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THE MCDONALD QUASI LINDLEY DISTRIBUTION AND ITS APPLICATIONS

F. Merovci, I. Elbatal, L. Puka

Abstract. In this paper, we present a new class of distributions called McDonald quasi Lindley distribution. This class of distributions contains several distributions such as beta-quasi Lindley, Kumaraswamy quasi Lindley and quasi Lindley as special cases. The hazard function, reverse hazard function, moments and mean residual life function are obtained. We estimate the parameters by maximum likelihood and provide the observed information matrix. The usefulness of the new distribution is illustrated with real data set that show that it is quite flexible in analyzing positive data instead of the McDonald quasi Lindley distribution and quasi Lindley distributions.

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1. Introduction and Motivation

Quasi Lindley distribution with parameters α and θ is defined by its probability density function (p.d.f)

$$g(x,\theta,\alpha)=\frac{\theta}{\alpha+1}(\alpha+\theta x)e^{-\theta x}\,,x>0\,,\theta>0\,,\alpha>-1. \tag{1}$$

It can easily be seen that at $\alpha=\theta$, the QLD equation (1) reduces to the Lindley distribution (1958) with probability density function

$$g(x,\theta)=\frac{\theta^2}{\theta+1}(1+x)e^{-\theta x},\quad x>0,\theta>0,$$

and at $\alpha=0$, it reduces to the gamma distribution with parameters $(2,\theta)$. The p.d.f. equation (1) can be shown as a mixture of exponential (θ) and gamma $(2,\theta)$ distributions as follows

$$g(x,\theta,\alpha)=pg_1(x)+(1-p)g_2(x)$$

where

$$p = \frac{1}{\alpha+1}, g_1(x) = \theta e^{-\theta x} \text{ and } g_2(x) = \theta^2 e^{-\theta x}$$

The cumulative distribution function (cdf) of QLD is obtained as

$$G(x,\theta) = 1 - e^{-\theta x}\left[1 + \frac{\theta x}{\alpha+1}\right], x > 0, \theta > 0, \alpha > -1. \quad (2)$$

where θ is scale parameter.

Ghitany et al. (2008a) have discussed various properties of this distribution and showed that in many ways equation (1) provides a better model for some applications than the exponential distribution. A discrete version of this distribution has been suggested by Deniz and Ojeda (2011) having its applications in count data related to insurance. Sankaran (1970) obtained the Lindley mixture of Poisson distribution. Ghitany et al. (2008b, c) obtained size-biased and zero-truncated version of Poisson-Lindley distribution and discussed their various properties and applications. Ghitany and Al-Mutairi (2009) discussed as various estimation methods for the discrete Poisson- Lindley distribution. Bakouch et al. (2012) obtained an extended Lindley distribution and discussed its various properties and applications. Mazucheli and Achcar (2011) discussed the applications of Lindley distribution to competing risks lifetime data. Rama and Mishra (2013) studied quasi Lindley distribution. Ghitany et al. (2011) developed a two-parameter weighted Lindley distribution and discussed its applications to survival data. Zakerzadah and Dolati (2010) obtained a generalized Lindley distribution and discussed its various properties and applications. Elbatal et al. (2014) obtained an extended Lindley distribution called transmuted Lindley-geometric distribution and discussed its various properties and applications. Merovci (2013) obtained an extended Lindley distribution called transmuted Lindley distribution and discussed its various properties and applications. Merovci and Sharma (2014) obtained the beta Lindley distribution and discussed its various properties and applications.

