

GENERALIZATION OF POPOVICIU INEQUALITY FOR HIGHER ORDER CONVEX FUNCTIONS VIA TAYLOR POLYNOMIAL

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ABSTRACT. We obtained useful identities via Taylor polynomial, by which the inequality of Popoviciu for convex functions is generalized for higher order convex functions. We investigate the bounds for the identities related to the generalization of the Popoviciu inequality using inequalities for the Čebyšev functional. Some results relating to the Grüss and Ostrowski type inequalities are constructed. Further, we also construct new families of exponentially convex functions and Cauchy-type means by looking at linear functionals associated with the obtained inequalities.

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1. INTRODUCTION AND PRELIMINARY RESULTS

The theory of convex functions has experienced a rapid development. This can be attributed to several causes: firstly, so many areas in modern analysis directly or indirectly involve the application of convex functions; secondly, convex functions are closely related to the theory of inequalities and many important inequalities are consequences of the applications of convex functions (see [10]).

Definition 1. A function $f : I \rightarrow \mathbb{R}$ is convex on I if

$$(x_3 - x_2) f(x_1) + (x_1 - x_3) f(x_2) + (x_2 - x_1) f(x_3) \geq 0 \quad (1)$$

holds for all $x_1, x_2, x_3 \in I$ such that $x_1 < x_2 < x_3$.

An important characterization of convex function is stated in [10, p. 2].

Theorem 1.1. *If f is a convex function defined on I and if $x_1 \leq y_1$, $x_2 \leq y_2$, $x_1 \neq x_2$, $y_1 \neq y_2$, then the following inequality is valid*

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} \leq \frac{f(y_2) - f(y_1)}{y_2 - y_1}. \quad (2)$$

If the function f is concave, then the inequality reverses.

Divided differences are found to be very helpful when we are dealing with functions having different degrees of smoothness. The following definition of divided difference is given in [10, p. 14].

Definition 2. *The m th-order divided difference of a function $f : [a, b] \rightarrow \mathbb{R}$ at mutually distinct points $x_0, \dots, x_m \in [a, b]$ is defined recursively by*

$$\begin{aligned} [x_i; f] &= f(x_i), \quad i = 0, \dots, m, \\ [x_0, \dots, x_m; f] &= \frac{[x_1, \dots, x_m; f] - [x_0, \dots, x_{m-1}; f]}{x_m - x_0}. \end{aligned} \quad (3)$$

It is easy to see that (3) is equivalent to

$$[x_0, \dots, x_m; f] = \sum_{i=0}^m \frac{f(x_i)}{q'(x_i)}, \quad \text{where } q(x) = \prod_{j=0}^m (x - x_j).$$

The following definition of a real valued convex function is characterized by m th-order divided difference (see [10, p. 15]).

Definition 3. *A function $f : [a, b] \rightarrow \mathbb{R}$ is said to be m -convex ($m \geq 0$) if and only if for all choices of $(m + 1)$ distinct points $x_0, \dots, x_m \in [a, b]$, $[x_0, \dots, x_m; f] \geq 0$ holds.*

If this inequality is reversed, then f is said to be m -concave. If the inequality is strict, then f is said to be a strictly m -convex (m -concave) function.

Remark 1.2. *Note that 0-convex functions are non-negative functions, 1-convex functions are increasing functions and 2-convex functions are simply the convex functions.*

The following theorem gives an important criteria to examine the m -convexity of a function f (see [10, p. 16]).

Theorem 1.3. *If $f^{(m)}$ exists, then f is m -convex if and only if $f^{(m)} \geq 0$.*

In 1965, T. Popoviciu introduced a characterization of convex function [11]. In 1976, Vasić and Stanković [12] (see also page 173 [10]) gave the weighted version. In recent years that inequality of Popoviciu is studied in [3, 6, 7, 8, 9].

Theorem 1.4. *Let $n, k \in \mathbb{N}$, $n \geq 3$, $2 \leq k \leq n - 1$, $[\alpha, \beta] \subset \mathbb{R}$, $\mathbf{x} = (x_1, \dots, x_n) \in [\alpha, \beta]^n$, $\mathbf{p} = (p_1, \dots, p_n)$ be a positive n -tuple such that $\sum_{i=1}^n p_i = 1$. Also let $f : [\alpha, \beta] \rightarrow \mathbb{R}$ be a convex function. Then*

$$p_{k,n}(\mathbf{x}, \mathbf{p}; f) \leq \frac{n-k}{n-1} p_{1,n}(\mathbf{x}, \mathbf{p}; f) + \frac{k-1}{n-1} p_{n,n}(\mathbf{x}, \mathbf{p}; f), \quad (4)$$

where

$$p_{k,n}(\mathbf{x}, \mathbf{p}; f) = p_{k,n}(\mathbf{x}, \mathbf{p}; f(x)) := \frac{1}{C_{k-1}^{n-1}} \sum_{1 \leq i_1 < \dots < i_k \leq n} \left(\sum_{j=1}^k p_{i_j} \right) f \left(\frac{\sum_{j=1}^k p_{i_j} x_{i_j}}{\sum_{j=1}^k p_{i_j}} \right)$$

is the linear functional with respect to f .

By inequality (4), we write

$$\Upsilon(\mathbf{x}, \mathbf{p}; f) := \frac{n-k}{n-1} p_{1,n}(\mathbf{x}, \mathbf{p}; f) + \frac{k-1}{n-1} p_{n,n}(\mathbf{x}, \mathbf{p}; f) - p_{k,n}(\mathbf{x}, \mathbf{p}; f). \quad (5)$$

Remark 1.5. *It is important to note that under the assumptions of Theorem 1.4, if the function f is convex then $\Upsilon(\mathbf{x}, \mathbf{p}; f) \geq 0$ and $\Upsilon(\mathbf{x}, \mathbf{p}; f) = 0$ for $f(x) = x$ or f is a constant function.*

The mean value theorems and exponential convexity of the linear functional $\Upsilon(\mathbf{x}, \mathbf{p}; f)$ are given in [6] for a positive n -tuple \mathbf{p} . Some special classes of convex functions are considered to construct the exponential convexity of $\Upsilon(\mathbf{x}, \mathbf{p}; f)$ in [6]. In [7] (see also [3]), the results related to $\Upsilon(\mathbf{x}, \mathbf{p}; f)$ are generalized with help of Green function and m -exponential convexity is proved instead of exponential convexity.

In section 2 of this paper, we use Taylor's formula to generalize the Popoviciu inequality. In section 3, the Čebyšev functional is used to find the bounds for new identities. Grüss and Ostrowski type inequalities related to generalized Popoviciu inequalities are constructed. In section 4, higher order convexity is used to produce exponential convexity of positive linear functionals coming from section 2. Last section is devoted to the respective Cauchy means. We employ the similar method as adopted in [5] for Steffensen's inequality.

