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INCLUSION RELATIONSHIPS PROPERTIES FOR CERTAIN CLASSES OF P-VALENT MEROMORPHIC FUNCTIONS

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Abstract. In this paper we investigate a family of linear operator defined on the space of p-valent meromorphic functions. By making use of this linear operator, we introduce and investigate some new subclasses of p-valent starlike, p-valent convex, p-valent close-to-convex and p-valent quasi-convex meromorphic functions. Also we establish some inclusion relationships associated with the aforementioned linear operator. Some interesting integral-preserving properties are also considered.

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1. Introduction

Let Σ_p denote the class of functions of the form

$$f(z) = z^{-p} + \sum_{n=1}^{\infty} a_{n-p} z^{n-p} \quad (p \in \mathbb{N} = \{1, 2, ...\}), \tag{1}$$

which are analytic and p-valent in the punctured unit disc $U^* = \{z : z \in \mathbb{C}$ and $0 < |z| < 1\} = U \backslash \{0\}$. A function $f \in \Sigma_p$ is said to be in the class $\Sigma S_p^*(\alpha)$ of meromorphic p-valent starlike functions of order α in U^* if and only if

$$Re\left(\frac{zf'(z)}{f(z)}\right) < -\alpha \quad (0 \leq \alpha < p; z \in U^*). \tag{2}$$

Also a function $f \in \Sigma_p$ is said to be in the class $\Sigma C_p(\alpha)$ of meromorphic p-valent convex functions of order α in U^* if and only if

$$Re\left(1 + \frac{zf''(z)}{f'(z)}\right) < -\alpha \quad (0 \leq \alpha < p; z \in U^*). \tag{3}$$

It is easy to observe from (2) and (3) that

$$f(z) \in \Sigma C_p(\alpha) \Leftrightarrow -\frac{zf^{'}(z)}{p} \in \Sigma S_p^*(\alpha)\,. \tag{4}$$

For a function $f \in \Sigma_p$, we say that $f \in \Sigma K_p(\beta,\alpha)$ if there exists a function $g \in \Sigma S_p^*(\alpha)$ such that

$$Re\left(\frac{zf^{'}(z)}{g(z)}\right) < -\beta \quad (0 \le \alpha, \beta < p; z \in U^*)\,. \tag{5}$$

Functions in the class $\Sigma K_p(\beta,\alpha)$ are called meromorphic p-valent close-to-convex functions of order β and type α. We also say that a function $f \in \Sigma_p$ is in the class $\Sigma K_p^*(\beta,\alpha)$ of meromorphic quasi-convex functions of order β and type α if there exists a function $g \in \Sigma C_p(\alpha)$ such that

$$Re\left(\frac{(zf^{'}(z))^{'}}{g^{'}(z)}\right) < -\beta \quad (0 \le \alpha, \beta < p; z \in U^*)\,. \tag{6}$$

It follows from (5) and (6) that

$$f(z) \in \Sigma K_p^*(\beta,\alpha) \Leftrightarrow -\frac{zf^{'}(z)}{p} \in \Sigma K_p(\beta,\alpha)\,. \tag{7}$$

Meromorphically multivalent functions have been extensively studied by several authors, see for example, Aouf and Hossen [3], Aouf [1], Joshi and Srivastava [9], Mogra [12], Owa et al. [13], Raina and Srivastava [14], Ibrahim [7] and Aouf and Xu [2].

For two functions $f_j(z) \in \Sigma_p (j = 1, 2)$ are given by

$$f_j(z) = \frac{1}{z^p} + \_n = 1^{\infty} a_{n-p,j} z^{n-p}$$

we define the Hadamard product (or convolution) of $f_1(z)$ and $f_2(z)$ in Σ_p by

$$(f_1 * f_2)(z) = \frac{1}{z^p} + \_n = 1^{\infty} a_{n-p,1} a_{n-p,2} z^{n-p} = (f_2 * f_1)(z).$$

El-Ashwah and Bulboaca [6] defined the linear operator as follows:

$$\mathcal{L}_{p,d}^s(z) = \frac{1}{z^p} + \_n = 1^{\infty} \left(\frac{d}{n+d}\right)^s z^{n-p}$$

$$(s \in \mathbb{C}; d \in \mathbb{C}^* = \mathbb{C}\backslash\{0,-1,-2,...\}; z \in U^*)$$

by setting

$$\mathcal{J}^{s}_{p,d}(z) = \frac{1}{z^p} + \_n = 1^{\infty}\left(\frac{n+d}{d}\right)^s z^{n-p}$$

and

$$\left(\mathcal{J}^{s,\lambda}_{p,d} * \mathcal{J}^{s}_{p,d}\right)(z) = \frac{1}{z^p\,(1-z)^{\lambda}} \qquad (\lambda > 0)\,,$$