Consider an arbitrary parent cdf $G(x)$. The probability density function (pdf) $f(x)$ of the new class of distributions called the Mc-Donald generalized distributions (denoted with the prefix " Mc" for short) is defined by

$$f(x,a,b,c) = \frac{c}{B(a,b)} g(x) G^{ac-1}(x) \left[1 - G^c(x)\right]^{b-1}, \quad (3)$$

where $a > 0, b > 0$ and $c > 0$ are additional shape parameters . (See Corderio et al. (2012) for additional details). Note that $g(x)$ is the pdf of parent distribution , $g(x) = \frac{dG(x)}{dx}$. Introduction of this additional shape parameters is specially to introduce skewness. Also, this allows us to vary tail weight. It is important to note that for $c = 1$ we obtain a sub-model of this generalization which is a beta generalization (see Eugene et al.(2002)) and for $a = 1$, we have the Kumaraswamy (Kw), [Kumaraswamy generalized distributions (see Cordeiro and Castro, (2010)). For random variable X with density function (3), we write $X \sim Mc - G(a, b, c)$. The probability density function (3) will be most tractable when $G(x)$ and $g(x)$ have simple analytic expressions. The corresponding cumulative function for this generalization is given by

$$F(x, a, b, c) = I_{G^c(x)}(a, b) = \frac{1}{B(a, b)} \int_0^{G(x)^c} w^{(1-a)}(1 - w)^{b-1} dw, \tag{4}$$

where $I_y(a, b) = \frac{1}{B(a,b)} \int_0^y w^{(1-a)}(1 - w)^{b-1} dw$ denotes the incomplete beta function (Gradshteyn & Ryzhik, 2000). Equation (4) can also be rewritten as follows

$$F(x, a, b, c) = \frac{G(x)^{ac}}{aB(a, b)}{}_2F_1(a, 1 - b; a + 1; G(x)^c), \tag{5}$$

where

$${}_2F_1(a, b; c; x) = B(b, c - b)^{-1} \int_0^1 \frac{t^{b-1}(1 - t)^{c-b-1}}{(1 - tx)^a} dt$$

is the well-known hypergeometric functions which are well established in the literature (see, Gradshteyn and Ryzhik (2000)).

Some mathematical properties of the cdf $F(x)$ for any Mc-G distribution defined from a parent $G(x)$ in equation (5), could, in principle, follow from the properties of the hypergeometric function, which are well established in the literature (Gradshteyn and Ryzhik, 2000, Sec. 9.1). One important benefit of this class is its ability to skewed data that cannot properly be fitted by many other existing distributions. Mc-G family of densities allows for higher levels of exibility of its tails and has a lot of applications in various fields including economics, finance, reliability, engineering, biology and medicine.

The hazard function (hf) and reverse hazard functions (rhf) of the Mc-G distribution are given by

$$h(x) = \frac{f(x)}{1 - F(x)} = \frac{cg(x)G^{ac-1}(x)\left[1 - G^c(x)\right]^{b-1}}{B(a, b)\left\{1 - I_{G^c(x)}(a, b)\right\}}, \tag{6}$$

and

$$\tau(x)=\frac{f(x)}{F(x)}=\frac{cg(x)G^{ac-1}(x)\left[1-G^{c}(x)\right]^{b-1}}{B(a,b)\left\{I_{G^{c}(x)}(a,b)\right\}}$$

respectively. Recently Cordeiro et al. (2012) introduced the The McDonald Normal Distribution. Now we introduce a new class of distribution, called Mc Quasi Lindley ($McQL$) distribution by taking $G(x)$ and $g(x)$ in (3) to be the cdf and pdf of (1) and (2). The pdf of the $McQL$ distribution is given by

$$\begin{aligned}f(x,\phi)=&\frac{c}{B(a,b)}\frac{\theta}{\alpha+1}(\alpha+\theta x)e^{-\theta x}\left[1-e^{-\theta x}\left[1+\frac{\theta x}{\alpha+1}\right]\right]^{ac-1}\\&\times\left[1-\left\{1-e^{-\theta x}\left[1+\frac{\theta x}{\alpha+1}\right]\right\}^{c}\right]^{b-1}\end{aligned}\tag{7}$$

where $\phi=(\alpha,\theta,a,b,c)$. The corresponding cdf of the $McQL$ distribution is given by

$$\begin{aligned}F(x)=I_{G^{c}(x)}(a,b)&=\frac{1}{B(a,b)}\int_{0}^{G(x)^{c}}w^{(1-a)}(1-w)^{b-1}dw\\&=\frac{1}{B(a,b)}\int_{0}^{\left[1-e^{-\theta x}\left[1+\frac{\theta x}{\alpha+1}\right]\right]^{c}}w^{(1-a)}(1-w)^{b-1}dw\\&=I_{\left[1-e^{-\theta x}\left[1+\frac{\theta x}{\alpha+1}\right]\right]^{c}}(a,b),\end{aligned}\tag{8}$$

