resp. (f_n)) converges weakly (resp. weak*) to 0 in E (resp. in E'). A subset A of a Riesz space is called solid whenever $|x| \leq |y|$ and $y \in A$ imply $x \in A$. The solid hull of a set A is the smallest solid set including A and is exactly the set

$$Sol(A) := \{x \in E : \exists y \in A \text{ } with \text{ } |x| \leq |y|\}.$$

We will use the term operator $T : E \longrightarrow F$ to mean a bounded linear mapping. It is positive if $T(x) \geq 0$ in F whenever $x \geq 0$ in E. Note that each positive linear mapping on a Banach lattice is continuous. If an operator $T : E \longrightarrow F$ is positive then, its adjoint $T' : F' \longrightarrow E'$ is likewise positive, where T' is defined by $T'(f)(x) = f(T(x))$ for each $f \in F'$ and for each $x \in E$.

Note that

$$E^a = \{x \in E : every \text{ } monoton \text{ } sequence \text{ } in \text{ } [0, |x|] \text{ } is \text{ } convergent\}$$

is the maximal closed order ideal in E on which the induced norm is order continuous. The following facts are basic and will be used in the rest of this paper very often (see [5], p. 212).

1. Every L-weakly compact subset A of a Banach lattice E is contained in E^a.
2. Every relatively compact subset of E^a is L-weakly compact. In particular, if E has an order continuous norm then every relatively compact subset of E is L-weakly compact so that every compact operator from X into E is L-weakly compact.

We refer the reader to [[1], [5]] for unexplained terminologies on Banach lattice theory and positive operators

3. Limited and L-weakly compact operators

Note that there exists an operator which is compact (resp. limited) but not L-weakly compact (resp. M-weakly compact). In fact, the operator $T : \ell^1 \longrightarrow \ell^\infty$ defined by

$$T((\lambda_n)_n) = (\sum_{n=1}^{\infty} \lambda_n)(1, 1, \ldots)$$

for all $(\lambda_n) \in \ell^1$. It is clear that T is a compact (and hence a limited) operator but it is neither L-weakly compact nor M-weakly compact [[1], p. 322]. Conversely, there exists an operator which is L-weakly compact (resp. M-weakly compact) but not limited (see Remark 1).

The following result characterizes pairs of Banach lattices E, F for which every limited (resp. compact) operator $T : E \longrightarrow F$ is L-weakly compact.

Theorem 1. *The following assertions are equivalent:*

1. *each limited operator $T : E \longrightarrow F$ is L-weakly compact;*
2. *each compact operator $T : E \longrightarrow F$ is L-weakly compact;*
3. *one of the following conditions is valid:*
   (a) $E = \{0\}$;
   (b) *the norm of F is order continuous.*

Proof. (1) $\Longrightarrow$ (2) Obvious.

(2) $\Longrightarrow$ (3) Assume by way of contradiction that $E \neq \{0\}$ and the norm of F is not order continuous. To finish the proof, we have to construct a compact operator $T : E \longrightarrow F$ which is not L-weakly compact. Since $E \neq \{0\}$, it follows from [12, Theorem 39.3] that there exist $a \in E^+$ and $\psi \in (E')^+$ such that $\|\psi\| = 1$ and $\psi(a) = \|a\| = 1$.

On the other hand, since the norm of F is not order continuous, it follows from [5, Theorem 2.4.2] that there exists an order bounded disjoint sequence $(u_n) \subseteq F^+$ which is not norm convergent to zero. We can assume that there is $u \in F^+$ with $0 \leq u_n \leq u$ for all n. As $|u_n| = u_n \longrightarrow 0$ for $\sigma(E, E')$ (see Remark of ([1], page 192), it follows from [4, Corollary 2.6], that there exists a bounded disjoint sequence $(f_n) \subset (F')^+$ such that $f_n(u_n) \geq \varepsilon$ for every n ($\varepsilon > 0$ fixed). Define the operator $T : E \longrightarrow F$ by $T(x) = \psi(x)u$ for each $x \in E$, and note that T is compact (because its rank is one). But is not L-weakly compact. In fact, since (f_n) is a bounded positive disjoint sequence in F' and

$$\|T'(f_n)\| = \|f_n(u)\psi\| \geq \|f_n(u_n)\psi\| \geq \varepsilon$$

for every n. This show that T' is not M-weakly compact. Hence by [5, Proposition 3.6.11] T is not L-weakly compact.