we, obtain the linear operator

$$\mathcal{J}^{s,\lambda}_{p,d}\,(z) = \frac{1}{z^p} + \_n = 1^{\infty}\left(\frac{d}{n+d}\right)^s \frac{(\lambda)\,n}{(1)\,n} z^{n-p},$$

which is defined by

$$\mathcal{J}^{s,\lambda}_{p,d} f\,(z) = \frac{1}{z^p} +_{n=1}^{\infty}\left(\frac{d}{n+d}\right)^s \frac{(\lambda)\,n}{(1)\,n} a_{n-p} z^{n-p}. \tag{8}$$

$$(\lambda > 0, s \in \mathbb{C}; d \in \mathbb{C}^*; z \in U^*),$$

where $f \in \Sigma_p$ is in the form (1) and $(\nu)_n$ denotes the Pochhammer symbol given by

$$(\nu)_n = \frac{\Gamma(\nu+n)}{\Gamma(\nu)} = \begin{cases} 1 & (n=0) \\ \nu(\nu+1)...(\nu+n-1) & (n \in \mathbb{N})\,. \end{cases}$$

It is readily verified from (8) that

$$z(\mathcal{J}^{s,\lambda}_{p,d} f(z))' = \lambda \mathcal{J}^{s,\lambda+1}_{p,d} f(z) - (\lambda+p)\mathcal{J}^{s,\lambda}_{p,d} f(z) \quad (\lambda > 0) \tag{9}$$

and

$$z(\mathcal{J}^{s+1,\lambda}_{p,d} f(z))' = d\mathcal{J}^{s,\lambda}_{p,d} f(z) - (d+p)\mathcal{J}^{s+1,\lambda}_{p,d} f(z)\,. \tag{10}$$

We now define the following subclasses of the meromorphic function class Σ_p by means of the linear operator $\mathcal{J}^{s,\lambda}_{p,d}(z)$ given by (8).

Definition 1. *In conjunction with (2) and (8),*

$$\Sigma S^{*s,\lambda}_{p,d}(\alpha) = \left\{f : f \in \Sigma_p \textit{ and } \mathcal{J}^{s,\lambda}_{p,d} f \in \Sigma S^*_p(\alpha)\,, 0 \le \alpha < p, p \in \mathbb{N}\right\}. \tag{11}$$

Definition 2. *In conjunction with (3) and (8),*

$$\Sigma C^{s,\lambda}_{p,d}(\alpha) = \left\{f : f \in \Sigma_p \textit{ and } \mathcal{J}^{s,\lambda}_{p,d} f \in \Sigma C_p(\alpha)\,, 0 \le \alpha < p, p \in \mathbb{N}\right\}. \tag{12}$$

Definition 3. *In conjunction with (5) and (8),*

$$\Sigma K_{p,d}^{s,\lambda}(\beta,\alpha)=\left\{f: f\in\Sigma_p \text{ and } \mathcal{J}_{p,d}^{s,\lambda}f\in\Sigma K_p(\beta,\alpha)\,, 0\leq\alpha,\beta<p, p\in\mathbb{N}\right\}. \quad (13)$$

Definition 4. *In conjunction with (6) and (8),*

$$\Sigma K_{p,d}^{*s,\lambda}(\beta,\alpha)=\left\{f: f\in\Sigma_p \text{ and } \mathcal{J}_{p,d}^{s,\lambda}f\in\Sigma K_p^{*}(\beta,\alpha)\,, 0\leq\alpha,\beta<p, p\in\mathbb{N}\right\}. \quad (14)$$

In order to establish our main results, we need the following lemma due to Miller and Mocanu [11].

Lemma 1. ***[11].*** *Let $\theta(u,v)$ be a complex-valued function such that*

$$\theta: D\to\mathbb{C},\ D\subset\mathbb{C}\times\mathbb{C}\qquad (\mathbb{C} \text{ is the complex plane})$$

and let $u=u_1+iu_2$ and $v=v_1+iv_2$. Suppose that $\theta(u,v)$ satisfies the following conditions :

(i) $\theta(u,v)$ is continuous in D;
(ii) $(1,0)\in D$ and $Re\{\theta(1,0)\}>0$;
(iii) for all $(iu_2,v_1)\in D$ such that

$$v_1\leq-\frac{1}{2}(1+u_2^2)\quad,\quad Re\{\theta(iu_2,v_1)\}\leq 0\,.$$

Let

$$q(z)=1+q_1z+q_2z^2+... \quad (15)$$

be analytic in U such that $(q(z),zq^{'}(z))\in D\,(z\in U)$. If

$$Re\left\{\theta(q(z),zq^{'}(z))\right\}>0\quad(z\in U)\,,$$

then

$$Re\{q(z)\}>0\quad(z\in U)\,.$$

2. The Main Results

In this section, we give several inclusion relationships for p-valent meromorphic function classes, which are associated with the linear operator $\mathcal{J}_{p,d}^{s,\lambda}$.

