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SINC AND THE NUMERICAL SOLUTION OF VOLTERRA-FREDHOLM INTEGRO-DIFFERENTIAL EQUATIONS

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Abstract. A numerical method based on sinc approximation is developed to solve linear and nonlinear Volterra-Fredholm integro-differential equations. Sinc approximation are typified by errors of the form $\mathrm{O}\left(e^{-k/h}\right)$, where $k > 0$ is a constant and h is a step size. The equations are reduced to systems of linear and nonlinear algebraic equations. Numerical examples are provided to illustrate the accuracy and computational efficiency of the method. The examples include convolution type, singular as well as singularly-perturbed problems.

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1. Introduction

Nonlinear phenomena, that appear in many applications in scientific fields, such as fluid dynamics, solid state physics, plasma physics, mathematical biology and chemical kinetics, can be modelled by partial differential equation, integral equations or by integro-differential equations (IDEs). For example, IDEs arise in the mathematical modelling of epidemic and various biological models[57]. They also model a lot of physical process such as nano-hydrodynamics [61], glass forming process [59], drop wise condensation [55], wind ripple in the desert [10] and studies of polymer chains [37].

The initial value problem for a nonlinear system of integro-differential equations was used to model the competition between tumor cells and the immune system [7]. In [1] a nonlinear problem involving fluid waves with an oceanographical application was reduced to a system of integro-differential equations. Nonlinear integro-differential equations are usually hard to solve analytically and exact solutions are scarce. Therefore, they are treated numerically or semi analytic numerical methods are used. Several numerical methods have been developed for solving integral and IDEs . Each of these methods has its inherent advantages and disadvantages.

The search for alternative, more general, easier and more accurate numerical methods is a continuous and ongoing process. The numerical solution of nonlinear integro-differential equations has two major aspects. First, the equations are discretized, generally by replacing them with a sequence of finite dimensional approximations following the discretization, the resulting finite dimensional problem may be solved by some type of iteration scheme such as Newton or quasi-Newton method. Existence theory for these equations was discussed in [46]. The upper and lower bounds of the solutions were studied in [9, 13, 15, 34]. Properties, Uniqueness and stability were discussed in [2, 31, 37].

The nonlinear Volterra-Fredholm integral equation was treated in [28] using homotopy perturbation method (HPM), in [11] using spline collocation, in [36, 63] using Taylor polynomials, in [30] using Trapezoidal Nystrom method and in [64] using Legendre wavelets. Existence and uniqueness of solutions to linear IDEs was discussed by Agarwal [3].

The solution of Linear Volterra-Fredholm integro-difference equation was presented in [62] using Taylor collocation. The decomposition method was used in [19] to treat high order linear Volterra-Fredholm IDEs. Recent paper by Maleknejad et al. [40] uses Berstein operational matrix to treat a system of high order linear V-F IDEs and includes other methods to solve these equations.

Solutions of nonlinear Volterra-IDEs up to 2010 were considered by the authors in [42]. A comparison between wavelet-Galerkin method and HPM was given in [28]. More recent contributions include the use of differential transform method [56]. For solutions of nonlinear Fredholm IDEs : Adomian's decomposition method [17, 51] was employed to treat coupled nonlinear system of Fredholm IDEs. In [18] the method was applied to a first order special nonlinear Fredholm IDEs in two variables. Optimal order spline methods were used in [22]. Neta [38] employed Galerkin's method to a special first order nonlinear Fredholm IDEs in two unknowns. In [21] orthogonal collocation was applied to a second order Fredholm IDE. Homotopy perturbation was employed in [25, 23]. Sine-cosine wavelet-Galerkin was used in [26] ,while Haar wavelet method was applied in [33].

The solution of both nonlinear Volterra and Fredholm IDEs was given in [52] using homotopy analysis method (HAM) while homotopy perturbation method (HPM) was used in [50] . The modified decomposition method was used in [60] and compared with the direct direct computation as well as the series solution method. Chebyshev polynomials were used in [14]. Use of HPM is given in [27]. An optimal control approach was used in [20].

