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SUBCLASSES OF MEROMORPHICALLY MULTIVALENT FUNCTIONS

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Abstract. In this paper, we consider some properties such as growth and distortion theorem, coefficient problems, radii of convexity and starlikeness and convex linear combinations for certain subclass of meromorphic p-valent functions with positive coefficients.

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1. Introduction

Let $\mathcal{A}_p$ denoted the class of functions $f(z)$ normalized by:

$$f(z) = \frac{1}{z^p} + \sum_{n=0}^{\infty} a_n z^n \quad (p \in N := 1, 2, 3, ...), \tag{1}$$

which are analytic and p- valent in the punctured unit disk $\mathcal{D} = \{z : 0 < |z| < 1\}$.

The function $f(z)$ in $\mathcal{A}_p$ is said to be meromorphically starlike of order α if and only if

$$Re(\frac{-zf'(z)}{f(z)}) > \alpha \qquad (z \in \mathcal{D}, p \in N) \tag{2}$$

for some α ($0 \leq \alpha < 1$). We denote by $S_p^*(\alpha)$ to the class of all meromorphically starlike functions of order α.Similarly, a function $f(z)$ in $\mathcal{A}_p$ is said to be meromorphically convex of order α if and only if

$$Re(-1 - \frac{zf''(z)}{f'(z)}) > \alpha \qquad (z \in \mathcal{D}, p \in N) \tag{3}$$

for some α ($0 \leq \alpha < 1$). We denote by $C_p(\alpha)$ to the class of all meromorphically convex functions of order α.
The functions of the form (1)was considred by Liu and Srivastava [11], and Raina and Srivastava [14].
Let S_p denoted the subclass of A_p consisting of functions of the form:

$$f(z) = z^{-p} + \sum_{n=p}^{\infty} |a_n| \, z^n \tag{4}$$

as studied by Mogra[12] and Liu and Srivastava[11]. For function f in the class A_p, we define a linear operator I^n by

$$I^0 f(z) = f(z)$$

,

$$I^1 f(z) = z(\frac{1}{z^0} f(z))' + \frac{p+1}{z^p}$$
$$I^2 f(z) = z^2(\frac{1}{z}(I^1 f(z)))' + \frac{p+2}{z^p},$$

and generally

$$I^k f(z) = z^k[\frac{1}{z^{k-1}}(I^{k-1} f(z))]' + \frac{(p+k)}{z^p}$$

$$\begin{aligned} I^k f(z) &= I(I^{k-1} f(z)) = z^{-p} + \sum_{k=0}^{\infty} P(n.k) a_n z^n \quad (f \in A_p, k \in N). \\ where \quad P(n,k) &= \frac{n!}{(n-k)!}. \end{aligned} \tag{5}$$

Then it is easily verified that

$$z(I^n f(z))' = I^{n+1} f(z) + nI^n f(z) + \frac{n-p-1}{z^p}, \quad (f \in A_p, k \in N_0, p \in N). \tag{6}$$

The linear operator I^k was considered, when $p = 1$, by M.Albehbah and M. Darus [1]. Also note, by similar approach in getting the differential operator I^k , was studied extensively by Ghanim and Darus ([21],[22]) and [23],also they presented several results .
With the help of the differential operator I^k , we define the class $S_p{}^*(k, \alpha)$ as follows:

Definition 1. *The function $f(z) \in \mathcal{A}_p$ is said to be a member of the class $S_p{}^*(\alpha, k)$ if it satisfies the following inequality:*

$$\left|\frac{z(I^k f(z))'}{I^k f(z)} + p\right| \leq \left|\frac{z(I^k f(z))'}{I^k f(z)} + 2\alpha - p\right| \qquad (k \in \boldsymbol{N}_0 = \boldsymbol{N} \cup 0) \tag{7}$$

for some α $(0 \leq \alpha < 1)$ and for all $z \in U$.

