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A SANDWICH THEOREM ON THE ϕ-LIKE FUNCTIONS INVOLVING $I_N \star \mathcal{L}_C(A, B)$ OPERATOR

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Abstract. In this paper, we introduce a new convolution operator $I_n \star \mathcal{L}_c(a, b)$. Several subordination and superordination results involving this operator are proved.

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1. Introduction

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1}$$

which are analytic in the open unit disk $U := \{z : \quad z \in \mathbb{C} \text{ and } |z| < 1\}$. Let $H(U)$ be the linear space of all analytic functions in U. For a positive integer number n and $a \in \mathbb{C}$, we let

$$H[a, n] := \left\{ f \in H(U) : \ f(z) = a + \sum_{k=n}^{\infty} a_k z^k \right\}.$$

Let $f, g \in A$, where f is given by (1) and g is defined by

$$g(z) = z + \sum_{k=2}^{\infty} b_k z^k.$$

Then the Hadamard product (or convolution) $f \star g$ of the functions f and g is defined by

$$(f \star g)(z) := z + \sum_{k=2}^{\infty} a_k b_k z^k = (g \star f)(z).$$

For two functions f and g, analytic in U, we say that the function f is subordinate to g in U, and we denote it by $f(z) \prec g(z)$, if there exists a Schwarz function w , which is analytic in U with $w(0) = 0$ and $|w(z)| < 1$ for $(z \in U)$, such that [1-15]

$$f(z) = g(w(z)), \quad (z \in U).$$

Indeed, it is known that

$$f(z) \prec g(z) \quad \Rightarrow \quad f(0) = g(0) \quad \texttt{and} \quad f(U) \subset g(U).$$

Furthermore, if the function g is univalent in U, then we have the following equivalence:

$$f(z) \prec g(z) \quad \Leftrightarrow \quad f(0) = g(0) \quad \texttt{and} \quad f(U) \subset g(U).$$

Let $\varphi : \mathbb{C}^2 \to \mathbb{C}$ and let h be univalent in U. If p is analytic in U and satisfies the differential subordination $\varphi(p(z), zp'(z)) \prec h(z)$ then p is called a solution of the differential subordination [12-18]. The univalent function q is called a dominant of the solutions of the differential subordination, $p \prec q$. If p and $\varphi(p(z), zp'(z))$ are univalent in U and satisfy the differential superordination $h(z) \prec \varphi(p(z), zp'(z))$ then p is called a solution of the differential superordination. An analytic function q is called subordinant of the solution of the differential superordination if $q \prec p$ [19-26].

Denote by $D^\alpha : A \to A$ the operator defined by

$$D^\alpha f(z) := \frac{z}{(1-z)^{\alpha+1}} \star f(z), \qquad \alpha > -1,$$

where $(\star)$ refers to the Hadamard product or convolution. Then implies that

$$D^n f(z) = \frac{z\left(z^{n-1} f^{(n)}(z)\right)}{n!}, \quad n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}.$$

We note that $D^0 f(z) = f(z)$ and $D' f(z) = zf'(z)$. The operator $D^n f$ is called Ruscheweyh derivative of n'th order of f [27-29]. Ali et al [2, 3] defined and studied an integral operator $I_n : A \to A$ analogous to $D^n f$ as follows: Let $f_n(z) = \frac{z}{(1-z)^{n+1}}$, $n \in \mathbb{N}_0$ and let $f_n^{(-1)}$ be defined such that

$$f_n(z) \star f_n^{(-1)}(z) = \frac{z}{(1-z)}. \tag{2}$$

Then

$$I_n f(z) = f_n(z) \star f_n^{(-1)}(z) = \left[\frac{z}{(1-z)^{n+1}}\right]^{(-1)} \star f(z).$$

Note that $I_0 f(z) = zf'(z)$ and $I_1 f(z) = f(z)$. The operator I_n is called the Noor Integral of n'th order of f. Using (1), (2) and a well-known identity for $D^n f$, we have

$$(n+1)I_n f(z) - nI_{n+1}(z) = z(I_{n+1}f(z))'. \tag{3}$$

Using hypergeometric functions ${}_2F_1$, (2) becomes

$$I_n f(z) = [z {}_2F_1(1,1;n+1,z)] \star f(z),$$

where ${}_2F_1(a,b;c,z)$ is defined by

$${}_2F_1(a,b;c,z) = 1 + \frac{ab}{c}\frac{z}{1!} + \frac{a(a+1)b(b+1)}{c(c+1)}\frac{z^2}{2!} + \cdots$$