The solution of nonlinear Volterra and Fredholm integro-differential equations was given in [4, 58] using homotopy analysis method (HAM), in [4] using modified Adomian decomposition ,in [12, 16, 40] using Taylor polynomials and in [21] using optimal order splines. The problem was treated in [6] using triangular function and its operational matrix and in [5] using operational matrix with block pulse functions (BPFs) to treat specific linear and nonlinear V-FIDEs. Orthogonal Legendre polynomials were used in [53] while orthogonal Chebyshev polynomials were used in [41].

In the present study, the basic ideas of previous works using sinc-collocation are developed and applied to linear and nonlinear Volterra-Fredholm integro-differential equations. Sinc approximation are typified by errors of the form $O\left(e^{-k/h}\right)$, where $k > 0$ is a constant and h is a step size. The equations have the form:

$$\sum_{i=0}^{n} \mu_i(x)\, u^{(i)}(x) = f(x) + \lambda_1 \int_a^x K_1(x,t)\, [u(t)]^{\nu}\, dt + \lambda_2 \int_a^b K_2(x,t)\, [u(t)]^{\sigma}\, dt, \quad x \in J = [a,b]$$

$$n\, u(a) = \gamma \qquad n\,(n-1)\, u(b) = \beta \tag{1}$$

where $K_1(x,t), K_2(x,t)$, $f(x)$, $u(x)$ and $\mu_i(x)$, $i = 0, 1, 2$, are analytic functions and λ_1 and λ_2 are parameters, and γ and β are real constant and $\nu \geq 1$, $\sigma \geq 1$. It will always be assumed that (1) possesses a unique solution $u \in C^n(J)$.

The organization of the paper is as follows. In Section 2, we describe the basic formulation in terms of sinc functions required for our subsequent development. In Section 3 we introduce the sinc-collocation method and show how the method is used to solve linear Volterra-Fredholm integro-differential equations with homogeneous boundary conditions. Section 4 is devoted to the solution of nonlinear Volterra-Fredholm integro-differential equations. Section 5 presents appropriate techniques to treat nonhomogeneous boundary conditions. The examples include convolution type, singular as well as singularly-perturbed problems in Section 6. The conclusion is given in Section 7.

2. Sinc function

Sinc methods have increasingly been recognized as powerful tools for attacking problems in applied physics and engineering. Previous treatments of integral and integro-differential equations using the sinc approximation include: Sinc-Nystrom method for numerical solution of one- dimensional, Cauchy singular integral equations on a smooth arc in the complex plane [8]. Ref [47] used the sinc method to treat first order linear FIDE. Sinc-collocation for linear FIE of second kind [48]. Ref. [45] used single and double exponential sinc-collocation to treat Volterra and Fredholm integral equations. Mohsen and El-Gamel[42] used sinc-collocation for second order linear FIDE and for linear and nonlinear VIDE [44].

The sinc-collocation approach for solving Fredholm IDEs presented by Mohsen and El-Gamel [42] and extended by the same authors to linear and nonlinear Volterra IDEs [44] is used here. A general review of sinc function approximation is given in [35, 43, 54]. Hence, only properties important to the present goals are outlined in this section.
If $f(x)$ is defined on the real line, then for $h > 0$ the Whittaker cardinal expansion of f is given by:

$$f_m(x) = \sum_{k=-N}^{N} f_k\, S(k,h)(x), \quad m = 2\,N+1$$

where $f_k = f(x_k)$, $x_k = h\,k$, and the mesh size is given by

$$h = \sqrt{\frac{\pi\, d}{\alpha\, N}}, \qquad 0 < \alpha \leq 1, \qquad d \leq \frac{\pi}{2} \tag{2}$$