It is easy to check that $S_p^*(0, \alpha)$ is the class of meromorphically starlike functions of order α and $S_p^*(0, 0)$ covers all classes of meromorphically starlike functions for all $z \in U$. Many important properties and characteristics of various interesting subclasses of the class $\mathcal{A}_p$ of meromorphically p-valent functions were investigated extensively by (among others) Aouf and Srivastava [3], Aouf and Hossen [2], Chen and Owa [4], Cho and Owa [5], Joshi and Srivastava [7], Kulkarni, Naik and Srivastava [8], Liu and Srivastava [9],[10], Mogra [12], Owa, Darwish and Aouf [13], Srivastava, Hossen and Aouf [16], Uralegaddi and Somanatha [18], [19], and Yang [20], (see also [17], [6]).

Let us write

$$S_p^*[k, \alpha] = S_p^*(k, \alpha) \cap S_p \tag{8}$$

where S_p is the class of functions of the form (4) that is analytic and $p-$ valent in U.

Next, we obtain the coefficient estimates for the classes $S_p^*[k, \alpha]$ and $S_p^*(k, \alpha)$.

2. Coefficient Estimates

Here we provide a sufficient condition for a function f analytic in U to be in $S_p^*(k, \alpha)$.

Theorem 1. *Let the function $f(z)$ be defined by (1). If*

$$\sum_{n=1}^{\infty} P(n, k)(n + \alpha)|a_n| \leq (p - \alpha) \qquad (k \in N_0) \tag{9}$$

where $(0 \leq \alpha < p)$, then $f(z) \in S_p^*(k, \alpha)$.

Proof. Suppose that (9) holds true $(0 \leq \alpha < p)$. Consider the expression

$$M(f, f') = \left|z(I^k f(z))' + pI^k f(z)\right| \leq \left|z(I^k f(z))' + (2\alpha - p)I^k f(z)\right|.$$

Then for $(0 < |z| = r < 1)$ we have

$M(f,f') = |\sum_{n=0}^{\infty} P(n,k)(n+p)a_n z^n| - \left| \frac{2(\alpha-p)}{z^p} + \sum_{n=0}^{\infty} P(n,k)(n+2\alpha-p)a_n z^n \right|.$

$$M(f,f') \leq \sum_{n=0}^{\infty} P(n,k)(n+p)|a_n|r^n - \frac{(2(p-\alpha)}{r^p}$$

$$-\sum_{n=1}^{\infty} P(n,k)(n+2\alpha-p)|a_n|r^{n+1}),$$

that is,

$$r^p M(f,f') \leq \sum_{n=1}^{\infty} P(n,k)(2)(n+\alpha)|a_n|r^n - (2(p-\alpha). \tag{10}$$

The inequality in(10)holds true for all $r(0 < r < 1)$. Therefore, by letting $r \to 1$ in(10), we obtain

$M(f,f') \leq \sum_{n=1}^{\infty}(2)P(n,k)(n+\alpha)|a_n| - (2(p-\alpha) \leq 0$

by the hypothesis in(9). Hence it follows that $\left|\frac{z(I^k f(z))'}{I^k f(z)} + p\right| < \left|\frac{z(I^k f(z))'}{I^k f(z)} + 2\alpha - p\right|$.

So that $f(z) \in S_p^*(k,\alpha)$. Hence the theorem is complete.

Corollary 2. *Let $k = \alpha = 0$ in Theorem 1, then we have*

$$\sum_{n=1}^{\infty} n|a_n| \leq p.$$

Corollary 3. *Set $k = 1$ and $\alpha = 0$ in Theorem 1, then we have*

$$\sum_{n=1}^{\infty} n^2|a_n| \leq p.$$

Next we will give a necessary and sufficient condition for a function $f \in S_p$ to be in the class $S_p^*[\alpha,k]$.