$(3; a) \Longrightarrow (1)$ Obvious.

$(3; b) \Longrightarrow (1)$ Let $T : E \longrightarrow F$ be a limited operator and let (f_n) be a disjoint sequence in $B_{F'}$. As the norm of F is order continuous then, it follows from [5, corollary 2.4.3] that (f_n) is a weak* null sequence. Now, as T is limited, $\|T'(f_n)\| \longrightarrow 0$. Then T' is M-weakly compact and hence T is L-weakly compact.

As a consequence, we obtain a characterization of the order continuity of the norm of a Banach lattice.

Corollary 2. *The following statements are equivalent:*

1. *every limited operator T from E into E is L-weakly compact;*

2. *every compact operator T from E into E is L-weakly compact;*

3. *the norm of E is order continuous.*

It should be noted that a subset $A \subset X$ is limited if, and only if, $f_n(x_n) \to 0$ for every sequence (x_n) in A and every weak* null sequence (f_n) in X'.

Now, we give the converse of Theorem 1. In fact, we give necessary conditions under which each L-weakly compact operator is limited.

Theorem 3. *If each L-weakly compact operator T from E into F is limited, then one of the following assertions is valid:*

1. *the norm of E' is order continuous;*

2. *each order bounded subset of F^a is limited.*

Proof. It suffices to prove that if the norm of E' is not order continuous, then each order bounded subset of F^a is limited i.e., For each $y \in (F^a)^+$, $(y_n) \subseteq [-y, y]$ and weak* null sequence $(g_n) \subseteq (F^a)'$, we have $g_n(y_n) \longrightarrow 0$.

Since the norm of E' is not order continuous, there exists a positive order bounded disjoint sequence $(x'_n) \subset E'$ satisfying $\|x'_n\| = 1$ for all n ([5, Theorem 2.4.2]). Let $x' \in (E')^+$ such that $0 \leq x'_n \leq x'$ for all n.

Now, consider the operators:

$$P : E \to \ell^1, x \longmapsto P(x) = (x'_n(x))_{n=1}^{\infty} \ ,$$

$$S : \ell^1 \to F, (\lambda_n)_{n=1}^{\infty} \longmapsto \sum_{n=1}^{\infty} \lambda_n y_n \ .$$

Since $\sum_{n=1}^{\infty} |x'_n(x)| \leq \sum_{n=1}^{\infty} x'_n|x| \leq x'|x|$ for each $x \in E$, the operator P take values in ℓ^1.

Let $T = S \circ P : E \longrightarrow \ell^1 \longrightarrow F$ such that $T(x) = \sum_{n=1}^{\infty} x'_n(x) y_n$ for each $x \in E$. Note that for all $x \in B_E$, we have

$$|T(x)| \leq \sum_{n=1}^{\infty} |x'_n(x)||y_n| \leq (\sum_{n=1}^{\infty} x'_n)|x|y \leq x'|x|y \leq \|x'\|y.$$

So that $T(B_E) \subseteq \|x'\|[-y, y]$, then T is L-weakly compact (because $y \in (F^a)^+$) and hence by our hypothesis T is limited. As (g_n) is a weak* null sequence in F', we have for every n

$$|T'(g_n)| = \sum_{i=1}^{\infty} |g_n(y_i)|x'_i \geq |g_n(y_n)|x'_n \geq 0.$$

Thus,

$$|g_n(y_n)| = |g_n(y_n)|\|x'_n\| \leq \|T'(g_n)\| \longrightarrow 0.$$

This ends the proof of the Theorem.