where N is suitably chosen and α depends on the asymptotic behavior of $f(x)$. The n-th derivative of the function f at the sampling points $x_k = k\,h$ can be approximated using a finite number of terms as:

$$f^{(n)}(x_k) \cong h^{-n} \sum_{k=-N}^{N} \delta_{j\,k}^{(n)}\, f_k$$

where

$$\delta_{j\,k}^{(n)} = \frac{d^n}{d\,x^n}\, S(j,h)(x)|_{x=x_k}$$

In particular,

$$\delta_{jk}^{(0)} = [S(j,h)(x)]\,|_{x=x_k} = \begin{cases} 1, & \mathrm{j} = k, \\ \\ 0, & \mathrm{j} \neq k, \end{cases} \tag{3}$$

$$\delta_{jk}^{(1)} = \frac{d}{d\,x}\,[S(j,h)\,(x)]\,|_{x=x_k} = \begin{cases} 0, & \mathrm{j} = k, \\ \\ \frac{(-1)^{k-j}}{k-j}, & \mathrm{j} \neq k, \end{cases} \tag{4}$$

and

$$\delta_{jk}^{(2)} = \frac{d^2}{d\,x^2}\,[S(j,h)\,(x)]\,|_{x=x_k} = \begin{cases} \frac{-\pi^2}{3}, & \mathrm{j} = k, \\ \\ \frac{-2(-1)^{k-j}}{(k-j)^2}, & \mathrm{j} \neq k. \end{cases} \tag{5}$$

We note that
$$\delta^{(0)}_{k\,j} = \delta^{(0)}_{j\,k}, \quad \delta^{(2)}_{k\,j} = \delta^{(2)}_{j\,k} \quad \text{and} \quad \delta^{(1)}_{k\,j} = -\delta^{(1)}_{j\,k}.$$
The interpolation formula for $f(x)$ over $[a,b]$ takes the form

$$f(x) \approx \sum_{k=-N}^{N} f_k \; S(k,h) \circ \phi(x), \tag{6}$$

where the basis functions on (a,b) are then given by

$$S(k,h) \circ \phi(x) = \operatorname{sinc}\left(\frac{\phi(x) - k\,h}{h}\right)$$

and the transformation function

$$\phi(x) = \ln\left(\frac{x-a}{b-x}\right) \tag{7}$$

transforms $[a,b]$ to the infinite range $[-\infty,\infty]$. The interpolation points $\{x_k\}$ are then given by:
$$x_k = \frac{a + b\,e^{kh}}{1 + e^{kh}}$$
The quadrature formula of $F(x)$ is given by

$$\int_a^b F(x)\,d\,x \approx h \sum_{k=-N}^{N} \frac{F(x_k)}{\phi'(x_k)}, \tag{8}$$

Corollary 1. *[54] Let N be a positive integer, let $\delta^{(-1)}_{k}$ be defined as*

$$\delta^{(-1)}_{k\,j} = \frac{1}{2} + \int_0^{k-j} \frac{\sin(\pi\,t)}{\pi\,t}\,d\,t. \tag{9}$$

and let h be defined as (2). Then there exists a constant C, which is independent of N, such that

$$\left| \int_a^x F(t)\,d\,t - h \sum_{j=-N}^{N} \delta^{(-1)}_{k\,j} \frac{F(x_j)}{\phi'(x_j)} \right| \le C\,e^{-(\pi\,d\,\alpha\,N)^{1/2}}. \tag{10}$$

then define $\boldsymbol{I}^{(-1)}$ by the Toeplitz matrix [29]

$$\boldsymbol{I}^{(-1)} = \left[\delta^{(-1)}_{k\,j}\right], \qquad k,j = -N,\ldots,N.$$

3. Linear Volterra-Fredholm Integro-differential Equations

For case $\nu = 1$, and $\sigma = 1$, we assume that $u(x)$, the solution of (1), is approximated by the finite expansion of sinc basis functions

$$u_m(x) = \sum_{j=-N}^{N} u_j \, S(j,h) \circ \phi(x), \qquad m = 2\,N + 1. \tag{11}$$