Theorem 4. *Let the function $f(z)$ be defined by (4) and let $f(z) \in S_p$. Then $f(z) \in S_p^*[\alpha,k]$ if and only if*

$$\sum_{n=p}^{\infty} P(n,k)(n+\alpha)|a_n| \leq (p-\alpha) \tag{11}$$

$(k \in N_0, n = p, p+1, p+2, ..., 0 \leq \alpha < 1)$.

Proof. In view of Theorem 1, it suffices to show that the 'only if' part. Assume that $f \in S_p^*[\alpha, k]$. Then

$$\left|\frac{\frac{z(I^k f(z))'}{I^k f(z)}+p}{\frac{z(I^k f(z))'}{I^k f(z)}+2\alpha-p}\right| = \left|\frac{\sum_{n=p}^{\infty} P(n,k)(n+p)a_n z^n}{\frac{2(\alpha-p)}{z^p}+\sum_{n=p}^{\infty} P(n,k)(n+2\alpha-p)a_n z^n}\right| < 1. \qquad (12)$$

Since $Re(z) \leq |z|$ for all z, it follows (12) that

$$Re\left\{\frac{\sum_{n=p}^{\infty} P(n,k)(n+p)a_n z^n}{\frac{2(\alpha-1)}{z^p}+\sum_{n=p}^{\infty} P(n,k)(n+2\alpha-p)a_n z^n}\right\} < 1, \qquad (z \in U). \qquad (13)$$

We now choose the values z on the real axis so that $\frac{z(I^k f(z))'}{I^k f(z)}$ is real. Upon clearing the denominator in (13) and letting $z \longrightarrow 1$ through real values, we obtain

$$\sum_{n=p}^{\infty} P(n,k)(n+p)a_n \leq 2(p-\alpha) - \sum_{n=1}^{\infty} P(n,k)(n+2\alpha-p)a_n, \qquad (14)$$

which immediately yield the required condition (9).

Our assertion in Theorem 4 is sharp for a function of the form:

$$f_n(z) = \frac{1}{z^p} + \frac{(p-\alpha)}{P(n,k)(n+\alpha)} z^n, \qquad (n = p, p+1, p+2, ..., ; k \in N_0). \qquad (15)$$

Corollary 5. *Let the function $f(z)$ be defined by (4) and let $f(z) \in S_p$. If $f \in S_p^*([k, \alpha]$. Then for fixed n , we have*

$$|a_n| \leq \frac{(p-\alpha)}{P(n,k)(n+\alpha)}. \qquad (16)$$

$(n = p, p+1, p+2, ..., ; k \in N_0)$.

The result(16) is sharp for functions $f_n(z)$ given by (15).

3. Covering Theorem

A growth and distortion property for functions $f(z)$ in the class $S_p^*[k, \alpha]$ is contained in the following theorem.

Theorem 6. *If the function $f(z)$ defined by (4) is in the class $S_p^*[k,\alpha]$, then for $0<|z|=r<1$ we have*

$$(\frac{(p+m-1)!}{(p-1)!}-\frac{(p-k)!(p-\alpha)}{(p-m)!(p+\alpha)}r^{2p})r^{-(p+m)}\leqslant|f^m(z)|$$

$$\leqslant(\frac{(p+m-1)!}{(p-1)!}+\frac{(p-k)!(p-\alpha)}{(p-m)!(p+\alpha)}r^{2p})r^{-(p+m)} \tag{17}$$

$(m=0,1,2,3,......,p-1)$.
These inequalities are sharp for the function f given by

$$f(z)=z^{-p}+\frac{(p-\alpha)}{P(p,k)(p+\alpha)}z^p. \tag{18}$$

Proof. Let $f\in S_p^*[k,\alpha]$. Then we find from Theorem 4 that

$$\frac{(p+\alpha)}{(p-k)!}\sum_{n=p}^{\infty}n!|a_n|\leqslant$$

$$\sum_{n=p}^{\infty}P(n,k)(n+\alpha)|a_n|\leq(p-\alpha)$$

which yields

$$\sum_{n=p}^{\infty}n!|a_n|\leqslant\frac{(p-k)!(p-\alpha)}{(p+\alpha)}. \tag{19}$$

Now, by differentiating f in (4) m times, we have

$$f^{(m)}(z)=(-1)^m\frac{(p+m-1)!}{(p-1)!}z^{-p-m}+\sum_{n=p}^{\infty}\frac{n!}{(n-m)!}\,|a_n|\,z^{n-m}. \tag{20}$$

Theorem 6 would readily follow from (19) and (20).