also, the cdf can be written as follows

$$F(x)=\frac{\left[1-e^{-\theta x}\left[1+\frac{\theta x}{\alpha+1}\right]\right]^{ac}}{aB(a,b)}\,{}_{2}F_{1}(a,1-b;a+1;\left[1-e^{-\theta x}\left[1+\frac{\theta x}{\alpha+1}\right]\right]^{c}),\tag{9}$$

where ${}_{2}F_{1}(a,b;c;x)=B(b,c-b)^{-1}\int_{0}^{1}\frac{t^{b-1}(1-t)^{c-b-1}}{(1-tx)^{a}}dt.$

Figure 1 and figure 2 illustrates some of the possible shapes of the pdf and cdf of McQL distribution for selected values of the parameters θ,α,a,b and c, respectively.

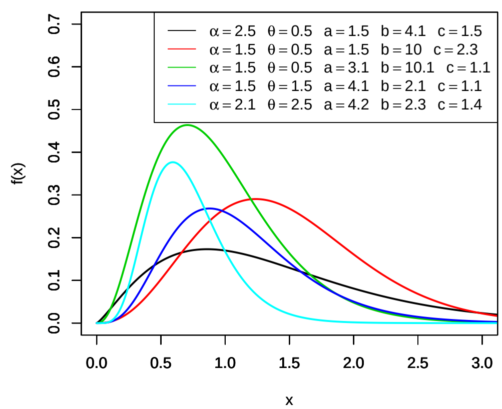

Figure 1: The pdf's of various McQL distributions.

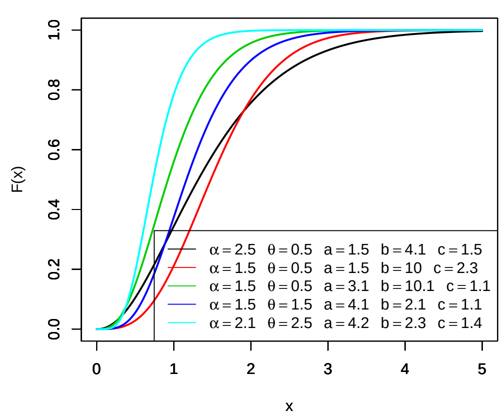

Figure 2: The cdf's of various McQL distributions.

The hazard rate function and reversed hazard rate function of the new distribution are given by

$$\begin{aligned} h(x) &= \frac{f(x)}{1-F(x)} \\ &= \frac{\theta(\alpha+\theta x)e^{-\theta x}\left[1-e^{-\theta x}\left[1+\frac{\theta x}{\alpha+1}\right]\right]^{ac-1}}{(\alpha+1)B(a,b)\left\{1-I_{\left[1-e^{-\theta x}\left[1+\frac{\theta x}{\alpha+1}\right]\right]^{c}}(a,b)\right\}} \\ &\times\left[1-\left\{1-e^{-\theta x}\left[1+\frac{\theta x}{\alpha+1}\right]\right\}^{c}\right]^{b-1} \end{aligned} \tag{10}$$

and

$$\begin{aligned} \tau(x) &= \frac{f(x)}{F(x)} \\ &= \frac{\theta(\alpha+\theta x)e^{-\theta x}\left[1-e^{-\theta x}\left[1+\frac{\theta x}{\alpha+1}\right]\right]^{ac-1}}{(\alpha+1)B(a,b)I_{\left[1-e^{-\theta x}\left[1+\frac{\theta x}{\alpha+1}\right]\right]^{c}}(a,b)} \\ &\times\left[1-\left\{1-e^{-\theta x}\left[1+\frac{\theta x}{\alpha+1}\right]\right\}^{c}\right]^{b-1}, \end{aligned} \tag{11}$$

respectively.