In the remark we will need the following Lemma,

Lemma 4. *Let A be a norm bounded subset of X. If for each $\varepsilon > 0$ there exists some limited subset A_ε in X such that $A \subset A_\varepsilon + \varepsilon B_X$ then, A is limited in X.*

Proof. Let $S : X \longrightarrow c_0$ be an operator and $\varepsilon > 0$, then by our hypothesis there exists a limited subset A_ε in X such that $A \subset A_\varepsilon + \varepsilon B_X$ hence, $S(A) \subset S(A_\varepsilon) + \varepsilon\|S\|B_{c_0}$. As A_ε is a limited set then, $S(A_\varepsilon)$ is relatively compact in c_0 [7, Theorem 2.3.] and hence by [1, Theorem 3.1], $S(A)$ is relatively compact in c_0. This shows by [7, Theorem 2.3.] that A is a limited set in X.

Remark 1.

1. *The first necessary condition of Theorem 3 is not sufficient. Indeed, let $E = \ell^\infty$ and $F = L^2[0,1]$. Since E' and F are not discrete, it follows from [8, Theorem 1] that there exist two operators $S, T : E \longrightarrow F$ such that $0 \leq S \leq T$ with T is compact (and hence is L-weakly compact because the norm of F is order continuous) and S is not compact (hence is not limited because F has the GP-property). Since the class of L-weakly compact operators satisfies the domination problem [1, Exercise 2 in Section 5.3], then S is L-weakly compact. We conclude that there exists a positive operator $S : E \longrightarrow F$ which is L-weakly compact (resp. M-weakly compact because E' and F have order continuous norms [5, Theorem 3.6.17]) but it is not limited. Although, the norm of E' is order continuous.*

2. *The second necessary condition of Theorem 3 is sufficient. In fact, if $T : E \longrightarrow F$ is L-weakly, then by [5, Proposition 3.6.2] for each $\varepsilon > 0$ there exists $x \in F^a$ such that $T(B_E) \subseteq [-y, y] + \varepsilon B_{F^a}$. Since each order bounded subset of F^a is limited it follows from Lemma 4 that T is limited.*

As consequence of Theorem 3 and Remark 1 we have the following characterization,

Corollary 5. *The following assertions are equivalent:*

1. *each L-weakly operator T from ℓ^1 into E is limited.*

2. *each order bounded subset of E^a is limited.*

4. Limited and M-weakly compact operators

In the following result, we characterize Banach lattice on which each positive limited operator is M-weakly compact.

Theorem 6. *Let F be a Dedekind $\sigma-$complete and F' has weak* sequentially continuous lattice operations. Then the following assertions are equivalent:*

1. *each positive limited operator $T : E \longrightarrow F$ is M-weakly compact;*
2. *one of the following conditions is valid:*
   - (a) *E is finite dimensional;*
   - (b) *$F = \{0\}$;*
   - (c) *The norms of E' and F are order continuous.*

Proof. (1) $\Longrightarrow$ (2) It suffices to establish the following two separate claims.

(a) if the norm on F is not order continuous then E is infinite dimensional,

(b) if the norm on E' is not order continuous then $F = \{0\}$.