Next, we determine the radii of meromorphically p-valent starlikeness and meromorphically p-valent convexity for functions f in the class $S_p^*[k,\alpha]$.

4. Radii Of Starlikeness and Convexity

Theorem 7. *If the function $f(z)$ defined by (4) is in the class $S_p^*[k,\alpha]$, then $f(z)$ is meromorphically starlike of order $\delta(0 \le \delta < 1)$ in $|z| < r_1$, where*

$$r_1 = r_1(k,\alpha,\delta) = inf\{\frac{P(n,k)(n+\alpha)(p-\delta)}{(p-\alpha)(n+2p-\delta)}\}^{\frac{1}{n+p}}. \tag{21}$$

The result is sharp for the function $f_n(z)$ given by (15).

Proof. It suffices to prove that

$$\left|\frac{zf'(z)}{f(z)}+p\right| \leqslant p-\delta, \tag{22}$$

for $|z| < r_1$. We have

$$\left|\frac{zf'(z)}{f(z)}+p\right| = \left|\frac{\sum_{n=p}^{\infty}(n+p)a_n z^n}{\frac{1}{z^p}+\sum_{n=p}^{\infty}a_n z^n}\right| \leqslant \frac{\sum_{n=p}^{\infty}(n+p)a_n\,|z|^{n+p}}{1-\sum_{n=p}^{\infty}a_n\,|z|^{n+p}}. \tag{23}$$

Hence (23) holds true

$$\sum_{n=p}^{\infty}(n+p)a_n\,|z|^{n+p} \leqslant (p-\delta)(1-\sum_{n=p}^{\infty}a_n\,|z|^{n+p}) \tag{24}$$

or

$$\frac{\sum_{n=p}^{\infty}(n+2p-\delta)a_n\,|z|^{n+p}}{(p-\delta)} \le 1. \tag{25}$$

With the aid of (11) and (25) is true if

$$\frac{(n+2p-\delta)a_n}{(p-\delta)}\,|z|^{n+p} \le \frac{P(n,k)(n+\alpha)}{(p-\alpha)}. \tag{26}$$

Solving (26) for $|z|$, we obtain

$$|z| \le \{\frac{P(n,k)(n+\alpha)(p-\delta)}{(p-\alpha)(n+2p-\delta)}\}^{\frac{1}{n+p}}. \tag{27}$$

This completes the proof of Theorem7.

Theorem 8. *If the function $f(z)$ defined by (4) is in the class $S_\omega^*(k,\alpha)$, then $f(z)$ is meromorphically convex of order $\delta(0 \le \delta < 1)$ in $|z| < r_2$, where*

$$r_2 = r_2(k,\alpha,\delta) = \inf\{\frac{\frac{n!}{(n-(k-1))!}(n+\alpha)(p-\delta)}{(p-\alpha)(n+2p-\delta)}\}^{\frac{1}{n+p}} \tag{28}$$

$n \le p$.
The result is sharp for the function $f_n(z)$ given by (15).

Proof. By using the technique employed in the proof of theorem 7, we can show that

$$\left|\frac{zf''(z)}{f'(z)}+p+1\right| \leq -\delta, \tag{29}$$

for $|z| < r_2$, with the aid of theorem 4. Thus we have the assertion of theorem 8.

5. Convex Linear Combinations

. Our next result involves linear combinations of the functions f of the type (15).