Let us define the real valued function

$$(x-t)_+ = \begin{cases} (x-t), & t \leq x, \\ 0, & t > x. \end{cases} \quad (6)$$

The well known Taylor's formula is as follows:

Let m be a positive integer and $\phi : [a, b] \rightarrow \mathbb{R}$ be such that $\phi^{(m-1)}$ is absolutely continuous, then for all $x \in [a, b]$ the Taylor's formula at the point $c \in [a, b]$ is

$$\phi(x) = T_{m-1}(\phi; c, x) + R_{m-1}(\phi; c, x) \quad (7)$$

where $T_{m-1}(\phi; c, x)$ is a Taylor's polynomial of degree $m - 1$, i.e.

$$T_{m-1}(\phi; c, x) = \sum_{k=0}^{m-1} \frac{\phi^{(k)}(c)}{k!} (x - c)^k$$

and the remainder is given by

$$R_{m-1}(\phi; c, x) = \frac{1}{(m-1)!} \int_c^x \phi^{(m)}(t) (x-t)^{m-1} dt.$$

Applying Taylor's formula at the points a and b respectively. We get

$$\phi(x) = \sum_{k=0}^{m-1} \frac{\phi^{(k)}(a)}{k!} (x-a)^k + \frac{1}{(m-1)!} \int_a^b \phi^{(m)}(t) ((x-t)_+^{m-1}) dt, \quad (8)$$

$$\phi(x) = \sum_{k=0}^{m-1} \frac{\phi^{(k)}(b)}{k!} (b-x)^k (-1)^k - \frac{1}{(m-1)!} \int_a^b (-1)^{m-1} \phi^{(m)}(t) ((t-x)_+^{m-1}) dt. \quad (9)$$

2. GENERALIZATION OF POPOVICIU'S INEQUALITY

Motivated by identity (5), we construct the following identities with help of (8) and (9) coming from Taylor's formula.

Theorem 2.1. *Let m be positive integer, $\phi : [\alpha, \beta] \rightarrow \mathbb{R}$ be such that $\phi^{(m-1)}$ is absolutely continuous and let $n, k \in \mathbb{N}$, $n \geq 3$, $2 \leq k \leq n - 1$, $[\alpha, \beta] \subset \mathbb{R}$, $\mathbf{x} = (x_1, \dots, x_n) \in [\alpha, \beta]^n$, $\mathbf{p} = (p_1, \dots, p_n)$ be a real n -tuple such that $\sum_{j=1}^k p_{i_j} \neq 0$*

for any $1 \leq i_1 < \dots < i_k \leq n$ and $\sum_{i=1}^n p_i = 1$. Also let $\frac{\sum_{j=1}^k p_{i_j} x_{i_j}}{\sum_{j=1}^k p_{i_j}} \in [\alpha, \beta]$ for any

$1 \leq i_1 < \dots < i_k \leq n$. Then we have the following identities:

$$\frac{n-k}{n-1} p_{1,n}(\mathbf{x}, \mathbf{p}; \phi(x)) + \frac{k-1}{n-1} p_{n,n}(\mathbf{x}, \mathbf{p}; \phi(x)) - p_{k,n}(\mathbf{x}, \mathbf{p}; \phi(x)) \quad (\text{T1})$$

$$= \sum_{z=2}^{m-1} \frac{\phi^{(z)}(\alpha)}{z!} \left(\frac{n-k}{n-1} p_{1,n}(\mathbf{x}, \mathbf{p}; (x-\alpha)^z) + \frac{k-1}{n-1} p_{n,n}(\mathbf{x}, \mathbf{p}; (x-\alpha)^z) - p_{k,n}(\mathbf{x}, \mathbf{p}; (x-\alpha)^z) \right)$$

$$\begin{aligned}
 & + \frac{1}{(m-1)!} \int_{\alpha}^{\beta} \left[\frac{n-k}{n-1} p_{1,n}(\mathbf{x}, \mathbf{p}; (x-t)_+^{m-1}) \right. \\
 & \qquad \qquad \qquad \left. + \frac{k-1}{n-1} p_{n,n}(\mathbf{x}, \mathbf{p}; (x-t)_+^{m-1}) - p_{k,n}(\mathbf{x}, \mathbf{p}; (x-t)_+^{m-1}) \right] \phi^{(m)}(t) dt.
 \end{aligned}$$

And

$$\frac{n-k}{n-1} p_{1,n}(\mathbf{x}, \mathbf{p}; \phi(x)) + \frac{k-1}{n-1} p_{n,n}(\mathbf{x}, \mathbf{p}; \phi(x)) - p_{k,n}(\mathbf{x}, \mathbf{p}; \phi(x)) \tag{T2}$$

$$= \sum_{z=2}^{m-1} \frac{(-1)^z \phi^{(z)}(\beta)}{z!} \left(\frac{n-k}{n-1} p_{1,n}(\mathbf{x}, \mathbf{p}; (\beta-x)^z) + \frac{k-1}{n-1} p_{n,n}(\mathbf{x}, \mathbf{p}; (\beta-x)^z) - p_{k,n}(\mathbf{x}, \mathbf{p}; (\beta-x)^z) \right)$$

$$\begin{aligned}
 & - \frac{(-1)^{m-1}}{(m-1)!} \int_{\alpha}^{\beta} \left(\frac{n-k}{n-1} p_{1,n}(\mathbf{x}, \mathbf{p}; (t-x)_+^{m-1}) \right. \\
 & \qquad \qquad \qquad \left. + \frac{k-1}{n-1} p_{n,n}(\mathbf{x}, \mathbf{p}; (t-x)_+^{m-1}) - p_{k,n}(\mathbf{x}, \mathbf{p}; (t-x)_+^{m-1}) \right) \phi^{(m)}(t) dt
 \end{aligned}$$

Proof. Using (8) and (9) in (4) respectively, we get the required results.

OR

We can also write (T1) and (T2) as follows:

$$\begin{aligned}
 & \Upsilon(\mathbf{x}, \mathbf{p}; \phi(x)) \\
 & = \sum_{z=2}^{m-1} \frac{\phi^{(z)}(\alpha)}{z!} \Upsilon(\mathbf{x}, \mathbf{p}; (x-\alpha)^z) + \frac{1}{(m-1)!} \int_{\alpha}^{\beta} \phi^{(m)}(t) \Upsilon(\mathbf{x}, \mathbf{p}; (x-t)_+^{m-1}) dt,
 \end{aligned}$$

$$\begin{aligned}
 & \Upsilon(\mathbf{x}, \mathbf{p}; \phi(x)) \\
 & = \sum_{z=2}^{m-1} \frac{(-1)^z \phi^{(z)}(\beta)}{z!} \Upsilon(\mathbf{x}, \mathbf{p}; (\beta-x)^z) - \frac{(-1)^{m-1}}{(m-1)!} \int_{\alpha}^{\beta} \phi^{(m)}(t) \Upsilon(\mathbf{x}, \mathbf{p}; (t-x)_+^{m-1}) dt.
 \end{aligned}$$

In the following theorem we obtain generalizations of Popoviciu's inequality for m -convex functions.