For two functions $f_j(z)$, $(j=1,2)$, given by

$$f_j(z) = z + \sum_{k=2}^{\infty} a_{kj} z^k, \qquad (j=1,2).$$

In terms of the Pochhammer symbol (or the shifted factorial), define $(k)_n$ by $(k)_0 = 1$, and $(k)_n = k(k+1)(k+2)\cdots(k+n-1)$, $(n \in \mathbb{N})$, and then define a function $\phi_c(a,b)$ by

$$\phi_c(a,b) = 1 + \sum_{n=1}^{\infty} \left(\frac{c}{c+n}\right) \frac{(a)_n}{(b)_n} \tag{4}$$

where $a \in \mathbb{R}$, $b \in \mathbb{R}\ \mathbb{Z}_0^-$; $c \in \mathbb{C}\ \mathbb{Z}_0^-$, $\left(\mathbb{Z}_0^- := \{0,-1,-2,\ldots\}\right)$. Corresponding to the function $\phi_c(a,b)$, given by (1), we introduce the following convolution operator,

$$\mathcal{L}_c(a,b) = \phi_c(a,b) \star \left(\frac{f(z)}{z}\right), \qquad (f \in A) \tag{5}$$

It is easy to see that

$$Z(\phi_c(a,b))' = a\phi_c(a+1,b) - a\phi_c(a,b) \tag{6}$$

Hence

$$Z(\mathcal{L}_c(a,b)f)'(z) = a\mathcal{L}_c(a+1,b)f(z) - a\mathcal{L}_c(a,b)f(z) \tag{7}$$

we define the Hadamard product (or convolution) of $I_n f(z)$ and $\mathcal{L}_c(a,b)f(z)$ by

$$\begin{aligned} I_n f(z) \star \mathcal{L}_c(a,b)f(z) &= \left[\frac{z}{(1-z)^{n+1}}\right]^{(-1)} \star f(z) \star \phi_c(a,b) \star \frac{f(z)}{z} \\ &= \left(\frac{f(z)}{z}\right)^2 (1-z)^{n+1} \star \phi_c(a,b) \\ &= \left[\left(\frac{f(z)}{z}\right)^2 (1-z)^{n+1}\right] \left[1 + \sum_{n=1}^{\infty} \left(\frac{c}{c+n}\right) \frac{(a)_n}{(b)_n}\right] \end{aligned} \tag{8}$$

Furthermore, we have

$$\frac{z\left[I_{n+1}\star\mathcal{L}_{c+1}(a,b)f(z)\right]'}{\phi\left[I_{n+1}\star\mathcal{L}_{c+1}(a,b)f(z)\right]}=(n+1)(c+1)\frac{z\left[I_n\star\mathcal{L}_c(a,b)f(z)\right]'}{\left[I_n\star\mathcal{L}_c(a,b)f(z)\right]}-(n+1)(c+1).$$

Definition 1. *Let ϕ be an analytic function in a domain containing $f(U)$, $\phi(0)=0$ and $\phi'(0)>0$. The function $[I_n\star\mathcal{L}_c(a,b)f]\in A$ is called ϕ-like if*

$$Re\frac{z[I_n\star\mathcal{L}_c(a,b)f(z)]'}{\phi[I_n\star\mathcal{L}_c(a,b)f(z)]}>0,\qquad (z\in U).\tag{9}$$

Definition 2. *Let ϕ be analytic function in a domain containing $f(U)$, $\phi(0)=0$, $\phi'(0)=1$ and $\phi(\omega)\neq 0$ for $\omega\in f(U)-0$. Let $q(z)$ be a fixed analytic function in U, $q(0)=1$. The function $[I_n\star\mathcal{L}_c(a,b)f]\in A$ is called ϕ-like with respect to*

$$\frac{z[I_n\star\mathcal{L}_c(a,b)f(z)]'}{\phi[I_n\star\mathcal{L}_c(a,b)f(z)]}\prec q(z),\qquad (z\in U).\tag{10}$$

2. Preliminaries

To derive our main results, we need the following definitions and lemmas.