Theorem 9. *Let*

$$f_{p-1} = z^{-p} \tag{30}$$

and

$$f_{n+p-1}(z) = \frac{1}{z^p} + \frac{(p-\alpha)}{P(n,k)(n+\alpha)} z^{n+p-1}, \qquad (n \geq p; k \in N_0). \tag{31}$$

Then $f(z) \in S_p^*[k,\alpha]$ *if and only if* f *can be expressed in the form*

$$f(z) = \sum_{n=p}^{\infty} \lambda_{n+p-1} f_{n+p-1}(z), \tag{32}$$

where $\lambda_{n+-1} \leq 0$ *and* $\sum_{n=p}^{\infty} \lambda_{n+p-1} = 1$.

Proof. From (32), it is easily seen that

$$\begin{aligned} f(z) &= \sum_{n=p}^{\infty} \lambda_{n+p-1} f_{n+p-1}(z) \\ &= \frac{1}{z^p} + \sum_{n=p}^{\infty} \frac{(p-\alpha)}{P(n,k)(n+\alpha)} \lambda_n z^n. \end{aligned} \tag{33}$$

Since

$$\begin{aligned} &\sum_{n=p}^{\infty} \frac{P(n,k)(n+\alpha)}{(p-\alpha)} \lambda_{n+p} \cdot \frac{(p-\alpha)}{P(n,k)(n+\alpha)} \\ &= \sum_{n=p}^{\infty} \lambda_{n+p} = 1 - \lambda_{p-1} \leq 1, \end{aligned}$$

it follows from Theorem 4 that the function $f(z) \in S_p^*[k,\alpha]$.

Conversely, let us suppose that $f(z) \in S_p^*[k, \alpha]$. Since

$$|a_{n+p}| \leq \frac{(p-\alpha)}{P(n,k)(n+\alpha)} \qquad (n \leq p; k \in N_0),$$

setting

$$\lambda_{n+p} = \frac{P(n,k)(n+\alpha)}{(p-\alpha)} \left|a_{n+p-1}\right|, \qquad (n \leq p; k \in N_0),$$

and

$$\lambda_{p-1} = 1 - \sum_{n=p}^{\infty} \lambda_{n+p},$$

it follows that $f(z) = \sum_{n=p}^{\infty} \lambda_{n+p-1} f_{n+p-1}(z)$.

This completes the proof of the theorem.

Finally, we prove the following theorem.

Theorem 10. . *The class $S_p^*[k, \alpha]$ is closed under convex linear combination.*

Proof. . Suppose that the function $f_1(z)$ and $f_2(z)$ defined by

$$f_j(z) = \frac{1}{z^p} + \sum_{n=p}^{\infty} \left|a_{n,j}\right| z^n, \qquad (j = 1, 2; z \in U) \tag{34}$$

are in the class $S_p^*[k, \alpha]$. Setting

$$f(z) = \mu f_1(z) + (1-\mu) f_2(z) \qquad (0 \leq \mu < 1). \tag{35}$$

From 35 we can write

$$f(z) = \frac{1}{z^p} + \sum_{n=p}^{\infty} \mu \left|a_{n,1}\right| + (1-\mu) \left|a_{n,2}\right| z^n, \tag{36}$$

$(0 \leq \mu < 1; z \in U)$.

Thus in view of Theorem 4, we have

$$\begin{gathered}\mu \sum_{n=p}^{\infty} [P(n,k)(n+\alpha)] \left|a_{n,1}\right| \\ +(1-\mu) \sum_{n=p}^{\infty} [P(n,k)(n+\alpha)] \left|a_{n,2}\right| \\ \leq \mu(p-\alpha) + (1-\mu)(p-\alpha),\end{gathered}$$

which shows that $f(z) \in S_p^*[k, \alpha]$. Hence the theorem is proved.

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References

[1] M.Albehbah and M.Darus, *On new subclasses of meromorphic univalent functions with positive coefficients*, Far East Journal of Mathematical Sciences. ,97(2015),481-493,.