Theorem 2.2. Let m be positive integer, $\phi : [\alpha, \beta] \rightarrow \mathbb{R}$ be such that $\phi^{(m-1)}$ is absolutely continuous and let $n, k \in \mathbb{N}$, $n \geq 3$, $2 \leq k \leq n - 1$, $[\alpha, \beta] \subset \mathbb{R}$, $\mathbf{x} = (x_1, \dots, x_n) \in [\alpha, \beta]^n$, $\mathbf{p} = (p_1, \dots, p_n)$ be a real n -tuple such that $\sum_{j=1}^k p_{i_j} \neq 0$

for any $1 \leq i_1 < \dots < i_k \leq n$ and $\sum_{i=1}^n p_i = 1$. Also let $\frac{\sum_{j=1}^k p_{i_j} x_{i_j}}{\sum_{j=1}^k p_{i_j}} \in [\alpha, \beta]$ for any $1 \leq i_1 < \dots < i_k \leq n$. Then

(i) If ϕ is m -convex function and

$$\Upsilon(\mathbf{x}, \mathbf{p}; (x - t)_+^{m-1}) \geq 0, \quad t \in [\alpha, \beta], \quad (10)$$

then

$$\Upsilon(\mathbf{x}, \mathbf{p}; \phi(x)) \geq \sum_{z=2}^{m-1} \frac{\phi^{(z)}(\alpha)}{z!} \Upsilon(\mathbf{x}, \mathbf{p}; (x - \alpha)^z) \quad (11)$$

(ii) If ϕ is m -convex function and

$$(-1)^{m-1} \Upsilon(\mathbf{x}, \mathbf{p}; (t - x)_+^{m-1}) \leq 0, \quad t \in [\alpha, \beta], \quad (12)$$

then

$$\Upsilon(\mathbf{x}, \mathbf{p}; \phi(x)) \geq \sum_{z=2}^{m-1} \frac{(-1)^z \phi^{(z)}(\beta)}{z!} \Upsilon(\mathbf{x}, \mathbf{p}; (\beta - x)^z) \quad (13)$$

Proof. Since $\phi^{(m-1)}$ is absolutely continuous on $[\alpha, \beta]$, $\phi^{(m)}$ exists almost everywhere. As ϕ is m -convex, applying Theorem 1.3, we have, $\phi^{(m)} \geq 0$ for all $x \in [\alpha, \beta]$. Hence we can apply Theorem 2.1 to obtain (11) and (13) respectively.

Corollary 2.3. Let $m \geq 2$ and all the assumptions of Theorem 2.1 be satisfied in addition with the condition that $\mathbf{p} = (p_1, \dots, p_m)$ be a positive m -tuple such that $\sum_{i=1}^m p_i = 1$. Then

- (i) If ϕ is m -convex function, then (11) holds. Moreover if $\phi^{(z)}(\alpha) \geq 0$ for $z = 2, \dots, m - 1$, the R.H.S. of (11) will be non negative.
- (ii) If m is even ϕ is m -convex function, then (13) holds. Moreover if $\phi^{(z)}(\beta) \geq 0$ for $z = 2, \dots, m - 2$, and $\phi^{(z)}(\beta) \leq 0$ for $z = 3, \dots, m - 1$, the R.H.S. of (13) will be non negative.
- (iii) If m is odd ϕ is m -convex function, then reverse of (13) holds. Moreover if $\phi^{(z)}(\beta) \leq 0$ for $z = 2, \dots, m - 1$, and $\phi^{(z)}(\beta) \geq 0$ for $z = 3, \dots, m - 2$, the R.H.S. of (13) will be non positive.

Proof. (i) Since

$$\frac{d}{dx^2}(x-t)_+^{m-1} = \begin{cases} (m-1)(m-2)(x-t)^{m-3} \geq 0, & \text{for } t \leq x, \\ 0, & t > x. \end{cases}$$

So $(x-t)_+^{m-1}$ is convex. As weights are positive, therefore using Remark 1.5, we get

$$\Upsilon(\mathbf{x}, \mathbf{p}; (x-t)_+^{m-1}) \geq 0.$$

By following Theorem 2.2, we get (11). Also $\Upsilon(\mathbf{x}, \mathbf{p}; (x-\alpha)^z) \geq 0$ for $z = 2, \dots, m-1$ (by Remark 1.5), so it is simple to get the non negativity of R.H.S. of (11) by using the given conditions.

Similarly, we can prove (ii) and (iii) respectively.

3. BOUNDS FOR IDENTITIES RELATED TO GENERALIZATION OF POPOVICIU'S INEQUALITY

For two Lebesgue integrable functions $f, h : [\alpha, \beta] \rightarrow \mathbb{R}$, we consider the Čebyšev functional

$$\Delta(f, h) = \frac{1}{\beta - \alpha} \int_{\alpha}^{\beta} f(t)h(t)dt - \frac{1}{\beta - \alpha} \int_{\alpha}^{\beta} f(t)dt \cdot \frac{1}{\beta - \alpha} \int_{\alpha}^{\beta} h(t)dt.$$

In [2] the authors proved the following theorems:

Theorem 3.1. *Let $f : [\alpha, \beta] \rightarrow \mathbb{R}$ be a Lebesgue integrable function and $h : [\alpha, \beta] \rightarrow \mathbb{R}$ be an absolutely continuous function with $(\cdot - \alpha)(\beta - \cdot)[h']^2 \in L[\alpha, \beta]$. Then we have the inequality*

$$|\Delta(f, h)| \leq \frac{1}{\sqrt{2}} [\Delta(f, f)]^{\frac{1}{2}} \frac{1}{\sqrt{\beta - \alpha}} \left(\int_{\alpha}^{\beta} (x - \alpha)(\beta - x)[h'(x)]^2 dx \right)^{\frac{1}{2}}. \quad (14)$$

The constant $\frac{1}{\sqrt{2}}$ in (14) is the best possible.

Theorem 3.2. *Assume that $h : [\alpha, \beta] \rightarrow \mathbb{R}$ is monotonic nondecreasing on $[\alpha, \beta]$ and $f : [\alpha, \beta] \rightarrow \mathbb{R}$ be an absolutely continuous with $f' \in L_{\infty}[\alpha, \beta]$. Then we have the inequality*

$$|\Delta(f, h)| \leq \frac{1}{2(\beta - \alpha)} \|f'\|_{\infty} \int_{\alpha}^{\beta} (x - \alpha)(\beta - x)[h'(x)]^2 dh(x). \quad (15)$$

The constant $\frac{1}{2}$ in (15) is the best possible.

In the sequel, we consider above theorems to derive generalizations of the results proved in the previous section. In order to avoid many notions let us denote

$$\mathfrak{R}(t) = \Upsilon(\mathbf{x}, \mathbf{p}; (x - t)_+^{m-1}), \quad t \in [\alpha, \beta], \quad (16)$$

$$\hat{\mathfrak{R}}(t) = (-1)^{m-1} \Upsilon(\mathbf{x}, \mathbf{p}; (t - x)_+^{m-1}), \quad t \in [\alpha, \beta]. \quad (17)$$

Consider the Čebyšev functionals $\Delta(\mathfrak{R}, \mathfrak{R})$ and $\Delta(\hat{\mathfrak{R}}, \hat{\mathfrak{R}})$ are given by:

$$\Delta(\mathfrak{R}, \mathfrak{R}) = \frac{1}{\beta - \alpha} \int_{\alpha}^{\beta} \mathfrak{R}^2(t) dt - \left(\frac{1}{\beta - \alpha} \int_{\alpha}^{\beta} \mathfrak{R}(t) dt \right)^2, \quad (18)$$

$$\Delta(\hat{\mathfrak{R}}, \hat{\mathfrak{R}}) = \frac{1}{\beta - \alpha} \int_{\alpha}^{\beta} \hat{\mathfrak{R}}^2(t) dt - \left(\frac{1}{\beta - \alpha} \int_{\alpha}^{\beta} \hat{\mathfrak{R}}(t) dt \right)^2. \quad (19)$$