Figure 3 illustrates some of the possible shapes of the hazard rate function of McQL distribution for selected values of the parameters θ, α, a, b and c.

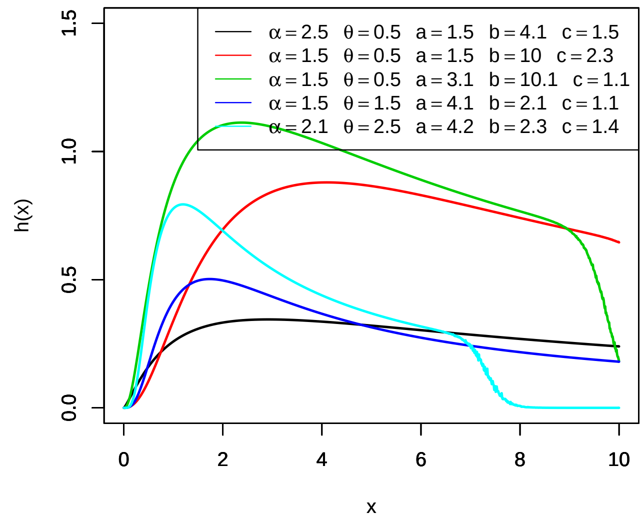

Figure 3: The pdf's of various McQL distributions.

The rest of the paper is organized as follows. In Section 2, we demonstrate that the $McQL$ density function can be expressed as a linear combination of the quasi Lindley distribution. This result is important to provide mathematical properties of the $McQL$ model directly from those properties of the quasi Lindley distribution. In Section 3. we discuss some important statistical properties of the $McQL$ distribution including quantile function, moments and moment generating function. The distribution of the order statistics is expressed in Section 4. We discuss in section 5 maximum likelihood estimation and calculate the elements of the observed information matrix. Section 6 provides applications to real data sets. Section 8 ends with some conclusions.

2. Expansion of Distribution

In this section, we present a series expansion of the $McQL$ cdf and pdf distribution depending if the parameter $b > 0$ is real non-integer or integer. First, if $|z| < 1$ and $b > 0$ is real non- integer, we have

$$(1-z)^{b-1} = \sum_{j=0}^{\infty} (-1)^j \binom{b-1}{j} z^j = \sum_{j=0}^{\infty} \frac{(-1)^j \Gamma(b)}{j!\Gamma(b-j)} z^j. \tag{12}$$

Using the expansion (12) in (8), the cdf of the $McQL$ distribution becomes

$$
\begin{aligned}
F(x) &= \frac{1}{B(a,b)} \int_0^{\left[1-e^{-\theta x}\left[1+\frac{\theta x}{\alpha+1}\right]\right]^c} w^{(1-a)}(1-w)^{b-1}dw \\
&= \frac{\Gamma(b)}{B(a,b)} \sum_{j=0}^{\infty} \frac{(-1)^j}{j!\Gamma(b-j)} \int_0^{G(x)^c} w^{a+j-1}dw \\
&= \sum_{j=0}^{\infty} \frac{(-1)^j \Gamma(b)}{B(a,b)j!\Gamma(b-j)(a+j)} \left[G(x,\alpha,\theta)\right]^{c(a+j)} \\
&= \sum_{j=0}^{\infty} q_j \; G(x,\theta c(a+j),\alpha)
\end{aligned}
$$

where

$$q_j = \frac{(-1)^j \Gamma(b)}{B(a,b)j!\Gamma(b-j)(a+j)}.$$

If $b > 0$ is an integer, then

$$F(x) = \sum_{j=0}^{b-1} q_j G(x,\theta c(a+j),\alpha). \tag{13}$$

Similarly, if $b > 0$ is real non- integer the pdf is given by

$$f(x) = \sum_{j=0}^{\infty} q_j g(x,\theta c(a+j),\alpha),$$

and

$$f(x) = \sum_{j=0}^{b-1} q_j g(x,\theta c(a+j),\alpha)$$

for $b > 0$ is an integer . This is a finite mixture of quasi Lindley distributions with parameters $\theta c(a+j)$ and α.