Unless otherwise mentioned we assume throughout the paper that $\lambda>0$,$s\in\mathbb{C},d\in\mathbb{C}^{*}$.

Theorem 2. *Let* $d_1 = Re(d) > \alpha - p, p \in \mathbb{N}$ *and* $0 \leq \alpha < p$. *Then*

$$\Sigma S_{p,d}^{*s,\lambda+1}(\alpha) \subset \Sigma S_{p,d}^{*s,\lambda}(\alpha) \subset \Sigma S_{p,d}^{*s+1,\lambda}(\alpha)\,.$$

Proof. (i) We begin by showing the second inclusion relationship:

$$\Sigma S_{p,d}^{*s,\lambda}(\alpha) \subset \Sigma S_{p,d}^{*s+1,\lambda}(\alpha)\,. \tag{16}$$

Let $f(z) \in \Sigma S_{p,d}^{*s,\lambda}(\alpha)$ and set

$$\frac{z(\mathcal{J}_{p,d}^{s+1,\lambda} f(z))'}{\mathcal{J}_{p,d}^{s+1,\lambda} f(z)} = -\alpha - (p-\alpha)q(z)\,, \tag{17}$$

where $q(z)$ is given by (15). By using the identity (10), we obtain

$$d\frac{\mathcal{J}_{p,d}^{s,\lambda} f(z)}{\mathcal{J}_{p,d}^{s+1,\lambda} f(z)} = -\alpha - (p-\alpha)q(z) + (d+p)\,. \tag{18}$$

Differentiating (18) logarithmically with respect to z, we obtain

$$\begin{aligned}\frac{z(\mathcal{J}_{p,d}^{s,\lambda} f(z))'}{\mathcal{J}_{p,d}^{s,\lambda} f(z)} &= \frac{z(\mathcal{J}_{p,d}^{s+1,\lambda} f(z))'}{\mathcal{J}_{p,d}^{s+1,\lambda} f(z)} + \frac{(p-\alpha)zq'(z)}{(p-\alpha)q(z)+\alpha-(d+p)} \\ &= -\alpha - (p-\alpha)q(z) + \frac{(p-\alpha)zq'(z)}{(p-\alpha)q(z)-d-p+\alpha}\,.\end{aligned}$$

Let

$$\theta(u,v) = (p-\alpha)u - \frac{(p-\alpha)v}{(p-\alpha)u+\alpha-d-p} \tag{19}$$

with $u = q(z) = u_1 + iu_2$ and $v = zq'(z) = v_1 + iv_2$. Then
(i) $\theta(u,v)$ is continuous in $D = \left(\mathbb{C}\backslash\left\{\frac{d+p-\alpha}{p-\alpha}\right\}\right) \times \mathbb{C}$;
(ii) $(1,0) \in D$ with $\{\theta(1,0)\} = p - \alpha > 0$.
(iii) for all $(iu_2, v_1) \in D$ such that $v_1 \leq -\frac{1}{2}\left(1+u_2^2\right)$ we have

$$\begin{aligned}Re\{\theta(iu_2,v_1)\} &= Re\left\{\frac{-(p-\alpha)v_1}{(p-\alpha)iu_2+\alpha-(d+p)}\right\} \\ &= \frac{(p-\alpha)\left(d_1+p-\alpha\right)v_1}{\left((p-\alpha)u_2+d_2\right)^2+(\alpha-d_1-p)^2} \\ &\leq -\frac{(p-\alpha)(d_1+p-\alpha)(1+u_2^2)}{2\left(\left((p-\alpha)u_2+d_2\right)^2+(d_1-p-\alpha)^2\right)} \\ &< 0\,,\end{aligned}$$

which shows that $\theta(u, v)$ satisfies the hypotheses of Lemma 1.1. Consequently, we easily obtain the inclusion relationship (16).

(ii) By using arguments similar to those detailed above, together with (9) and $\theta(u, v)$ is continuous in $D = \left(\mathbb{C}\backslash\left\{\frac{\lambda+p-\alpha}{p-\alpha}\right\}\right) \times \mathbb{C}$, we can also prove the left part of Theorem 2.1, that is, that

$$\Sigma S^{*s,\lambda+1}_{p,d}(\alpha) \subset \Sigma S^{*s,\lambda}_{p,d}(\alpha) \ . \tag{20}$$

Combining the inclusion relationships (16) and (20), we complete the proof of Theorem 2.1.