Definition 3. *A function $L(z,t)$, $(z\in U,\quad t\geq 0)$ is said to be a subordination chain if $L(0,t)$ is analytic and univalent in U for all $t\geq 0$, $L(z,0)$ is continuously differentiable on [0,1) for all $z\in U$ and $L(z,t_1)\prec L(z,t_2)$ for all $0\leq t_1\leq t_2$.*

Remark 1. *Denote by Q the set of all functions f that are analytic and injective on $\bar{U}-E(f)$, where*

$$E(f)=\{\varepsilon\in\partial U:\quad \lim_{z\to\varepsilon}f(z)=\infty\},$$

and such that $f'(\varepsilon)\neq 0$ for $\varepsilon\in\partial U-E(f)$. The subclass of Q for which $f(0)=a$, $(a\in\mathbb{C})$, is denoted by $Q(a)$.

Lemma 1. *The function $L(z,t):\ U\times[0,\infty)\to\mathbb{C}$ of the form*

$$L(z,t)=a_1(t)z+a_2(t)z^2+\cdots,\qquad (a_1(t)\neq 0;\ t\geq 0),$$

and $\lim_{t\to\infty}|a_1(t)|=\infty$ is a subordination chain if and only if

$$Re\left(\frac{z\partial L/\partial z}{\partial L/\partial t}\right)>0,\qquad (z\in U;\ t\geq 0).$$

Proof. See [11].

Lemma 2. *Suppose that the function $H: \mathbb{C}^2 \to \mathbb{C}$ satisfies the condition $Re(H(is,t)) \leq 0$ for all real s and for all*

$$t \leq -\frac{n(1+s^2)}{2}, \qquad (n \in \mathbb{N}).$$

If the function

$$p(z) = 1 + p_n z^n + p_{n+1} z^{n+1} + \cdots,$$

is analytic in U and $Re(H(p(z), zp'(z)) > 0$, $(z \in U)$, then $Re(p(z)) > 0$, $(z \in U)$.

Proof. See [11].

Lemma 3. *Let $k, \gamma \in \mathbb{C}$ with $k \neq 0$ and let $h \in H(U)$ with $h(0) = c$. If $Re(kh(z) + \gamma) > 0$, $(z \in U)$, then the solution of the following differential equation:*

$$q(z) + \frac{zq'(z)}{kq(z) + \gamma} = h(z), \qquad (z \in U,\ q(0) = c),$$

is analytic in U and satisfies the inequality given by $Re(kq(z) + \gamma) > 0$, $(z \in U)$.

Proof. See [11].

Lemma 4. *Let $p \in Q(a)$ and*

$$q(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots, \qquad (q \neq a,\ n \in \mathbb{N}).$$

be analytic in U. If q is not subordinate to p, then there exists two points

$$z_0 = r_0 e^{i\theta} \in U, \qquad \texttt{and} \quad \varepsilon_0 \in \partial U / E(f),$$

such that $q(U_{r0}) \subset p(U)$, $q(z_0) = p(\varepsilon_0)$ and $z_0 q'(z_0) = m_0 \varepsilon_0 p'(\varepsilon_0)$, $(m \geq n)$.

Proof. See [11].

Lemma 5. *Let $q \in H[a,1]$ and $\phi : \mathbb{C}^2 \to \mathbb{C}$. Also set*

$$\phi(q(z), zq'(z)) \equiv h(z), \qquad (z \in U).$$

Let

$$L(z,t) := \phi(q(z), tzq'(z)),$$

be a subordination chain and $p \in H[a,1]\ Q(a)$. Then $h(z) \prec \phi(p(z), zp'(z))$ implies that $q(z) \prec p(z)$. Furthermore, if $\phi(q(z), zq'(z)) = h(z)$ has a univalent solution $q \in Q(a)$, then q is the best subordinate.

Proof. See [11].