Theorem 3.3. *Let m be positive integer, $\phi : [\alpha, \beta] \rightarrow \mathbb{R}$ be such that $\phi^{(m)}$ is absolutely continuous with $(\cdot - \alpha)(\beta - \cdot)[\phi^{(m+1)}]^2 \in L[\alpha, \beta]$ and let $n, k \in \mathbb{N}$, $n \geq 3$, $2 \leq k \leq n - 1$, $[\alpha, \beta] \subset \mathbb{R}$, $\mathbf{x} = (x_1, \dots, x_n) \in [\alpha, \beta]^n$, $\mathbf{p} = (p_1, \dots, p_n)$ be a real n -tuple such that $\sum_{j=1}^k p_{i_j} \neq 0$ for any $1 \leq i_1 < \dots < i_k \leq n$ and $\sum_{i=1}^n p_i = 1$. Also*

let $\frac{\sum_{j=1}^k p_{i_j} x_{i_j}}{\sum_{j=1}^k p_{i_j}} \in [\alpha, \beta]$ for any $1 \leq i_1 < \dots < i_k \leq n$ and $\mathfrak{R}, \hat{\mathfrak{R}}$ be defined by (16), (17)

respectively . Then

(i)

$$\begin{aligned} \Upsilon(\mathbf{x}, \mathbf{p}; \phi(x)) &= \sum_{z=2}^{m-1} \frac{\phi^{(z)}(\alpha)}{z!} \Upsilon(\mathbf{x}, \mathbf{p}; (x - \alpha)^z) \\ &\quad + \frac{\phi^{(m-1)}(\beta) - \phi^{(m-1)}(\alpha)}{(\beta - \alpha)(m-1)!} \int_{\alpha}^{\beta} \mathfrak{R}(t) dt + \mathfrak{R}_m^1(\alpha, \beta; \phi), \end{aligned} \quad (20)$$

where the remainder $\mathfrak{R}_m^1(\alpha, \beta; \phi)$ satisfies the bound

$$|\mathfrak{R}_m^1(\alpha, \beta; \phi)| \leq \frac{\sqrt{\beta - \alpha}}{\sqrt{2}(m-1)!} [\Delta(\mathfrak{R}, \mathfrak{R})]^{1/2} \left| \int_{\alpha}^{\beta} (t - \alpha)(\beta - t) [\phi^{(m+1)}(t)]^2 dt \right|^{1/2}. \quad (21)$$

(ii)

$$\begin{aligned} \Upsilon(\mathbf{x}, \mathbf{p}; \phi(x)) &= \sum_{z=2}^{m-1} \frac{(-1)^z \phi^{(z)}(\beta)}{z!} \Upsilon(\mathbf{x}, \mathbf{p}; (\beta - x)^z) \\ &\quad + \frac{\phi^{(m-1)}(\beta) - \phi^{(m-1)}(\alpha)}{(\alpha - \beta)(m-1)!} \int_{\alpha}^{\beta} \hat{\mathfrak{R}}(t) dt - \mathfrak{R}_m^2(\alpha, \beta; \phi). \end{aligned} \quad (22)$$

where the remainder $\mathfrak{R}_m^2(\alpha, \beta; \phi)$ satisfies the bound

$$|\mathfrak{R}_m^2(\alpha, \beta; \phi)| \leq \frac{\sqrt{\beta - \alpha}}{\sqrt{2}(m-1)!} [\Delta(\hat{\mathfrak{R}}, \hat{\mathfrak{R}})]^{\frac{1}{2}} \left| \int_{\alpha}^{\beta} (t - \alpha)(\beta - t) [\phi^{(m+1)}(t)]^2 dt \right|^{\frac{1}{2}} \quad (23)$$

Proof. (i) If we apply Theorem 3.1 for $f \mapsto \mathfrak{R}$ and $h \mapsto \phi^{(m)}$, we get

$$\begin{aligned} & \left| \frac{1}{\beta - \alpha} \int_{\alpha}^{\beta} \mathfrak{R}(t) \phi^{(m)}(t) dt - \frac{1}{\beta - \alpha} \int_{\alpha}^{\beta} \mathfrak{R}(t) dt \cdot \frac{1}{\beta - \alpha} \int_{\alpha}^{\beta} \phi^{(m)}(t) dt \right| \\ & \leq \frac{1}{\sqrt{2}} [\Delta(\mathfrak{R}, \mathfrak{R})]^{\frac{1}{2}} \frac{1}{\sqrt{\beta - \alpha}} \left| \int_{\alpha}^{\beta} (t - \alpha)(\beta - t) [\phi^{(m+1)}(t)]^2 dt \right|^{\frac{1}{2}}. \end{aligned}$$

Hence, we have

$$\frac{1}{(m-1)!} \int_{\alpha}^{\beta} \mathfrak{R}(t) \phi^{(m)}(t) dt = \frac{\phi^{(m-1)}(\beta) - \phi^{(m-1)}(\alpha)}{(\beta - \alpha)(m-1)!} \int_{\alpha}^{\beta} \mathfrak{R}(t) dt + \mathfrak{R}_m^1(\alpha, \beta; \phi),$$

where the remainder $\mathfrak{R}_m^1(\alpha, \beta; \phi)$ satisfies the estimation (21). Now from identity T_1 , we obtain (20).

(ii) Similar to the above part.

The following Grüss type inequalities can be obtained by using Theorem 3.2

Theorem 3.4. *Let m be positive integer, $\phi : [\alpha, \beta] \rightarrow \mathbb{R}$ be such that $\phi^{(m)}$ is absolutely continuous function and $\phi^{(m+1)} \geq 0$ on $[\alpha, \beta]$ and let the functions $\mathfrak{R}, \hat{\mathfrak{R}}$ be defined by (16), (17) respectively. Then, we have*

(i) *the representation (20) and the remainder $\mathfrak{R}_m^1(\alpha, \beta; \phi)$ satisfies the estimation*

$$|\mathfrak{R}_m^1(\alpha, \beta; \phi)| \leq \frac{\beta - \alpha}{(m-1)!} \|\mathfrak{R}'\|_{\infty} \left[\frac{\phi^{(m-1)}(\beta) + \phi^{(m-1)}(\alpha)}{2} - \frac{\phi^{(m-2)}(\beta) - \phi^{(m-2)}(\alpha)}{\beta - \alpha} \right]. \quad (24)$$

(i) *The representation (22) and the remainder $\mathfrak{R}_m^2(\alpha, \beta; \phi)$ satisfies the estimation*

$$|\mathfrak{R}_m^2(\alpha, \beta; \phi)| \leq \frac{\beta - \alpha}{(m-1)!} \|\hat{\mathfrak{R}}'\|_{\infty} \left[\frac{\phi^{(m-1)}(\beta) + \phi^{(m-1)}(\alpha)}{2} - \frac{\phi^{(m-2)}(\beta) - \phi^{(m-2)}(\alpha)}{\beta - \alpha} \right]. \quad (25)$$