Theorem 3. *Let* $d_1 = Re(d) > \alpha - p, p \in \mathbb{N}$ *and* $0 \le \alpha < p$. *Then*

$$\Sigma C^{s,\lambda+1}_{p,d}(\alpha) \subset \Sigma C^{s,\lambda}_{p,d}(\alpha) \subset \Sigma C^{s+1,\lambda}_{p,d}(\alpha).$$

Proof. Let $f \in \Sigma C^{s,\lambda}_{p,d}(\alpha)$. Then, by Definition 1.2, we have

$$\mathcal{J}^{s,\lambda}_{p,d} f(z) \in \Sigma C_p(\alpha)\,.$$

Furthermore, in view of the relationship (4), we find that

$$-\frac{z}{p}(\mathcal{J}^{s,\lambda}_{p,d} f(z))' \in \Sigma S^*_p(\alpha)\,,$$

that is, that

$$\mathcal{J}^{s,\lambda}_{p,d}\left(-\frac{zf'(z)}{p}\right) \in \Sigma S^*_p(\alpha)\,.$$

Thus, by Definition 1.1 and Theorem 2.1, we have

$$-\frac{z}{p}f'(z) \in \Sigma S^{*s,\lambda}_{p,d}(\alpha) \subset \Sigma S^{*s+1,\lambda}_{p,d}(\alpha)\ ,$$

which implies that

$$\Sigma C^{s,\lambda}_{p,d}(\alpha) \subset \Sigma C^{s+1,\lambda}_{p,d}(\alpha)\,.$$

The left part of Theorem 2.2 can be proved by using similar arguments. The proof of Theorem 2.2 is thus completed.

Theorem 4. *Let* $d_1 = Re(d) > \alpha - p, p \in \mathbb{N}$ *and* $0 \le \alpha,\ \beta < p$. *Then*

$$\Sigma K^{s,\lambda+1}_{p,d}(\beta,\alpha) \subset \Sigma K^{s,\lambda}_{p,d}(\beta,\alpha) \subset \Sigma K^{s+1,\lambda}_{p,d}(\beta,\alpha)\,.$$

Proof. Let us begin by proving that

$$\Sigma K_{p,d}^{s,\lambda}(\beta,\alpha) \subset \Sigma K_{p,d}^{s+1,\lambda}(\beta,\alpha). \tag{21}$$

Let $f(z) \in \Sigma K_{p,d}^{s,\lambda}(\beta,\alpha)$. Then there exists a function $\Psi(z) \in \Sigma S_p^*(\alpha)$ such that

$$Re\left(\frac{z(\mathcal{J}_{p,d}^{s,\lambda}f(z))'}{\Psi(z)}\right) < -\beta \quad (z \in U^*)\,.$$

We put

$$\mathcal{J}_{p,d}^{s,\lambda}g(z) = \Psi(z)\ ,$$

so that we have

$$g(z) \in \Sigma S_{p,d}^{*s,\lambda}(\alpha) \text{ and } Re\left(\frac{z(\mathcal{J}_{p,d}^{s,\lambda}f(z))'}{\mathcal{J}_{p,d}^{s,\lambda}g(z)}\right) < -\beta \quad (z \in U^*)\,.$$

We next put

$$\frac{z(\mathcal{J}_{p,d}^{s+1,\lambda}f(z))'}{\mathcal{J}_{p,d}^{s+1,\lambda}g(z)} = -\beta-(p-\beta)q(z)\,, \tag{22}$$

where $q(z)$ is given by (15). Thus, by using the identity (10), we obtain

$$\begin{aligned}
\frac{z(\mathcal{J}_{p,d}^{s,\lambda}f(z))'}{\mathcal{J}_{p,d}^{s,\lambda}g(z)} &= \frac{(\mathcal{J}_{p,d}^{s,\lambda}(zf'(z)))}{\mathcal{J}_{p,d}^{s,\lambda}g(z)} \\
&= \frac{z\left[\mathcal{J}_{p,d}^{s+1,\lambda}(zf'(z))\right]'+(d+p)\mathcal{J}_{p,d}^{s+1,\lambda}(zf'(z))}{z(\mathcal{J}_{p,d}^{s+1,\lambda}g(z))'+(d+p)\mathcal{J}_{p,d}^{s+1,\lambda}g(z)} \\
&= \frac{\dfrac{z\left[\mathcal{J}_{p,d}^{s,\lambda}(zf'(z))\right]'}{\mathcal{J}_{p,d}^{s+1,\lambda}g(z)}+(d+p)\dfrac{\mathcal{J}_{p,d}^{s+1,\lambda}(zf'(z))}{\mathcal{J}_{p,d}^{s+1,\lambda}g(z)}}{\dfrac{z(\mathcal{J}_{p,d}^{s+1,\lambda}g(z))'}{\mathcal{J}_{p,d}^{s+1,\lambda}g(z)}+(d+p)}\,.
\end{aligned}$$