3. Main Results

We begin by presenting our first subordination property given by Theorem 6, below. For convenience, let

$$A_0 := \{f \in A : \ [I_n \star \mathcal{L}_c(a,b)]f(z) \neq 0, \qquad (z \in U)\}.$$

Theorem 6. *Let $f, g \in A$ and $a \in \mathbb{C}$, $Re(nc) > 0$. Further let*

$$Re\left(1 + \frac{z\varphi''(z)}{\varphi'(z)}\right) > -\delta, \quad \left(z \in U, \ \varphi(z) := \frac{z[I_n \star \mathcal{L}_c(a,b)g(z)]'}{\phi[I_n \star \mathcal{L}_c(a,b)g(z)]}\right), \tag{11}$$

where

$$\delta := \frac{1 + (nc)^2 - |1 - (nc)^2|}{4Re(nc)}. \tag{12}$$

Then the subordination

$$\frac{z[I_n \star \mathcal{L}_c(a,b)f(z)]'}{\phi[I_n \star \mathcal{L}_c(a,b)f(z)]} \prec \frac{z[I_n \star \mathcal{L}_c(a,b)g(z)]'}{\phi[I_n \star \mathcal{L}_c(a,b)g(z)]},$$

implies that

$$\frac{z[I_{n+1} \star \mathcal{L}_{c+1}(a,b)f(z)]'}{\phi[I_{n+1} \star \mathcal{L}_{c+1}(a,b)f(z)]} \prec \frac{z[I_{n+1} \star \mathcal{L}_{c+1}(a,b)g(z)]'}{\phi[I_{n+1} \star \mathcal{L}_{c+1}(a,b)g(z)]}.$$

Furthermore, the function $\frac{z[I_{n+1} \star \mathcal{L}_{c+1}(a,b)g(z)]'}{\phi[I_{n+1} \star \mathcal{L}_{c+1}(a,b)g(z)]}$ is the best dominant.

Proof. Let the functions F, G and Q be defined by

$$F := \frac{z[I_{n+1} \star \mathcal{L}_{c+1}(a,b)f(z)]'}{\phi[I_{n+1} \star \mathcal{L}_{c+1}(a,b)f(z)]}, \quad G := \frac{z[I_{n+1} \star \mathcal{L}_{c+1}(a,b)g(z)]'}{\phi[I_{n+1} \star \mathcal{L}_{c+1}(a,b)g(z)]},$$

$$Q := 1 + \frac{z\varphi''(z)}{\varphi'(z)}. \tag{13}$$

We assume here, without loss of generality, that G is analytic and univalent on $\bar{U}$ and $G'(\varepsilon) \neq 0$, $(|\varepsilon| = 1)$. If not, then we replace F and G by $F(\rho z)$ and $G(\rho z)$, respectively, with $0 < \rho < 1$. These new functions have the desired properties on $\bar{U}$, and we can use them in the proof of our result. Therefore, the result would follow by letting $\rho \to 1$. We first show that $Re(Q(z)) > 0$, $(z \in U)$. By virtue of (1) and the definitions of G, we know that

$$\varphi(z) = G(z) + \frac{1}{nc} zG'(z). \tag{14}$$

Differentiating both sides of (14) with respect to z yields

$$\varphi'(z) = \left(1 + \frac{1}{nc}\right) G(z) + \frac{1}{nc} zG''(z). \tag{15}$$

Combining (13) and (15), we easily get

$$1 + \frac{z\varphi''(z)}{\varphi'(z)} = Q(z) + \frac{zQ'(z)}{Q(z) + nc} = h(z), \quad (z \in U). \tag{16}$$

It follows from (11) and (16) that

$$Re(h(z) + nc) > 0, \quad (z \in U). \tag{17}$$

Moreover, by Lemma 2.5, we conclude that the differential equation (16) has a solution $Q \in H(U)$ with $h(0) = Q(0) = 1$. Let $H(u, v) := u + \frac{v}{u+nc} + \delta$, where δ is given by (12). From (16) and (17), we obtain

$$Re(H(Q(z), zQ'z)) > 0, \qquad (z \in U).$$

To verify the condition that

$$Re(H(is, t)) \le 0, \qquad \left(s \in \mathbb{R};\ t \le -\frac{n(1 + s^2)}{2}\right), \tag{18}$$

we proceed it as follows:

$$Re(H(is, t)) = Re\left(is + \frac{t}{is + nc} + \sigma\right) = \frac{tn}{|is + nc|^2} + \sigma \le -\frac{\psi(n, s)}{2|is + nc|^2}\ ,$$