Proof. (i) Applying Theorem 3.2 for $f \mapsto \mathfrak{R}$ and $h \mapsto \phi^{(m)}$, we get

$$\begin{aligned} & \left| \frac{1}{\beta - \alpha} \int_{\alpha}^{\beta} \mathfrak{R}(t) \phi^{(m)}(t) dt - \frac{1}{\beta - \alpha} \int_{\alpha}^{\beta} \mathfrak{R}(t) dt \cdot \frac{1}{\beta - \alpha} \int_{\alpha}^{\beta} \phi^{(m)}(t) dt \right| \\ & \leq \frac{1}{2(\beta - \alpha)} \|\mathfrak{R}'\|_{\infty} \int_{\alpha}^{\beta} (t - \alpha)(\beta - t) \phi^{(m+1)}(t) dt. \quad (26) \end{aligned}$$

Since

$$\begin{aligned} \int_{\alpha}^{\beta} (t - \alpha)(\beta - t)\phi^{(m+1)}(t)dt &= \int_{\alpha}^{\beta} [2t - (\alpha + \beta)]\phi^{(m)}(t)dt \\ &= (\beta - \alpha)[\phi^{(m-1)}(\beta) + \phi^{(m-1)}(\alpha)] - 2(\phi^{(m-2)}(\beta) - \phi^{(m-2)}(\alpha)). \end{aligned}$$

Therefore, using identity T_1 and the inequality (26), we deduce (24).

(ii) Similar to the above proof.

Now we intend to give the Ostrowski type inequalities related to generalizations of Popoviciu's inequality.

Theorem 3.5. *Suppose all the assumptions of Theorem 2.1 hold. Moreover, assume (p, q) is a pair of conjugate exponents, that is $1 \leq p, q \leq \infty$, $1/p + 1/q = 1$. Let $|\phi^{(m)}|^p : [\alpha, \beta] \rightarrow \mathbb{R}$ be a R -integrable function for some $m \geq 2$. Then, we have*

(i)

$$\begin{aligned} |\Upsilon(\mathbf{x}, \mathbf{p}; \phi(x)) - \sum_{z=2}^{m-1} \frac{\phi^{(z)}(\alpha)}{z!} \Upsilon(\mathbf{x}, \mathbf{p}; (x - \alpha)^z)| \\ \leq \frac{1}{(m-1)!} \|\phi^{(m)}\|_p \left(\int_{\alpha}^{\beta} |\Upsilon(\mathbf{x}, \mathbf{p}; (x-t)_+^{m-1})|^q dt \right)^{1/q}. \end{aligned} \quad (27)$$

The constant on the R.H.S. of (27) is sharp for $1 < p \leq \infty$ and the optimal for $p = 1$.

(ii)

$$\begin{aligned} |\Upsilon(\mathbf{x}, \mathbf{p}; \phi(x)) - \sum_{z=2}^{m-1} \frac{(-1)^z \phi^{(z)}(\beta)}{z!} \Upsilon(\mathbf{x}, \mathbf{p}; (\beta - x)^z)| \\ \leq \frac{1}{(m-1)!} \|\phi^{(m)}\|_p \left(\int_{\alpha}^{\beta} |(-1)^{m-1} \Upsilon(\mathbf{x}, \mathbf{p}; (t-x)_+^{m-1})|^q dt \right)^{1/q}. \end{aligned} \quad (28)$$

The constant on the R.H.S. of (28) is sharp for $1 < p \leq \infty$ and the best possible for $p = 1$.

Proof. (i) Let us denote

$$\mathfrak{V} = \frac{1}{(m-1)!} \Upsilon(\mathbf{x}, \mathbf{p}; (x-t)_+^{m-1}), \quad t \in [\alpha, \beta].$$

Using identity T_1 and applying Hölder's inequality, we obtain

$$\begin{aligned} \left| \Upsilon(\mathbf{x}, \mathbf{p}; \phi(x)) - \sum_{z=2}^{m-1} \frac{\phi^{(z)}(\alpha)}{z!} \Upsilon(\mathbf{x}, \mathbf{p}; (x - \alpha)^z) \right| \\ = \left| \int_{\alpha}^{\beta} \mathfrak{W}(t) \phi^{(m)}(t) dt \right| \leq \|\phi^{(m)}\|_p \left(\int_{\alpha}^{\beta} |\mathfrak{W}(t)|^q dt \right)^{1/q}. \end{aligned}$$

For the proof of the sharpness of the constant $\left(\int_{\alpha}^{\beta} |\mathfrak{W}(t)|^q dt \right)^{1/q}$, let us define the function ϕ for which the equality in (27) is obtained.

For $1 < p \leq \infty$ take ϕ to be such that

$$\phi^{(m)}(t) = \operatorname{sgn} \mathfrak{W}(t) |\mathfrak{W}(t)|^{\frac{1}{p-1}}.$$

For $p = \infty$ take $\phi^{(m)}(t) = \operatorname{sgn} \mathfrak{W}(t)$.

For $p = 1$, we prove that

$$\left| \int_{\alpha}^{\beta} \mathfrak{W}(t) \phi^{(m)}(t) dt \right| \leq \max_{t \in [\alpha, \beta]} |\mathfrak{W}(t)| \left(\int_{\alpha}^{\beta} \phi^{(m)}(t) dt \right) \quad (29)$$

is the best possible inequality. Suppose that $|\mathfrak{W}(t)|$ attains its maximum at $t_0 \in [\alpha, \beta]$. To start with first we assume that $\mathfrak{W}(t_0) > 0$. For δ small enough we define $\phi_{\delta}(t)$ by

$$\phi_{\delta}(t) = \begin{cases} 0, & \alpha \leq t \leq t_0, \\ \frac{1}{\delta m!} (t - t_0)^m, & t_0 \leq t \leq t_0 + \delta, \\ \frac{1}{m!} (t - t_0)^{m-1}, & t_0 + \delta \leq t \leq \beta. \end{cases}$$

Then for δ small enough

$$\left| \int_{\alpha}^{\beta} \mathfrak{W}(t) \phi^{(m)}(t) dt \right| = \left| \int_{t_0}^{t_0 + \delta} \mathfrak{W}(t) \frac{1}{\delta} dt \right| = \frac{1}{\delta} \int_{t_0}^{t_0 + \delta} \mathfrak{W}(t) dt.$$

Now from inequality (29), we have

$$\frac{1}{\delta} \int_{t_0}^{t_0 + \delta} \mathfrak{W}(t) dt \leq \mathfrak{W}(t_0) \int_{t_0}^{t_0 + \delta} \frac{1}{\delta} dt = \mathfrak{W}(t_0).$$

Since

$$\lim_{\delta \rightarrow 0} \frac{1}{\delta} \int_{t_0}^{t_0 + \delta} \mathfrak{W}(t) dt = \mathfrak{W}(t_0),$$

the statement follows. The case when $\mathfrak{B}(t_0) < 0$, we define $\phi_\delta(t)$ by

$$\phi_\delta(t) = \begin{cases} \frac{1}{m!}(t - t_0 - \delta)^{m-1}, & \alpha \leq t \leq t_0, \\ \frac{-1}{\delta m!}(t - t_0 - \delta)^m, & t_0 \leq t \leq t_0 + \delta, \\ 0, & t_0 + \delta \leq t \leq \beta, \end{cases}$$

and rest of the proof is the same as above.