Indeed, Assume that E is infinite dimensional and that the norm of F is not order continuous. By [10, Proposition 0.2.11] there exists a positive disjoint sequence (x_n) of E such that $\|x_n\| \geq \varepsilon$. As $\|x_n\| = \sup\{f(x_n) : f \in (B_{E'})^+\}$, for each n there exists $f_n \in (B_{E'})^+$ such that $f_n(x_n) \geq \varepsilon$. Applying [10, Proposition 0.3.11] and its proof, we find a positive disjoint sequence (g_n) of E' such that $g_n \leq f_n$, $g_n(x_n) = f_n(x_n)$ for all n and $g_n(x_m) = 0$ for $n \neq m$. Consider the positive operator $P : E \longrightarrow \ell^\infty$ defined by $P(x) = (g_n(x))_n$ and note that $P(B_E) \subseteq B_{\ell^\infty}$. On the other hand, since the norm of F is not order continuous, it follows from [5, Theorem 2.4.2] that there exists an order bounded disjoint sequence $(y_n) \subseteq F^+$ which is not norm convergent to zero. We can assume that there is $y \in F^+$ with $0 \leq y_n \leq y$ for all n. It follows from the proof of [11, Theorem 117.3] that the operator

$$S : \ell^\infty \longrightarrow F, (\lambda_k) \longmapsto \sum_{k=1}^{\infty} \lambda_k y_k$$

defines a positive operator from ℓ^∞ into F where the convergence is in the sense of the order.

Now, we consider the operator $T = S \circ P : E \longrightarrow F$ defined by

$$T(x) = \sum_{k=1}^{\infty} g_k(x) y_k \quad for\ each \quad x \in E$$

is well defined and limited. Indeed, note that for all $(\lambda_k) \in B_{\ell^\infty}$ and from the disjointness of the sequence (y_n), we have

$$|S((\lambda_k))| \leq \sum_{k=1}^{\infty} |\lambda_k||y_k| \leq (\sup_k |\lambda_k|) \sum_{k=1}^{\infty} |y_k| \leq y.$$

Since $T(B_E) = S(P(B_E)) \subseteq [-y, y]$ and the order interval $[-y, y]$ is limited ([6, Proposition 3.1]), then $T(B_E)$ is limited so that T is limited. But T is not M-weakly compact. To see this, note that $\|T(x_n)\| = \|S \circ P(x_n)\| = \|S(e_n)\| = \|y_n\| \nrightarrow 0$.

To prove claim (b), we suppose that the norm on E' is not order continuous and $F \neq \{0\}$. By [11, Theorem 116.1] there is a norm bounded disjoint sequence (u_n) of positive elements in E which does not converge weakly to zero. Hence, we may assume that $\|u_n\| \leq 1$ for all n and also that for some $0 \leq \phi \in E'$ and some $\epsilon > 0$ we have $\phi(u_n) > \epsilon$ for all n. Then it follows from [11, Theorem 116.3] that the components ϕ_n of ϕ in the carriers C_{u_n} form an order bounded disjoint sequence in $(E')^+$ such that

$$\phi_n(u_n) = \phi(u_n) \text{ for all } n \text{ and } \phi_n(u_m) = 0 \text{ if } n \neq m.$$

Now, we define a positive operator $P : E \longrightarrow \ell^1$ by

$$P(x) = (\tfrac{\phi_n(x)}{\phi(u_n)})_{n=1}^{\infty} \text{ for all } x \in E.$$

Since $\sum_{n=1}^{\infty} |\frac{\phi_n(x)}{\phi(u_n)}| \leq \frac{1}{\epsilon} \sum_{n=1}^{\infty} \phi_n(|x|) \leq \frac{1}{\epsilon} |\phi(|x|)|$ holds for each $x \in E$, the operator P is well defined. On the other hand, as $F \neq \{0\}$, there exists a non-null element $u \in F^+$. We consider the positive operator defined by $S : \ell^1 \longrightarrow E$ defined by

$$S((\lambda_n)) = (\sum_{n=1}^{\infty} \lambda_n) u \text{ for all } (\lambda_n) \in \ell^1.$$

Now, we consider the composed operator $T = S \circ P : E \longrightarrow \ell^1 \longrightarrow E$ defined by

$$T(x) = (\sum_{n=1}^{\infty} \tfrac{\phi_n(x)}{\phi(u_n)}) u \text{ for all } x \in E.$$

Note that T is compact (and hence is limited). Since (u_n) is a disjoint sequence in B_E and $\|T(u_n)\| = \|S \circ P(u_n)\| = \|S(e_n)\| = \|u\|$ for all n, it follows that T is not M-weakly compact, and this gives a contradiction with our hypothesis (1).