Application of (8) to the first kernel integral in (1) gives

$$\int_a^b K_1(x,t)\, u(t)\, d\,t \approx h \sum_{j=-N}^{N} \frac{K_1(x,t_j)}{\phi'(t_j)}\, u_j, \tag{12}$$

Application of (10) to the second kernel integral in (1) gives

$$\int_a^x K_2(x,t)\, u(t)\, d\,t \approx h \sum_{j=-N}^{N} \delta_{k\,j}^{(-1)} \frac{K_2(x,x_j)}{\phi'(x_j)}\, u_j, \tag{13}$$

where u_j denotes an approximate value of $u(x_j)$. If we replace the integration terms on the right-hand side of (1) with the right-hand side of (12) and (13) we have

$$\sum_{j=-N}^{N} \left[\sum_{i=0}^{n} \mu_i(x) \frac{d^i}{d\,x^i} S(j,h) \circ \phi(x) - h\,\lambda_1 \frac{K_1(x,t_j)}{\phi'(t_j)} - \lambda_2\, h\, \delta_{k\,j}^{(-1)} \frac{K_2(x,x_j)}{\phi'(x_j)} \right] u_j = f(x). \tag{14}$$

Setting

$$\frac{d^i}{d\phi^i}[S(j,h) \circ \phi(x)] = S_j^{(i)}(x), \qquad 0 \le i \le 2, \tag{15}$$

we note

$$\begin{aligned} \frac{d}{dx}[S(j,h) \circ \phi(x)] &= S_j^{(1)}(x)\, \phi'(x) \\ \frac{d^2}{dx^2}[S(j,h) \circ \phi(x)] &= S_j^{(2)}(x)\, \left[\phi'(x)\right]^2 + S_j^{(1)}(x)\, \phi''(x). \end{aligned} \tag{16}$$

Using (15) and (16) and substituting $x = x_k$ in (14) and applying the collocation to it, we eventually obtain the following theorem

Theorem 2. *If the assumed approximate solution of the problem (1) is (11), then the discrete sinc-collocation system for the determination of the unknown coefficients*

$\{u_j, -N \le j \le N\}$ *is given by*

$$\sum_{j=-N}^{N} \left[\sum_{i=0}^{n} g_i(x_k) \frac{(-1)^i \delta_{kj}^{(i)}}{h^i} - h\,\lambda_1 \frac{K_1(x_k,t_j)}{\phi'(t_j)} - h\,\lambda_2\; \delta_{kj}^{(-1)} \frac{K_2(x_k,x_j)}{\phi'(x_j)} \right] u_j = f_k,$$
$$k = -N, -N+1, \ldots, N \tag{17}$$

where for $n = 0, 1$ *and* 2 *we have*

$$g_0(x_k) = \mu_0(x_k), \qquad g_2(x_k) = \mu_2(x_k)\; \left[\phi'(x_k)\right]^2,$$
$$g_1(x_k) = \mu_1(x_k)\,\phi'(x_k) + \mu_2(x_k)\,\phi''(x_k).$$

To obtain a matrix representation of the equations in (17), recall the notation of Toeplitz matrices [29]. Let $\mathbf{D}(g(x_j))$ denote the $m \times m$ diagonal matrix with

$$\mathbf{D}(g(x))_{ij} = \begin{cases} g(x_i) & i = j, \\ 0 & i \neq j. \end{cases}$$