(ii) Similar to first part.

4. MEAN VALUE THEOREMS AND m -EXPONENTIAL CONVEXITY

We recall some definitions and basic results from [1], [4] and [5] which are required in sequel.

Definition 4. A function $\phi : I \rightarrow \mathbb{R}$ is m -exponentially convex in the Jensen sense on I if

$$\sum_{i,j=1}^m \xi_i \xi_j \phi\left(\frac{x_i + x_j}{2}\right) \geq 0,$$

hold for all choices $\xi_1, \dots, \xi_m \in \mathbb{R}$ and all choices $x_1, \dots, x_m \in I$. A function $\phi : I \rightarrow \mathbb{R}$ is m -exponentially convex if it is m -exponentially convex in the Jensen sense and continuous on I .

Definition 5. A function $\phi : I \rightarrow \mathbb{R}$ is exponentially convex in the Jensen sense on I if it is m -exponentially convex in the Jensen sense for all $m \in \mathbb{N}$.

A function $\phi : I \rightarrow \mathbb{R}$ is exponentially convex if it is exponentially convex in the Jensen sense and continuous.

Proposition 4.1. If $\phi : I \rightarrow \mathbb{R}$ is an m -exponentially convex in the Jensen sense, then the matrix $\left[\phi\left(\frac{x_i + x_j}{2}\right)\right]_{i,j=1}^n$ is a positive semi-definite matrix for all $n \in \mathbb{N}, n \leq m$. Particularly,

$$\det \left[\phi\left(\frac{x_i + x_j}{2}\right) \right]_{i,j=1}^n \geq 0$$

for all $n \in \mathbb{N}, n = 1, 2, \dots, m$.

Remark 4.2. It is known that $\phi : I \rightarrow \mathbb{R}$ is a log-convex in the Jensen sense if and only if

$$\alpha^2 \phi(x) + 2\alpha\beta \phi\left(\frac{x+y}{2}\right) + \beta^2 \phi(y) \geq 0,$$

holds for every $\alpha, \beta \in \mathbb{R}$ and $x, y \in I$. It follows that a positive function is log-convex in the Jensen sense if and only if it is 2-exponentially convex in the Jensen sense.

A positive function is log-convex if and only if it is 2-exponentially convex.

Remark 4.3. *By the virtue of Theorem 2.2, we define the positive linear functionals with respect to m -convex function ϕ as follows*

$$\Gamma_1(\phi) := \Upsilon(\mathbf{x}, \mathbf{p}; \phi(x)) - \sum_{z=2}^{m-1} \frac{\phi^{(z)}(\alpha)}{z!} \Upsilon(\mathbf{x}, \mathbf{p}; (x - \alpha)^z) \geq 0, \quad (30)$$

$$\Gamma_2(\phi) := \Upsilon(\mathbf{x}, \mathbf{p}; \phi(x)) - \sum_{z=2}^{m-1} \frac{(-1)^z \phi^{(z)}(\beta)}{z!} \Upsilon(\mathbf{x}, \mathbf{p}; (\beta - x)^z) \geq 0. \quad (31)$$

Lagrange and Cauchy type mean value theorems related to defined functionals are given in the following theorems.

Theorem 4.4. *Let $\phi : [\alpha, \beta] \rightarrow \mathbb{R}$ be such that $\phi \in C^m[\alpha, \beta]$. If the inequalities in (10) ($i = 1$) and (12) ($i = 2$) hold, then there exist $\xi_i \in [\alpha, \beta]$ such that*

$$\Gamma_i(\phi) = \phi^{(m)}(\xi_i) \Gamma_i(\varphi), \quad i = 1, 2 \quad (32)$$

where $\varphi(x) = \frac{x^m}{m!}$ and $\Gamma_i(\cdot)$ ($i = 1, 2$) are defined by (30)-(31).

Proof. Similar to the proof of Theorem 4.1 in [5].

Theorem 4.5. *Let $\phi, \psi : [\alpha, \beta] \rightarrow \mathbb{R}$ be such that $\phi, \psi \in C^m[\alpha, \beta]$. If the inequalities in (10) ($i = 1$) and (12) ($i = 2$) hold, then there exist $\xi_i \in [\alpha, \beta]$ such that*

$$\frac{\Gamma_i(\phi)}{\Gamma_i(\psi)} = \frac{\phi^{(m)}(\xi_i)}{\psi^{(m)}(\xi_i)}, \quad i = 1, 2 \quad (33)$$

provided that the denominators are non-zero and $\Gamma_i(\cdot)$ ($i = 1, 2$) are defined by (30)-(31).

Proof. Similar to the proof of Corollary 4.2 in [5].

Theorem 4.5 enables us to define Cauchy means, because if

$$\xi_i = \left(\frac{\phi^{(m)}}{\psi^{(m)}} \right)^{-1} \left(\frac{\Gamma_i(\phi)}{\Gamma_i(\psi)} \right), \quad i = 1, 2$$

which means that ξ_i ($i = 1, 2$) are means for given functions ϕ and ψ .

Next we construct the non trivial examples of m -exponentially and exponentially convex functions from positive linear functionals $\Gamma_i(\cdot)$ ($i = 1, 2$). In the sequel I and J are intervals in \mathbb{R} .

Theorem 4.6. *Let $\Omega = \{\phi_t : t \in J\}$, where J is an interval in \mathbb{R} , be a family of functions defined on an interval I in \mathbb{R} such that the function $t \mapsto [x_0, \dots, x_m; \phi_t]$ is m -exponentially convex in the Jensen sense on J for every $(m + 1)$ mutually different points $x_0, \dots, x_m \in I$. Then for the linear functionals $\Gamma_i(\phi_t)$ ($i = 1, 2$) as defined by (30) – (31), the following statements are valid:*

- (i) *The function $t \rightarrow \Gamma_i(\phi_t)$ is m -exponentially convex in the Jensen sense on J and the matrix $[\Gamma_i(\phi_{\frac{t_j+t_l}{2}})]_{j,l=1}^n$ is a positive semi-definite for all $n \in \mathbb{N}, n \leq m$, $t_1, \dots, t_n \in J$. Particularly,*

$$\det[\Gamma_i(\phi_{\frac{t_j+t_l}{2}})]_{j,l=1}^n \geq 0 \text{ for all } n \in \mathbb{N}, n = 1, 2, \dots, m.$$

- (ii) *If the function $t \rightarrow \Gamma_i(\phi_t)$ is continuous on J , then it is m -exponentially convex on J .*

Proof. (i) For $\xi_j \in \mathbb{R}$ and $t_j \in J, j = 1, \dots, m$, we define the function

$$h(x) = \sum_{j,l=1}^m \xi_j \xi_l \phi_{\frac{t_j+t_l}{2}}(x).$$

Using the assumption that the function $t \mapsto [x_0, \dots, x_m; \phi_t]$ is m -exponentially convex in the Jensen sense, we have

$$[x_0, \dots, x_m, h] = \sum_{j,l=1}^m \xi_j \xi_l [x_0, \dots, x_m; \phi_{\frac{t_j+t_l}{2}}] \geq 0,$$

which in turn implies that h is a m -convex function on J , therefore from Remark 4.3 we have $\Gamma_i(h) \geq 0, i = 1, 2$. The linearity of $\Gamma_i(\cdot)$ gives

$$\sum_{j,l=1}^m \xi_j \xi_l \Gamma_i(\phi_{\frac{t_j+t_l}{2}}) \geq 0.$$

We conclude that the function $t \mapsto \Gamma_i(\phi_t)$ is m -exponentially convex on J in the Jensen sense.