where

$$\psi(n, s) := (n - 2\delta)s^2 - 4\sigma ns - 2\sigma n^2 + n. \tag{19}$$

For δ given by (12), we note that the coefficient of s^2 in the quadratic expression $\psi(n, s)$ given by (19) is positive or equal to zero. Furthermore, we observe that the quadratic expression $\psi(n, s)$ by s in (19) is a perfect square, which implies that (18) holds. Thus, by Lemma 2.4, we conclude that $Re(Q(z)) > 0$, $(z \in U)$. Let $f \in H(U)$, then f is convex if and only if $f'(0) \neq 0$ and $Re\{1+(f''(z))/(f'(z))\} > 0$, $z \in U$. Now by the definition of Q, we know that G is convex. To prove $F \prec G$, let the function L be defined by

$$L(z, t) := G(z) + \frac{t}{n} zG'(z), \qquad (z \in U;\ 0 \le t < \infty). \tag{20}$$

Since G is convex and $n > 0$, then

$$\frac{\partial L(z,t)}{\partial z}|_{z=0} = G'(0)\left(1+\frac{t}{n}\right) \neq 0, \qquad (z \in U;\ 0 \leq t < \infty),$$

and

$$Re\left(\frac{z\partial L/\partial z}{\partial L/\partial t}\right) = Re(n + tQ(z)) > 0, \qquad (z \in U).$$

Therefore, by Lemma 2.3, we deduce that P is a subordination chain. It follows from the definition of subordination chain that $\varphi(z) = G(z) + \frac{1}{n}zG'(z) = L(z,0)$ and $L(z,0) \prec L(z,t)$, $(0 \leq t < \infty)$, which implies that

$$L(\varepsilon,t) \notin L(U,0) = \varphi(U), \qquad (\varepsilon \in U;\ 0 \leq t < \infty). \tag{21}$$

If F is not subordinate to G, by Lemma 2.6, we know that there exist two points $z_0 \in U$ and $\varepsilon_0 \in \frac{\partial U}{E(f)}$ such that

$$F(z_0) = G(\varepsilon_0) \quad \texttt{and} \quad z_0F(z_0) = t\varepsilon_0G'(\varepsilon_0), \qquad (0 \leq t < \infty). \tag{22}$$

Hence, by virtue of (1) and (22), we have

$$L(\varepsilon_0,t) = G(\varepsilon_0) + \frac{t}{n}\varepsilon_0G'(\varepsilon_0) = F(z_0) + \frac{1}{n}z_0F'(z_0) = \frac{I_{n+1}f(z_0)}{z_0} \in \varphi(U).$$

This contradicts to (21). Thus, we deduce that $F \prec G$. Considering $F = G$, we see that the function G is the best dominant.

By similarly applying the method of proof of Theorem 3.1, as well as (1), we easily get the following result.

Corollary 7. *Let $f, g \in A$ and $n > -1$. Further let*

$$Re\left(1+\frac{z\chi''(z)}{\chi'(z)}\right) > -\bar{\omega}, \qquad \left(z \in U;\ \chi(z) := \frac{I_n g(z)}{z}\right),$$

where

$$\bar{\omega} := \frac{1+(n+1)^2 - \left|1-(n+1)^2\right|}{4(n+1)}. \tag{23}$$

Then the subordination $\frac{I_nf(z)}{z} \prec \frac{I_ng(z)}{z}$, implies that $\frac{I_{n+1}f(z)}{z} \prec \frac{I_{n+1}g(z)}{z}$. Furthermore, the function $\frac{I_{n+1}g(z)}{z}$ is the best dominant.

If f is subordinate to F, then F is superordinate to f. We now derive the following superordination result.

Theorem 8. *Let $f, g \in A_p$ and $n > 0$. Further let*

$$Re\left(1 + \frac{z\varphi''(z)}{\varphi'(z)}\right) > -\delta, \qquad \left(z \in U;\ \varphi(z) := \frac{I_{n+1}g(z)}{z}\right), \tag{24}$$

where δ is given by (12). If the function $\frac{I_{n+1}f(z)}{z}$ is univalent in U and $\frac{I_n f(z)}{z} \in Q$, then the subordination

$$\frac{I_{n+1}g(z)}{z} \prec \frac{I_{n+1}f(z)}{z},$$

implies that

$$\frac{I_n g(z)}{z} \prec \frac{I_n f(z)}{z}.$$

Furthermore, the function $\frac{I_n g(z)}{z}$ is the best subordinate.