Let $\mathbf{u}$ be the m-vector with j-th component given by u_j , and $\mathbf{1}$ is an m-vector each of whose components is 1. In this notation the system in (17) takes the matrix form

$$\mathbf{A}\,\mathbf{u} = \Theta, \tag{18}$$

where

$$\Theta = \mathbf{D}\,(f)\;\mathbf{1},$$
$$\mathbf{u} = [u_{-N}, u_{-N+1}, \ldots, u_N]^{\tau},$$

and

$$\mathbf{A} = \sum_{i=0}^{n} \frac{1}{h^i}\,\mathbf{I}^{(i)}\,\mathbf{D}\,(g_i) - h\,\lambda_1 \frac{K_1(x_k,t_j)}{\phi'(t_j)} - h\,\lambda_2 \left(\mathbf{I}^{(-1)}\,\mathbf{D}\left(\frac{1}{\phi'(x_j)}\right)\right) \circ\; \mathbf{k},$$

where

$$\mathbf{k} = [K_2(x_k,x_j)],$$

The notation " $\circ$ " denotes the Hadamard matrix multiplication and

$$\mathbf{I}^{(i)} = \left[\delta_{kj}^{(i)}\right], \quad i = -1, 0, 1, 2, \qquad k, j = -N, \ldots, N.$$

Now we have a linear system of m equations for the m unknown coefficients, namely, $\{u_j\}_{j=-N}^{N}$. We can obtain the coefficient of the approximate solution by solving this linear system by Q-R method. The solution $\mathbf{u} = (u_{-N}, \ldots, u_N)^{\tau}$ gives the coefficients in the approximate sinc-collocation solution $u_m(x)$ of $u(x)$.

4. Non-Linear Volterra-Fredholm Integro-differential Equations

The main objective of this section is to find the numerical solutions of the nonlinear Volterra-Fredholm integro-differential equation (1), namely, for case $\nu > 1$ and $\sigma > 1$, we assume that $u(x)$, the solution of (1), is approximated by the finite expansion of sinc basis functions

$$u_m(x) = \sum_{j=-N}^{N} u_j \, S(j,h) \circ \phi(x), \qquad m = 2\,N + 1.$$

Application of equation (10) to the kernel integral in (1) gives the following lemma

Lemma 3. *The following relation holds*

$$\int_a^{x_k} K_2(x,t)\, u^{\nu}(t)\, d\,t \approx h \sum_{j=-N}^{N} \delta_{k\,j}^{(-1)} \frac{K_2(x_k, x_j)}{\phi'(x_j)}\, u_j^{\nu}, \tag{19}$$

and

$$\int_a^{b} K_1(x,t)\, u^{\sigma}(t)\, d\,t \approx h \sum_{j=-N}^{N} \frac{K_1(x, t_j)}{\phi'(t_j)}\, u_j^{\sigma}, \tag{20}$$

where u_j denotes an approximate value of $u(x_j)$.

If we replace the integration terms on the right-hand side of (1) with the right-hand side of (19) and (20) we have

$$\sum_{j=-N}^{N} \left[\sum_{i=0}^{n} \mu_i(x) \frac{d^i}{d\,x^i} S(j,h) \circ \phi(x) \right] u_j - h\,\lambda_1 \sum_{j=-N}^{N} \frac{K_1(x_x, t_j)}{\phi'(t_j)}\, u_j^{\sigma} \\ - h\,\lambda_2 \sum_{j=-N}^{N} \delta_{k\,j}^{(-1)} \frac{K_2(x_k, x_j)}{\phi'(x_j)}\, u_j^{\nu} = f(x). \tag{21}$$

Using (15) and (16) and substituting $x = x_k$ in (21) and applying the collocation to it, we eventually obtain the following theorem

Theorem 4. *If the assumed approximate solution of the problem (1) is (11), then the discrete sinc-collocation system for the determination of the unknown coefficients $\{u_j, -N \le j \le N\}$ is given by*

$$\sum_{j=-N}^{N} \left[\sum_{i=0}^{n} g_i(x_k) \frac{(-1)^i \delta_{k\,j}^{(i)}}{h^i} \right] u_j - h\,\lambda_1 \sum_{j=-N}^{N} \frac{K_1(x_x, t_j)}{\phi'(t_j)}\, u_j^{\sigma} \\ - h\,\lambda_2 \sum_{j=-N}^{N} \delta_{k\,j}^{(-1)} \frac{K_2(x_k, x_j)}{\phi'(x_j)}\, u_j^{\nu} = f_k, \qquad k = -N, -N+1, \ldots, N \tag{22}$$