Since $g(z) \in \Sigma S_{p,d}^{*s,\lambda}(\alpha)$, by Theorem 2.1, we can put

$$\frac{z(\mathcal{J}_{p,d}^{s+1,\lambda}(g(z)))'}{\mathcal{J}_{p,d}^{s+1,\lambda}g(z)} = -\alpha-(p-\alpha)G(z)\,,$$

where

$$G(z) = g_1(x,y)+ig_2(x,y) \text{ and } Re(G(z)) = g_1(x,y) > 0 \quad (z \in U)\,.$$

Then

$$\frac{z(\mathcal{J}^{s,\lambda}_{p,d}f(z))'}{\mathcal{J}^{s,\lambda}_{p,d}g(z)} = \frac{\dfrac{z\left[\mathcal{J}^{s,\lambda}_{p,d}(zf'(z))\right]'}{\mathcal{J}^{s,\lambda}_{p,d}g(z)} - (d+p)\left[\beta+(p-\beta)q(z)\right]}{-\alpha-(p-\alpha)G(z)+(d+p)} \,. \tag{23}$$

We thus find from (22) that

$$z(\mathcal{J}^{s+1,\lambda}_{p,d}f(z))' = -\mathcal{J}^{s+1,\lambda}_{p,d}g(z)\left[\beta+(p-\beta)q(z)\right] \,. \tag{24}$$

Differentiating both sides of (24) with respect to z, we obtain

$$\frac{z\left[\mathcal{J}^{s+1,\lambda}_{p,d}zf'(z))\right]'}{\mathcal{J}^{s+1,\lambda}_{p,d}g(z)} = -(p-\beta)zq'(z) + \left[\alpha+(p-\alpha)G(z)\right]\left[\beta+(p-\beta)q(z)\right] \,. \tag{25}$$

By substituting (25) into (23), we have

$$\frac{z(\mathcal{J}^{s,\lambda}_{p,d}f(z))'}{\mathcal{J}^{s,\lambda}_{p,d}g(z)} + \beta = -\left\{(p-\beta)q(z) - \frac{(p-\beta)zq'(z)}{(p-\alpha)G(z)+\alpha-(d+p)}\right\} \,.$$

Taking $u = q(z) = u_1 + iu_2$ and $v = zq'(z) = v_1 + iv_2$, we define the function $\Phi(u,v)$ by

$$\Phi(u,v) = (p-\beta)u - \frac{(p-\beta)v}{(p-\alpha)G(z)+\alpha-(d+p)} \,, \tag{26}$$

where $(u,v) \in D = (\mathbb{C}\backslash D^*) \times \mathbb{C}$ and

$$D^* = \left\{z : z \in \mathbb{C} \text{ and } Re(G(z)) = g_1(x,y) \geq 1 + \frac{Re(d)}{p-\alpha}\right\} \,.$$

Then it follows from (26) that

(i) $\Phi(u,v)$ is continuous in D;

(ii) $(1,0) \in D$ and $Re\left\{\Phi(1,0)\right\} = p-\beta > 0$;

(iii) for all $(iu_2, v_1) \in D$ such that $v_1 \leq -\frac{1}{2}(1+u_2^2)$, we have

$$Re\left\{\Phi(iu_2,v_1)\right\} = Re\left\{-\frac{(p-\beta)v_1}{(p-\alpha)G(z)+\alpha-d-p}\right\}$$

$$\begin{aligned}
&= \frac{(p-\beta)v_1\left[d_1+p-\alpha-(p-\alpha)g_1(x,y)\right]}{\left[(p-\alpha)g_1(x,y)+\alpha-d_1-p\right]^2+\left[(p-\alpha)g_2(x,y)-d_2\right]^2} \\
&\leq -\frac{(p-\beta)(1+u_2^2)\left[d_1+p-\alpha-(p-\alpha)g_1(x,y)\right]}{2\left[(p-\alpha)g_1(x,y)+\alpha-d_1-p\right]^2+2\left[(p-\alpha)g_2(x,y)-d_2\right]^2} \\
&< 0 \,,
\end{aligned}$$

which shows that $\Phi(u,v)$ satisfies the hypotheses of Lemma 1.1. Thus, in light of (22), we easily deduce the inclusion relationship (21).

The remainder of our proof of Theorem 2.3 would make use of the identity (9) in an analogous manner and assume that

$$D^{*}=\left\{z : z\in\mathbb{C} \text{ and } Re(G(z))=g_1(x,y)\geq 1+\frac{\lambda}{p-\alpha}\right\}.$$

We, therefore, choose to omit details involved.