The remaining part follows from Proposition 4.1.

- (ii) If the function $t \rightarrow \Gamma_i(\phi_t)$ is continuous on J , then it is m -exponentially convex on J by definition.

The following corollary is an immediate consequence of the above theorem

Corollary 4.7. Let $\Omega = \{\phi_t : t \in J\}$, where J is an interval in \mathbb{R} , be a family of functions defined on an interval I in \mathbb{R} , such that the function $t \mapsto [x_0, \dots, x_m; \phi_t]$ is exponentially convex in the Jensen sense on J for every $(m+1)$ mutually different points $x_0, \dots, x_m \in I$. Then for the linear functionals $\Gamma_i(\phi_t)$ ($i = 1, 2$) as defined by (30) – (31), the following statements hold:

- (i) The function $t \rightarrow \Gamma_i(\phi_t)$ is exponentially convex in the Jensen sense on J and the matrix $[\Gamma_i(\phi_{\frac{t_j+t_l}{2}})]_{j,l=1}^n$ is a positive semi-definite for all $n \in \mathbb{N}, n \leq m, t_1, \dots, t_n \in J$. Particularly,

$$\det[\Gamma_i(\phi_{\frac{t_j+t_l}{2}})]_{j,l=1}^n \geq 0 \text{ for all } n \in \mathbb{N}, n = 1, 2, \dots, m.$$

- (ii) If the function $t \rightarrow \Gamma_i(\phi_t)$ is continuous on J , then it is exponentially convex on J .

Corollary 4.8. Let $\Omega = \{\phi_t : t \in J\}$, where J is an interval in \mathbb{R} , be a family of functions defined on an interval I in \mathbb{R} , such that the function $t \mapsto [x_0, \dots, x_m; \phi_t]$ is 2–exponentially convex in the Jensen sense on J for every $(m+1)$ mutually different points $x_0, \dots, x_m \in I$. Let $\Gamma_i(\cdot)$, $i = 1, 2$ be linear functionals defined by (30)-(31). Then the following statements hold:

- (i) If the function $t \mapsto \Gamma_i(\phi_t)$ is continuous on J , then it is 2–exponentially convex function on J . If $t \mapsto \Gamma_i(\phi_t)$ is additionally strictly positive, then it is also log-convex on J . Furthermore, the following inequality holds true:

$$[\Gamma_i(\phi_s)]^{t-r} \leq [\Gamma_i(\phi_r)]^{t-s} [\Gamma_i(\phi_t)]^{s-r}, \quad i = 1, 2$$

for every choice $r, s, t \in J$, such that $r < s < t$.

- (ii) If the function $t \mapsto \Gamma_i(\phi_t)$ is strictly positive and differentiable on J , then for every $p, q, u, v \in J$, such that $p \leq u$ and $q \leq v$, we have

$$\mu_{p,q}(\Gamma_i, \Omega) \leq \mu_{u,v}(\Gamma_i, \Omega), \tag{34}$$

where

$$\mu_{p,q}(\Gamma_i, \Omega) = \begin{cases} \left(\frac{\Gamma_i(\phi_p)}{\Gamma_i(\phi_q)} \right)^{\frac{1}{p-q}}, & p \neq q, \\ \exp \left(\frac{d}{dp} \frac{\Gamma_i(\phi_p)}{\Gamma_i(\phi_p)} \right), & p = q, \end{cases} \tag{35}$$

for $\phi_p, \phi_q \in \Omega$.

Proof. (i) This is an immediate consequence of Theorem 4.6 and Remark 4.2.

- (ii) Since $p \mapsto \Gamma_i(\phi_p)$ is positive and continuous, by (i) we have that $t \mapsto \Gamma_i(\phi_t)$ is log-convex on J , that is, the function $t \mapsto \log \Gamma_i(\phi_t)$ is convex on J . Hence we get

$$\frac{\log \Gamma_i(\phi_p) - \log \Gamma_i(\phi_q)}{p - q} \leq \frac{\log \Gamma_i(\phi_u) - \log \Gamma_i(\phi_v)}{u - v}, \quad (36)$$

for $p \leq u, q \leq v, p \neq q, u \neq v$. So, we conclude that

$$\mu_{p,q}(\Gamma_i, \Omega) \leq \mu_{u,v}(\Gamma_i, \Omega).$$

Cases $p = q$ and $u = v$ follow from (36) as limit cases.

5. APPLICATIONS TO CAUCHY MEANS

In this section, we present some families of functions which fulfil the conditions of Theorem 4.6, Corollary 4.7 and Corollary 4.8. This enables us to construct a large families of functions which are exponentially convex. Explicit form of this functions is obtained after we calculate explicit action of functionals on a given family.

Example 5.1. Let us consider a family of functions

$$\Omega_1 = \{\phi_t : \mathbb{R} \rightarrow \mathbb{R} : t \in \mathbb{R}\}$$

defined by

$$\phi_t(x) = \begin{cases} \frac{e^{tx}}{t^m}, & t \neq 0, \\ \frac{x^m}{m!}, & t = 0. \end{cases}$$

Since $\frac{d^m \phi_t}{dx^m}(x) = e^{tx} > 0$, the function ϕ_t is m -convex on \mathbb{R} for every $t \in \mathbb{R}$ and $t \mapsto \frac{d^m \phi_t}{dx^m}(x)$ is exponentially convex by definition. Using analogous arguing as in the proof of Theorem 4.6 we also have that $t \mapsto [x_0, \dots, x_m; \phi_t]$ is exponentially convex (and so exponentially convex in the Jensen sense). Now, using Corollary 4.7 we conclude that $t \mapsto \Gamma_i(\phi_t)$ ($i = 1, 2$) are exponentially convex in the Jensen sense. It is easy to verify that this mapping is continuous (although the mapping $t \mapsto \phi_t$ is not continuous for $t = 0$), so it is exponentially convex. For this family of functions, $\mu_{t,q}(\Gamma_i, \Omega_1)$ ($i = 1, 2$), from (35), becomes

$$\mu_{t,q}(\Gamma_i, \Omega_1) = \begin{cases} \left(\frac{\Gamma_i(\phi_t)}{\Gamma_i(\phi_q)} \right)^{\frac{1}{t-q}}, & t \neq q, \\ \exp \left(\frac{\Gamma_i(id \cdot \phi_t)}{\Gamma_i(\phi_t)} - \frac{m}{t} \right), & t = q \neq 0, \\ \exp \left(\frac{1}{m+1} \frac{\Gamma_i(id \cdot \phi_0)}{\Gamma_i(\phi_0)} \right), & t = q = 0, \end{cases}$$

where “ id ” is the identity function. By Corollary 4.8 $\mu_{t,q}(\Gamma_i, \Omega_1)$ ($i = 1, 2$) are monotone functions in parameters t and q .