Let $\mathbf{u}$ be the m-vector with j-th component given by u_j , and let $\mathbf{u}^\nu$ be the m-vectors with j-th component given by u_j^n and $\mathbf{1}$ is an m-vector each of whose components is 1. In this notation the system in (22) takes the matrix form

$$\mathbf{A}\,\mathbf{u} + \mathbf{B}\,\mathbf{u}^\sigma + \mathbf{C}\,\mathbf{u}^\nu = \Theta, \tag{23}$$

where

$$\Theta = \mathbf{D}\,(f)\;\mathbf{1},$$

$$\mathbf{u} = [u_{-N}, u_{-N+1}, \ldots, u_N]^\tau\,,$$

$$\mathbf{A} = \sum_{i=0}^{n} \frac{1}{h^i}\,\mathbf{I}^{(i)}\,\mathbf{D}\,(g_i)\,.$$

$$\mathbf{C} = -h\,\lambda\,\frac{K_1(x_k, t_j)}{\phi'(t_j)},$$

$$\mathbf{B} = -h\,\lambda\,\left(\mathbf{I}^{(-1)}\,\mathbf{D}\left(\frac{1}{\phi'(x_j)}\right)\right) \circ \mathbf{k},$$

$$\mathbf{k} = K_2(x_k, x_j),$$

and

$$\mathbf{I}^{(i)} = \left[\delta_{k\,j}^{(i)}\right], \quad \text{for} \quad i = -1, 0, 1, 2.$$

Now we have a nonlinear system of m equations for the m unknown coefficients, namely, $\{u_j\}_{j=-N}^{N}$. We can obtain the coefficients of the approximate solution by solving this nonlinear system using *Newton's method*.

Starting from an initial estimate $\mathbf{u}_0$, the corrections are made using

$$\mathbf{u}_{j+1} = \mathbf{u}_j + J^{-1}(\mathbf{u}_j)\left\{\Theta - \mathbf{A}\,\mathbf{u}_j - \mathbf{B}\,\mathbf{u}_j^\sigma - \mathbf{C}\,\mathbf{u}_j^\nu\right\}$$

$$\mathbf{J}(\mathbf{u}_j) = \mathbf{A} + \sigma\,\mathbf{B}\,\mathbf{u}_j^{\sigma-1} + \nu\,\mathbf{C}\,\mathbf{u}_j^{\nu-1}.$$

Here, $\mathbf{u}_j$ is the current iterate, and $\mathbf{u}_{j+1}$is the new iterate. A common numerical practice is to stop the Newton iteration whenever the distance between two iterates is less than a given tolerance, i.e when

$$\|\mathbf{u}_{j+1} - \mathbf{u}_j\| \le \epsilon,$$

where the Euclidean norm is used. The solution $\mathbf{u}$ gives the coefficients in the approximate sinc-collocation solution $u_m(x)$ of $u(x)$.

5. Treatment of Boundary Condition

In the previous section the development of the sinc-collocation technique for homogeneous boundary conditions provided a practical approach since, the sinc functions composed with the various conformal maps, $S(j,h)\circ\phi$, are zero at the endpoints of the interval. For $n=2$, if the boundary conditions are nonhomogeneous, then these conditions need be converted to homogeneous conditions via an interpolation by a known function . Using the transformation

$$y(x) = u(x) - \Lambda(x),$$

where $\Lambda(x)$ is the interpolating polynomial that satisfies $\Lambda(a)=\frac{\gamma}{2}$ and $\Lambda(b)=\frac{\beta}{2}$