Theorem 5. *Let $d_1=Re(d)>\alpha-p, p\in\mathbb{N}, 0\leq\alpha$ and $\beta<p$. Then*

$$\Sigma K_{p,d}^{*s,\lambda+1}(\beta,\alpha)\subset\Sigma K_{p,d}^{*s,\lambda}(\beta,\alpha)\subset\Sigma K_{p,d}^{*s+1,\lambda}(\beta,\alpha).$$

Proof. Just as we derived Theorem 2.2 as a consequence of Theorem 2.1 by using the equivalence (4), we can also prove Theorem 2.4 by using Theorem 2.3 in conjunction with the equivalence (7).

3. A set of integral-preserving properties

In this section, we present several integral-preserving properties of the meromorphic function classes introduced here. We first recall a familiar integral operator $J_{c,p}$ defined by (see [10])

$$J_{c,p}(f)(z)=\frac{c}{z^{c+p}}\int\limits_0^z t^{c+p-1}f(t)dt \quad (c>0;p\in\mathbb{N};f\in\Sigma_p), \tag{27}$$

which satisfies the following relationship :

$$z\left(\mathcal{J}_{p,d}^{s,\lambda}J_{c,p}(f)(z)\right)^{'}=c\mathcal{J}_{p,d}^{s,\lambda}f(z)-(c+p)\mathcal{J}_{p,d}^{s,\lambda}J_{c,p}(f)(z)\ . \tag{28}$$

In order to obtain the integral-preserving properties involving the integral operator $J_{c,p}$, we also need the following lemma which is popularly known as Jack's lemma.

Lemma 6. *[8]. Let $\omega(z)$ be a non-constant function analytic in U^{*} with $\omega(0)=0$. If $|\omega(z)|$ attains its maximum value on the circle $|z|=r<1$ at z_0, then*

$$z_0\omega^{'}(z_0)=\zeta\omega(z_0),$$

where $\zeta\geq 1$ is a real number.

Theorem 7. *Let $c > 0, p \in \mathbb{N}$ and $0 \leq \alpha < p$. If $f(z) \in \Sigma S^{*s,\lambda}_{p,d}(\alpha)$, then*

$$J_{c,p}(f)(z) \in \Sigma S^{*s,\lambda}_{p,d}(\alpha).$$

Proof. Suppose that $f(z) \in \Sigma S^{*s,\lambda}_{p,d}(\alpha)$ and let

$$\frac{z\left(\mathcal{J}^{s,\lambda}_{p,d} J_{c,p}(f)(z)\right)'}{\mathcal{J}^{s,\lambda}_{p,d} J_{c,p}(f)(z)} = -\frac{p + (p-2\alpha)\omega(z)}{1-\omega(z)}\,, \tag{29}$$

where $\omega(0) = 0$. Then, by using (28) and (29), we have

$$\frac{\mathcal{J}^{s,\lambda}_{p,d} f(z)}{\mathcal{J}^{s,\lambda}_{p,d} J_{c,p}(f)(z)} = \frac{c - (c+2p-2\alpha)\omega(z)}{c(1-\omega(z))}\,. \tag{30}$$

Differentiating (30) logarithmically with respect to z, we obtain

$$\begin{aligned} \frac{z\left(\mathcal{J}^{s,\lambda}_{p,d} f(z)\right)'}{\mathcal{J}^{s,\lambda}_{p,d} f(z)} &= -\frac{p + (p-2\alpha)\omega(z)}{1-\omega(z)} + \frac{z\omega'(z)}{1-\omega(z)} \\ &\quad - \frac{(c+2p-2\alpha)z\omega'(z)}{c-(c+2p-2\alpha)\omega(z)}\,, \end{aligned} \tag{31}$$

so that

$$\begin{aligned} \frac{z\left(\mathcal{J}^{s,\lambda}_{p,d} f(z)\right)'}{\mathcal{J}^{s,\lambda}_{p,d} f(z)} + \alpha &= \frac{(\alpha-p)(1+\omega(z))}{1-\omega(z)} + \frac{z\omega'(z)}{1-\omega(z)} \\ &\quad - \frac{(c+2p-2\alpha)z\omega'(z)}{c-(c+2p-2\alpha)\omega(z)}\,. \end{aligned} \tag{32}$$