Proof. Suppose that the functions F and G and Q are defined by (13). By applying the similar method as in the proof of Theorem 3.1, we get $Re(Q(z)) > 0$, $(z \in U)$. Next, to arrive at our desired result, we show that $G \prec F$. For this, we suppose that the function L be defined by (20). Since $n > 0$ and G is convex, by applying a similar method as in Theorem 3.1, we deduce that L is subordination chain. Therefore, by Lemma 2.7, we conclude that $G \prec F$. Moreover, since the differential equation

$$\varphi(z) = G(z) + \frac{1}{n}zG'(z) = \phi(G(z), zG'(z)),$$

has a univalent solution G, it is the best subordinate.

Applying a similar proof as in Theorem 3.2, and using (1), the following results are easily obtained.

Corollary 9. *Let $A_p = \{f \in H(U) : \ f(z) = a + \sum_{k=p}^{\infty} a_k z^k\}$, $f, g \in A_p$ and $n > 0$. Further let*

$$Re\left(1 + \frac{z\chi''(z)}{\chi'(z)}\right) > -\bar{\omega}, \qquad \left(z \in U;\ \chi(z) := \frac{I_n g(z)}{z}\right),$$

where $\bar{\omega}$ is given by (23). If the function $\frac{I_n f(z)}{z}$ is univalent in U and $\frac{I_{n+1}f(z)}{z} \in Q$, then the subordination

$$\frac{I_n g(z)}{z} \prec \frac{I_n f(z)}{z},$$

implies that

$$\frac{I_{n+1}g(z)}{z} \prec \frac{I_{n+1}f(z)}{z}.$$

Furthermore, the function $\frac{I_{n+1}g(z)}{z}$ is the best subordinate.

Combining the above mentioned subordination and super ordination results involving the operator I_n, the following "sandwich-type results" are derived.

Corollary 10. *Let $f, g_k \in A$, $(k = 1, 2)$ and $n > 0$. Further let*

$$Re\left(1 + \frac{z\varphi''(z)}{\varphi'(z)}\right) > -\delta, \quad \left(z \in U;\ \varphi(z) := \frac{I_{n+1}g_k(z)}{z},\ k = 1, 2\right), \tag{25}$$

where δ is given by (12). If the function $\frac{I_{n+1}f(z)}{z}$ is univalent in U and $\frac{I_n f(z)}{z} \in Q$, then the subordination chain

$$\frac{I_{n+1}g_1(z)}{z} \prec \frac{I_{n+1}f(z)}{z} \prec \frac{I_{n+1}g_2(z)}{z},$$

implies that

$$\frac{I_n g_1(z)}{z} \prec \frac{I_n f(z)}{z} \prec \frac{I_n g_2(z)}{z}.$$

Furthermore, the functions $\frac{I_n g_1(z)}{z}$ and $\frac{I_n g_2(z)}{z}$ are, respectively, the best subordinate.

Corollary 11. *Let $f, g_k \in A$, $(k = 1, 2)$ and $n > 0$. Further let*

$$Re\left(1 + \frac{z\chi_k''(z)}{\chi_k'(z)}\right) > -\bar{\omega}, \quad \left(z \in U;\ \chi_k(z) := \frac{I_n g_k(z)}{z},\ k = 1, 2\right), \tag{26}$$

where $\bar{\omega}$ is given by (12). If the function $\frac{I_n f(z)}{z}$ is univalent in U and $\frac{I_{n+1}f(z)}{z} \in Q$, then the subordination chain

$$\frac{I_n g_1(z)}{z} \prec \frac{I_n f(z)}{z} \prec \frac{I_n g_2(z)}{z},$$

implies that

$$\frac{I_{n+1}g_1(z)}{z} \prec \frac{I_{n+1}f(z)}{z} \prec \frac{I_{n+1}g_2(z)}{z}.$$

Furthermore, the functions $\frac{I_{n+1}g_1(z)}{z}$ and $\frac{I_{n+1}g_2(z)}{z}$ are, respectively, the best subordinate.

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