$$\Lambda(x) = \frac{(b-x)}{2\,(b-a)}\,\gamma + \frac{(x-a)}{2\,(b-a)}\,\beta$$

to the problem (1) yields the differential equation

$$\begin{aligned}\sum_{i=0}^{2}\mu_i(x)\,y^{(i)}(x) &= \hat{f}(x) + \lambda_1\int_a^b \mathbf{K}_1(x,t)\left(\sum_{r=0}^{\sigma-1}\binom{\nu}{r}[y(t)]^{\sigma-r}\,[\Lambda(t)]^r\right)dt\\ &\quad + \lambda_2\int_a^x \mathbf{K}_2(x,t)\left(\sum_{r=0}^{\nu-1}\binom{\nu}{r}[y(t)]^{\nu-r}\,[\Lambda(t)]^r\right)dt \qquad x\in J=[a,b]\\ y(a)=0 \qquad & y(b)=0\end{aligned} \tag{24}$$

where

$$\begin{aligned}\hat{f}(x) = f(x) - \frac{\beta-\gamma}{2\,(b-a)}\,\mu_1(x) &- \left(\frac{(\beta-\gamma)\,x - a\,\beta + b\,\gamma}{2\,(b-a)}\right)\mu_0(x)\\ &+ \lambda_1\int_a^b \mathbf{K}_1(x,t)\left(\frac{(\beta-\gamma)\,t+\gamma\,b-a\,\beta}{2\,(b-a)}\right)[\Lambda(t)]^{\sigma}\,dt\\ &\quad + \lambda_2\int_a^x \mathbf{K}_2(x,t)\left(\frac{(\beta-\gamma)\,t+\gamma\,b-a\,\beta}{2\,(b-a)}\right)[\Lambda(t)]^{\nu}\,dt\end{aligned}$$

The resulting discrete system for the coefficients in the approximate sinc solution

$$y_m(x) = \sum_{j=-N}^{N} y_j\,S(j,h)\circ\phi(x) + \Lambda(x), \tag{25}$$

is exactly the system in (17), with f in that system replaced by $\hat{f}$. Notice that if $\gamma=\beta=0$, the problem reduces to the homogeneous case.

6. Numerical Examples

In this section, we report the numerical results of some examples solved by the sinc-collocation method described in this paper. We calculated with 32 digits of accuracy with MATLAB programming.

In the examples, the maximum absolute error at sinc points x_i is taken as

$$\|E_{sinc-col}\| = \max_{0\leq i\leq N} \left|u_{\text{exact}}\ (x_i) - u_{\text{sinc-collocation}}(x_i)\right|,$$

Example 1: [32] Let us first consider the linear Volterra-Fredholm integro-differential equation given by

$$u = f(x) + \int_0^x \cos(x-t)\, u(t)\, d\,t + \int_0^1 \sin(x+t)\, u(t)\, d\,t \qquad 0 \leq x,\, t \leq 1.$$

where

$$f(x) = \frac{1}{2}\left[e^x - e\,\sin(x-1) - e\,\cos(x-1)\right] - \cos(x),$$

subject to the boundary conditions

$$u(0) = 1, \qquad u(1) = e,$$

whose exact solution is

$$u(x) = e^x.$$

The maximum absolute error,$\|E_{sinc-col}\|$, is reported in **Table 5.1** as N increases from $N = 10$ to $N = 60$.

Table 5.1 Maximum absolute errors for Example 1

N	$\|E_{sinc-col}\|$
10	8.0966E-05
20	2.8788E-06
30	6.6664E-08
40	2.2772E-08
50	3.1328E-09
60	5.1405E-10

Maximum absolute error are tabulated in **Table 5.2** for sinc-collocation together with the analogous results of Laeli Dastjerdi and Maalek Ghaini [32].