Now, assuming that $\max\limits_{|z|\leq|z_0|} |\omega(z)| = |\omega(z_0)| = 1\ (z_0 \in U)$ and applying Jack's lemma, we have

$$z_0\omega'(z_0) = \zeta\omega(z_0) \quad (\zeta \geq 1)\,. \tag{33}$$

If we set $\omega(z_0) = e^{i\theta}\ (\theta \in R)$ in (32) and observe that

$$Re\left\{\frac{(\alpha-p)(1+\omega(z_0))}{1-\omega(z_0)}\right\} = 0\,,$$

then we obtain

$$\begin{aligned} Re\left\{\frac{z_0\left(\mathcal{J}^{s,\lambda}_{p,d}f(z_0)\right)'}{\mathcal{J}^{s,\lambda}_{p,d}f(z_0)}+\alpha\right\} &= Re\left\{\frac{z_0\omega'(z_0)}{1-\omega(z_0)}-\frac{(c+2p-2\alpha)z_0\omega'(z_0)}{c-(c+2p-2\alpha)\omega(z_0)}\right\} \\ &= Re\left\{-\frac{2(p-\alpha)\zeta e^{i\theta}}{(1-e^{i\theta})\left[c-(c+2p-2\alpha)e^{i\theta}\right]}\right\} \\ &= \frac{2\zeta(p-\alpha)(c+p-\alpha)}{c^2-2c(c+2p-2\alpha)\cos\theta+(c+2p-2\alpha)^2} \\ &\geq 0\,, \end{aligned}$$

which obviously contradicts the hypothesis $f(z)\in\Sigma S^{*s,\lambda}_{p,d}(\alpha)$. Consequently, we can deduce that $|\omega(z)|<1\,(z\in U)$, which, in view of (29), proves the integral-preserving property asserted by Theorem 3.1.

Theorem 8. *Let $c>0, p\in\mathbb{N}$ and $0\leq\alpha<p$. If $f(z)\in\Sigma C^{s,\lambda}_{p,d}(\alpha)$, then*

$$J_{c,p}(f)(z)\in\Sigma C^{s,\lambda}_{p,d}(\alpha)\,.$$

Proof. By applying Theorem 3.1, it follows that

$$\begin{aligned} f(z) &\in \Sigma C^{s,\lambda}_{p,d}(\alpha)\Leftrightarrow\frac{-zf'(z)}{p}\in\Sigma S^{*s,\lambda}_{p,d}(\alpha) \\ &\Rightarrow J_{c,p}\left(\frac{-zf'(z)}{p}\right)\in\Sigma S^{*s,\lambda}_{p,d}(\alpha) \\ &\Leftrightarrow -\frac{z}{p}\left(J_{c,p}f(z)\right)'\in\Sigma S^{*s,\lambda}_{p,d}(\alpha) \\ &\Rightarrow J_{c,p}(f)(z)\in\Sigma C^{s,\lambda}_{p,d}(\alpha)\,, \end{aligned}$$

which proves Theorem 3.2.

Theorem 9. *Let $c>0, p\in\mathbb{N}$ and $0\leq\alpha,\beta<p$. If $f(z)\in\Sigma K^{s,\lambda}_{p,d}(\beta,\alpha)$, then*

$$J_{c,p}(f)(z)\in\Sigma K^{s,\lambda}_{p,d}(\beta,\alpha)\,.$$

Proof. Suppose that $f(z)\in\Sigma K^{s,\lambda}_{p,d}$. Then, by Definition 1.3, there exists a function $g(z)\in\Sigma S^{*s,\lambda}_{p,d}$ such that

$$Re\left(\frac{z\left(\mathcal{J}^{s,\lambda}_{p,d}f(z)\right)'}{\mathcal{J}^{s,\lambda}_{p,d}g(z)}\right)<-\beta\quad(z\in U^*)\,.$$

Thus, upon setting

$$\frac{z\left(\mathcal{J}_{p,d}^{s,\lambda}J_{c,p}f(z)\right)'}{\mathcal{J}_{p,d}^{s,\lambda}J_{c,p}g(z)}+\beta=-(p-\beta)q(z)\,, \tag{34}$$

where $q(z)$ is given by (15), we find from (28) that

$$\begin{aligned}\frac{z\left(\mathcal{J}_{p,d}^{s,\lambda}f(z)\right)'}{\mathcal{J}_{p,d}^{s,\lambda}g(z)} &= -\frac{\mathcal{J}_{p,d}^{s,\lambda}(-zf'(z))}{\mathcal{J}_{p,d}^{s,\lambda}g(z)}\\ &= -\frac{(c+p)\mathcal{J}_{p,d}^{s,\lambda}J_{c,p}(-zf'(z))+z(\mathcal{J}_{p,d}^{s,\lambda}J_{c,p}(-zf'(z)))'}{(c+p)\mathcal{J}_{p,d}^{s,\lambda}J_{c,p}g(z)+z(\mathcal{J}_{p,d}^{s,\lambda}J_{c,p}(g(z)))'}\\ &= -\frac{\frac{z(\mathcal{J}_{p,d}^{s,\lambda}J_{c,p}(-zf'(z)))'}{\mathcal{J}_{p,d}^{s,\lambda}J_{c,p}g(z)}+(c+p)\frac{\mathcal{J}_{p,d}^{s,\lambda}J_{c,p}(-zf'(z))}{\mathcal{J}_{p,d}^{s,\lambda}J_{c,p}g(z)}}{\frac{z(\mathcal{J}_{p,d}^{s,\lambda}J_{c,p}g(z))'}{\mathcal{J}_{p,d}^{s,\lambda}J_{c,p}g(z)}+(c+p)}\,.\end{aligned}$$