[2] M.K. Aouf and H. M. Hossen, *New criteria for meromorphic p-valent starlike functions*, Tsukuba J. Math.,17(1993),481-486.

[3] M.K. Aouf and H. M. Srivastava, *A new criterion for meromorphically p-valent convex functions of order alpha*, Math. Sci. Res., Hot-Line 1 (8) (1997), 7-12.

[4] M. P. Chen, H. Irmak and H. M. Srivastava, *Some families of multivalently analytic functions with negative coefficients*, J. Math. Anal. Appl., 214(1997) ,674-690.

[5] N. E. Cho and S. Owa, *On certain classes of meromorphically p -valent starlike functions, In New Developments in Univalent Function Theory*, Special Issue of Surikaiseki kenkygusho Kokyuroku, 821(1993), 159-165.

[6] A. W. Goodman, *Univalent functions and non analytic curves*, Proc. Amer.Math. Soc., 8(1957),598-601.

[7] S. B. Joshi and H. M. Srivastava, *A certain family of meromorphically multivalent functions*, Computers Math. Applic., 38 (3/4)(1999),201-211.

[8] S. R. Kulkarni, U. H. Naik and H. M. Srivastava, *Certain class of meromorphically p-valent quasi-convex functions*, Pan Amer. Math. J., 8(1) (1998), 57-64.

[9] J. L. Liu and H. M. Srivastava, *A linear operator and associated families of meromorphically multivalent functions* , J. Math. Anal. Appl., 259(2001) ,566-581.

[10] J. L. Liu and H. M. Srivastava, *Some convolution conditions for starlikeness and convexity of meromorphically multivalent functions*, Appl. Math. Lett., 16(1)(2003),13-16.

[11] J. L. Liu and H. M. Srivastava ,*Subclasses of Meromorphically Multivalent Functions Associated with a Certain Linear Operator*, Mathematical and Computer Modelling., 39(2004),35-44.

[12] M. L. Mogra, *Meromorphic multivalent functions with positive coefficients I and II* , Math. Japan. 35(1990),1-11 and 1089-1098.

[13] S. Owa, H. E. Darwish and M. K. Aouf, *Meromorphic multivalent functions with positive and fixed second coefficients*, Math. Japan., 46(1997),231-236.

[14] R. K. Raina and H. M. Srivastava, *A new class of meromorphically multivalent functions with applications to generalized hypergeometric functions*, Mathematical and Computer Modelling., 43 (2006),350-356.

[15] H. M. Srivastava and J. Patel, *Applications of differential subordination to certain subclasses of meromorphically multivalent functions*, J. Inequal. Pure and Appl. Math., 6(3)Art. (88)(2005).

[16] H. M. Srivastava, H. M. Hossen and M. K. Aouf, *A unified presentation of some classes of meromorphically multivalent functions*, Computers Math. Applic., 38(11/12)(1999), 63-70.

[17] H. M. Srivastava and S. Owa ,Editors *Current Topics in Analytic Function Theory*, World Scientific, Singapore.,(1992).

[18] B. A. Uralegaddi and C. Somanatha, *Certain classes of meromorphic multivalent functions*, Tamkang J. Math., 23(1992),223-231.

[19] B. A. Uralegaddi and C. Somanatha, *New criteria for meromorphic starlike univalent functions*, Bull. Austral. Math. Soc., 43(1991), 137-140.

[20] D. G. Yang, *On new subclasses of meromorphic p-valent functions*, J. Math. Res. Exposition., 15(1995),3-17.

[21] F.Ghanim and M.Darus, *On certain class of analytic function with fixed second positive coefficient*, Int. J. Math. Anal., 2(2008) ,55-66.

[22] F.Ghanim , M.Darus and S.Sivasubramanian, *On new subclass of analytic univalent function*, International Journal of Pure and Applied Mathematics., 40(2007), 307.

[23] F.Ghanim and M.Darus, *Subclasses Of Meromorphically Multivalent functions*, Acta Universitatis Apulensis., 23(2010),201-212.

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