$(2; a) \Longrightarrow (1)$ and $(2; b) \Longrightarrow (1)$ are obvious.

$(2; c) \Longrightarrow (1)$ It follow from $(2; b) \Longrightarrow (1)$ of Theorem 1 and [5, Theorem 3.6.17].

Remark 2. *The conditions "F is Dedekind σ-complete" and "F' has weak* sequentially continuous lattice operations " are not an accessory in the above theorem. Indeed, each operator $T : \ell^\infty \longrightarrow c$ is weakly compact (see the proof of [9, Proposition 1]). Since ℓ^∞ is an AM-space, T is M-weakly compact [1, Theorem 5.62]. Yet none of the three possible conditions listed holds.*

In [6], the authors considered a weak version of the class of limited operators, so called order limited operators. Recall that an operator T from E into X is said to be order limited, if T carries each order bounded subset in E to a limited one in X, equivalently, $|T'(f_n)| \longrightarrow 0$ for $\sigma(E', E)$ for each sequence $(f_n) \subset X'$ such that $f_n \longrightarrow 0$ for $\sigma(X', X)$ [6, Theorem 3.3 (3)].

Note that, there exists an order limited operator which is not limited. Indeed, the identity operator of the Banach lattice l^1 is order limited, but is not limited. In our last major result, we show that each operator between Banach lattices is limited whenever it is both order limited and M-weakly compact.

Theorem 7. *Each operator $T : E \longrightarrow F$ is limited whenever it is both order limited and M-weakly compact.*

Proof. Consider an operator $T : E \longrightarrow F$ which is order limited and M-weakly compact.

Let $(f_n) \subset F'$ be a weak* null sequence. We shall show that$\|T'(f_n)\| \longrightarrow 0$. By [4, Corollary 2.7], it suffices to prove that $|T'(f_n)| \longrightarrow 0$ for $\sigma(E', E)$ and $(T'(f_n))(x_n) \longrightarrow 0$ for every disjoint and norm bounded sequence $(x_n) \subset E^+$. Indeed:

- as T is order limited then, $|T'(f_n)| \longrightarrow 0$ for $\sigma(E', E)$.

- as $f_n \longrightarrow 0$ for $\sigma(F', F)$, (f_n) is norm bounded Hence and since T is M-weakly, we obtain $|T'(f_n)(x_n)| = |f_n(T(x_n))| \leq \|f_n\|\|T(x_n)\| \longrightarrow 0$. This complete the proof.

An operator $T : E \longrightarrow F$ is regular if $T = T_1 - T_2$ where T_1 and T_2 are positive operators from E into F.

As a consequence of Theorem 7, we give some sufficient conditions under which, the class of L-weakly compact, M-weakly compact and limited operators to coincide.

Corollary 8. *If E' and F have order continuous norms. Then, for each regular order limited operator $T : E \longrightarrow F$ the following statements are equivalent:*

1. *T is L-weakly compact;*
2. *T is M-weakly compact;*
3. *T is limited.*

Proof. (1) $\Longrightarrow$ (2) Follows from [5, Theorem 3.6.17].
(2) $\Longrightarrow$ (3) Follows from Theorem 7.
(3) $\Longrightarrow$ (1) Follows from Theorem 1.

Note that if E' has weak* sequentially continuous lattice operations, then every operator T from E into an arbitrary Banach space is order limited. The following consequence of Theorem 7 gives a sufficient condition under which each M-weakly compact operator is limited.

Corollary 9. *If E' has weak* sequentially continuous lattice operations. Then, each M-weakly compact operator from E into F is limited.*

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