Since $g(z)\in\Sigma S_{p,d}^{*s,\lambda}$, we know from Theorem 3.1 that $J_{c,p}g(z)\in\Sigma S_{p,d}^{*s,\lambda}$. So we can set

$$\frac{z(\mathcal{J}_{p,d}^{s,\lambda}J_{c,p}g(z))'}{\mathcal{J}_{p,d}^{s,\lambda}J_{c,p}g(z)}+\alpha=-(p-\alpha)G(z)\,, \tag{35}$$

where

$$G(z)=g_1(x,y)+ig_2(x,y) \text{ and } Re(G(z))=g_1(x,y)>0 \quad (z\in U)\,.$$

Then we have

$$\frac{z(\mathcal{J}_{p,d}^{s,\lambda}f(z))'}{\mathcal{J}_{p,d}^{s,\lambda}g(z)}=\frac{\frac{z(\mathcal{J}_{p,d}^{s,\lambda}J_{c,p}(-zf'(z)))'}{\mathcal{J}_{p,d}^{s,\lambda}J_{c,p}g(z)}+(c+p)\left[\beta+(p-\beta)q(z)\right]}{\alpha+(p-\alpha)G(z)-(c+p)}\,. \tag{36}$$

We also find from (34) that

$$z(\mathcal{J}_{p,d}^{s,\lambda}J_{c,p}f(z))'=(-\mathcal{J}_{p,d}^{s,\lambda}J_{c,p}g(z))\left[\beta+(p-\beta)q(z)\right]\,. \tag{37}$$

Differentiating both sides of (37) with respect to z, we obtain

$$\begin{aligned}z\left[z\left(\mathcal{J}_{p,d}^{s,\lambda}J_{c,p}f(z)\right)'\right]' &= -z(\mathcal{J}_{p,d}^{s,\lambda}J_{c,p}g(z))'\left[\beta+(p-\beta)q(z)\right]\\ &\quad -(p-\beta)zq'(z)\mathcal{J}_{p,d}^{s,\lambda}J_{c,p}g(z)\,,\end{aligned} \tag{38}$$

that is,

$$\frac{z\left[z\left(I_{p,\mu}^{m}(\lambda,\ell)J_{c,p}f(z)\right)'\right]'}{I_{p,\mu}^{m}(\lambda,\ell)J_{c,p}g(z)} = -(p-\beta)zq'(z)$$
$$+\left[\alpha+(p-\alpha)G(z)\right]\left[\beta+(p-\beta)q(z)\right] \ . \tag{39}$$

Substituting (39) into (36), we find that

$$\frac{z\left(\mathcal{J}_{p,d}^{s,\lambda}f(z)\right)'}{\mathcal{J}_{p,d}^{s,\lambda}g(z)}+\beta = -(p-\beta)q(z)+\frac{(p-\beta)zq'(z)}{(p-\alpha)G(z)+\alpha-(c+p)} \,. \tag{40}$$

Then, by setting

$$u = q(z) = u_1 + iu_2 \ \text{ and } \ v = zq'(z) = v_1 + iv_2 \ ,$$

we can define the function $\theta(u,v)$ by

$$\theta(u,v) = (p-\beta)u - \frac{(p-\beta)v}{(p-\alpha)G(z)+\alpha-(c+p)} \,.$$

The remainder of our proof of Theorem 3.3 is similar to that of Theorem 2.3, so we choose to omit the analogous details involved.

Theorem 10. *Let $c>0, p\in\mathbb{N}$ and $0\le\alpha$, $\beta<p$. If $f(z)\in\Sigma K_{p,d}^{*s,\lambda}(\beta,\alpha)$, then*

$$J_{c,p}(f)(z)\in\Sigma K_{p,d}^{*s,\lambda}(\beta,\alpha)\,.$$

Proof. Just as we derived Theorem 3.2 from Theorem 3.1, we easily deduce the integral-preserving property asserted by Theorem 3.4 from Theorem 3.3.

Remark 1. *Putting $p=1$, replacing $a=\lambda(\lambda>0)$ and $c=1$ in the above results we obtain results obtained by El-Ashwah [4].*

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