Table 5.2 Maximum absolute error for Example 1

sinc-collocation	Results of [32]
5.1405E-10	8.0E-07

Example 2: Now we turn to a singular linear Volterra-Fredholm integro-differential equation of the convolution type

$$u''+\frac{1}{x}\,u'+\frac{1}{x^2}\,u = f(x)+\int_0^x \sin(x-t)\,u(t)\,d\,t+\int_0^1 \cos(x-t)\,u(t)\,d\,t \qquad 0 \le x,\, t \le 1.$$

where

$$f(x) = \frac{2}{x} - 7 - x + x^2 - \sin(x) + 3\,\cos(x) + \cos(x-1) + 2\,\sin(x-1),$$

subject to the boundary conditions

$$u(0) = 0, \qquad u(1) = 0,$$

whose exact solution is

$$u(x) = x - x^2.$$

The maximum absolute error, $\|E_{sinc-col}\|$, is reported in **Table 5.3** as N increases from $N = 10$ to $N = 40$.

Table 5.3 Maximum absolute errors for Example 2

N	$\|E_{sinc-col}\|$
5	0.002004
10	9.4886E-005
20	1.1318E-006
30	3.9761E-008
40	3.6348E-009

Example 3: Now we turn to a singularly perturbed Volterra-Fredholm integro-differential equation

$$\epsilon\,u'' + 2\,u' +\; u = f(x) + \int_0^x u(t)\,d\,t + \int_0^1 u(t)\,d\,t \qquad 0 \le x,\, t \le 1.$$

where

$$f(x) = e^{-\frac{x}{\epsilon}}\left(1+\epsilon-\frac{1}{\epsilon}\right) + \epsilon\,\left(e^{-\frac{1}{\epsilon}} - 2\right)$$

subject to the boundary conditions

$$u(0) = 1, \qquad u(1) = e^{-\frac{1}{\epsilon}},$$

whose exact solution is

$$u(x) = e^{-\frac{x}{\epsilon}}.$$

The maximum absolute error, $\|E_{sinc-col}\|$, is reported in **Table 5.4** for different ϵ as N increases from $N = 30$ to $N = 70$.

Table 5.4 $\|E_{sinc-col}\|$ for Example 3 for different ϵ

N	$\epsilon = 10^{-2}$	$\epsilon = 10^{-4}$	$\epsilon = 10^{-6}$
30	4.49440E-005	1.99630E-004	0.01972
40	7.53207E-006	1.88998E-005	0.00152
50	3.39109E-006	3.43889E-006	1.52588E-004
70	1.70045E-006	4.33292E-007	2.82403E-006

Example 4: Our last example is the nonlinear Volterra-Fredholm integral equation

$$u(x) = f(x) + \int_0^x u^2(t)\, dt + \int_0^1 u(t)\, dt \qquad 0 \leq x,\, t \leq 1.$$

where

$$f(x) = x - x^2 - \left(\frac{x^4}{4} - \frac{3}{5}\, x^5 + \frac{1}{2}\, x^6 - \frac{1}{7}\, x^7\right),$$

whose exact solution is

$$u(x) = x - x^2.$$

The maximum absolute error, $\|E_{sinc-col}\|$, is reported in **Table 5.5** as N increases from $N = 10$ to $N = 100$.

Table 5.5 Maximum absolute errors for Example 4

N	$\|E_{sinc-col}\|$
10	4.3820E-006
20	1.3169E-007
30	8.1748E-009
50	7.2002E-011
100	2.7117E-014

7. Discussion of Results and Conclusions

In this paper, we calculated the approximate solutions of the linear and nonlinear Volterra-Fredholm integro-differential equations by using sinc-collocation method.

Numerical examples including regular, singular, convolution type as well as singularly perturbed problems are presented. As expected, the accuracy increases as the number of terms N in the sinc expansion increases.

The sinc-collocation method is a simple method with high accuracy for solving a large variety of nonlinear Volterra-Fredholm integral and integro-differential problems. So it may be easily applied by researchers and engineers familiar with the sinc method.

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