Since

$$\left(\frac{\frac{d^m f_t}{dx^m}}{\frac{d^m f_q}{dx^m}} \right)^{\frac{1}{t-q}} (\log x) = x,$$

using Theorem 4.5 it follows that:

$$M_{t,q}(\Gamma_i, \Omega_1) = \log \mu_{t,q}(\Gamma_i, \Omega_1), \quad i = 1, 2$$

satisfies

$$\alpha \leq M_{t,q}(\Gamma_i, \Omega_1) \leq \beta, \quad i = 1, 2.$$

Hence $M_{t,q}(\Gamma_i, \Omega_1)$ ($i = 1, 2$) are monotonic means.

Example 5.2. *Let us consider a family of functions*

$$\Omega_2 = \{g_t : (0, \infty) \rightarrow \mathbb{R} : t \in \mathbb{R}\}$$

defined by

$$g_t(x) = \begin{cases} \frac{x^t}{t(t-1)\dots(t-m+1)}, & t \notin \{0, 1, \dots, m-1\}, \\ \frac{x^j \log x}{(-1)^{m-1-j} j!(m-1-j)!}, & t = j \in \{0, 1, \dots, m-1\}. \end{cases}$$

Since $\frac{d^m g_t}{dx^m}(x) = x^{t-m} > 0$, the function g_t is m -convex for $x > 0$ and $t \mapsto \frac{d^m g_t}{dx^m}(x)$ is exponentially convex by definition. Arguing as in Example 5.1 we get that the mappings $t \mapsto \Gamma_i(g_t)$ ($i = 1, 2$) are exponentially convex. Hence, for this family of functions $\mu_{p,q}(\Gamma_i, \Omega_2)$ ($i = 1, 2$), from (35), are equal to

$$\mu_{t,q}(\Gamma_i, \Omega_2) = \begin{cases} \left(\frac{\Gamma_i(g_t)}{\Gamma_i(g_q)} \right)^{\frac{1}{t-q}}, & t \neq q, \\ \exp \left((-1)^{m-1} (m-1)! \frac{\Gamma_i(g_0 g_t)}{\Gamma_i(g_t)} + \sum_{k=0}^{m-1} \frac{1}{k-t} \right), & t = q \notin \{0, 1, \dots, m-1\}, \\ \exp \left((-1)^{m-1} (m-1)! \frac{\Gamma_i(g_0 g_t)}{2\Gamma_i(g_t)} + \sum_{\substack{k=0 \\ k \neq t}}^{m-1} \frac{1}{k-t} \right), & t = q \in \{0, 1, \dots, m-1\}. \end{cases}$$

Again, using Theorem 4.5 we conclude that

$$\alpha \leq \left(\frac{\Gamma_i(g_t)}{\Gamma_i(g_q)} \right)^{\frac{1}{t-q}} \leq \beta, \quad i = 1, 2. \quad (37)$$

Hence $\mu_{t,q}(\Gamma_i, \Omega_2)$ ($i = 1, 2$) are means and their monotonicity is followed by (34).

Example 5.3. Let

$$\Omega_3 = \{\zeta_t : (0, \infty) \rightarrow \mathbb{R} : t \in (0, \infty)\}$$

be a family of functions defined by

$$\zeta_t(x) = \begin{cases} \frac{t^{-x}}{(-\log t)^m}, & t \neq 1; \\ \frac{x^m}{(m)!}, & t = 1. \end{cases}$$

Since $\frac{d^m \zeta_t}{dx^m}(x) = t^{-x}$ is the Laplace transform of a non-negative function (see [13]) it is exponentially convex. Obviously ζ_t are m -convex functions for every $t > 0$. For this family of functions, $\mu_{t,q}(\Gamma_i, \Omega_3)$ ($i = 1, 2$), in this case for $[\alpha, \beta] \subset \mathbb{R}^+$, from (35) becomes

$$\mu_{t,q}(\Gamma_i, \Omega_3) = \begin{cases} \left(\frac{\Gamma_i(\zeta_t)}{\Gamma_i(\zeta_q)}\right)^{\frac{1}{t-q}}, & t \neq q; \\ \exp\left(-\frac{\Gamma_i(id, \zeta_t)}{t\Gamma_i(\zeta_t)} - \frac{m}{t \log t}\right), & t = q \neq 1; \\ \exp\left(-\frac{1}{m+1} \frac{\Gamma_i(id, \zeta_1)}{\Gamma_i(\zeta_1)}\right), & t = q = 1, \end{cases}$$

where id is the identity function. By Corollary 4.8 $\mu_{p,q}(\Gamma_i, \Omega_3)$ ($i = 1, 2$) are monotone functions in parameters t and q .

Using Theorem 4.5 it follows that

$$M_{t,q}(\Gamma_i, \Omega_3) = -L(t, q) \log \mu_{t,q}(\Gamma_i, \Omega_3); \quad i = 1, 2.$$

satisfy

$$\alpha \leq M_{t,q}(\Gamma_i, \Omega_3) \leq \beta; \quad i = 1, 2.$$

This shows that $M_{t,q}(\Gamma_i, \Omega_3)$ ($i = 1, 2$) are means. Because of the inequality (34), these mean are monotonic. $L(t, q)$ is logarithmic mean defined by

$$L(t, q) = \begin{cases} \frac{t-q}{\log t - \log q}, & t \neq q; \\ t, & t = q. \end{cases}$$

Example 5.4. Let

$$\Omega_4 = \{\gamma_t : (0, \infty) \rightarrow \mathbb{R} : t \in (0, \infty)\}$$

be a family of functions defined by

$$\gamma_t(x) = \frac{e^{-x\sqrt{t}}}{(-\sqrt{t})^m}.$$

Since $\frac{d^m \gamma_t}{dx^m}(x) = e^{-x\sqrt{t}}$ is the Laplace transform of a non-negative function (see [13]) it is exponentially convex. Obviously γ_t are m -convex function for every $t > 0$. For this family of functions, $\mu_{t,q}(\Gamma_i, \Omega_4)$ ($i = 1, 2$), in this case for $[\alpha, \beta] \subset \mathbb{R}^+$, from (35) becomes

$$\mu_{t,q}(\Gamma_i, \Omega_4) = \begin{cases} \left(\frac{\Gamma_i(\gamma_t)}{\Gamma_i(\gamma_q)}\right)^{\frac{1}{t-q}}, & t \neq q; \\ \exp\left(-\frac{\Gamma_i(id.\gamma_t)}{2\sqrt{t}\Gamma_i(\gamma_t)} - \frac{m}{2t}\right), & t = q; \end{cases} \quad i = 1, 2.$$

By Corollary 4.8, these are monotone functions in parameters t and q . Using Theorem 4.5 it follows that

$$M_{t,q}(\Gamma_i, \Omega_4) = -\left(\sqrt{t} + \sqrt{q}\right) \ln \mu_{t,q}(\Gamma_i, \Omega_4); \quad i = 1, 2.$$

satisfy

$$\alpha \leq M_{t,q}(\Gamma_i, \Omega_4) \leq \beta; \quad i = 1, 2.$$

This shows that $M_{t,q}(\Gamma_i, \Omega_4)$ ($i = 1, 2$) are means. Because of the above inequality (34), these means are monotonic.

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