2.1. Submodels

The McDonald quasi Lindley distribution is very flexible model that approaches to different distributions when its parameters are changed. The $McQL$ distribution contains as special- models the following well known distributions. If X is a random

variable with pdf (7) or cdf(8) we use the notation $X \sim McQL(\alpha, \theta a, b, c)$ then we have the following cases
1- For $\alpha = \theta$, $McQL$ distribution reduces to the McDonald Lindley Distribution.
2- If $\alpha = 0$ then (1.8) becomes the McDonald gamma distribution with parameter (2,θ).
3- For $a = b = c = 1$, then (1.8) reduces to the quasi Lindley distribution which is introduced by Rama et al (2013).
4- For $a = c = 1$ we get the Kumaraswamy quasi Lindley distribution.
5- Kumaraswamy gamma $(2, \theta)$ distrtribution arises as a special case of $McQL$ by taking $a = c = 1$,and $\alpha = 0$.
6- Applying $\alpha = \theta$ and $c = 1$ we can obtain the Lindley distribution.

3. Statistical Properties

This section is devoted to studying statistical properties of the $McQL$ distribution, specifically quantile function, moments, and moment generating function.

3.1. Quantile Function

The $McQL$ quantile function , say $Q(u) = F^{-1}(u)$, is straightforward to be computed by inverting (8), we have

$$e^{-\theta x}\left[1 + \frac{\theta x}{\alpha + 1}\right] = 1 - Q_{(a,b)}(u)^{c})$$

we can easily generate X by taking u as a uniform random variable in $(0, 1)$.

3.2. Moments

In this subsection we discuss the k_{th} non-central moment for $McQL$ distribution. Moments are necessary and important in any statistical analysis, especially in applications. It can be used to study the most important features and characteristics of a distribution (e.g., tendency, dispersion, skewness and kurtosis).

Theorem 1. *If X has $McQL(\phi, x)$,$\phi = (\alpha, \theta, a, b, c)$ then the k_{th} non-central moment of X is given by the following*

$$\mu_k^{'}(x) = E(X^K) = w_{i,j,k}\left[\frac{\alpha\Gamma(r + i + 1)}{(\theta(k + 1))^{r+i+1}} + \frac{\theta\Gamma(r + i + 2)}{(\theta(k + 1))^{r+i+2}}\right]. \qquad (14)$$

where

$$w_{i,j,k} = \sum_{i,j,k}^{\infty}(-1)^{j+k}\binom{b-1}{j}\binom{c(a+j)-1}{k}\binom{k}{i}.$$

Proof. Let X be a random variable with density function (7). The r_{th} non-central moment of the $McQL$ distribution is given by

$$\mu_r^{'}(x) = E(X^{r)} = \int_0^\infty x^r f(x,\phi)dx$$
$$= \frac{c}{B(a,b)}\frac{\theta}{\alpha+1}\int_0^\infty x^r(\alpha+\theta x)e^{-\theta x}\left[1-e^{-\theta x}\left[1+\frac{\theta x}{\alpha+1}\right]\right]^{ac-1}$$
$$\times\left[1-\left\{1-e^{-\theta x}\left[1+\frac{\theta x}{\alpha+1}\right]\right\}^c\right]^{b-1}dx \qquad (15)$$

using the fact that

$$\left[1-\left\{1-e^{-\theta x}\left[1+\frac{\theta x}{\alpha+1}\right]\right\}^c\right]^{b-1} = \sum_{j=0}^\infty(-1)^j\binom{b-1}{j}\left\{1-e^{-\theta x}\left[1+\frac{\theta x}{\alpha+1}\right]\right\}^{cj}$$

then

$$\mu_r^{'}(x) = \sum_{j=0}^\infty(-1)^j\binom{b-1}{j}\frac{c}{B(a,b)}\frac{\theta}{\alpha+1}\int_0^\infty x^r(\alpha+\theta x)e^{-\theta x}\left[1-e^{-\theta x}\left[1+\frac{\theta x}{\alpha+1}\right]\right]^{c(a+j)-1}dx.$$

again

$$\left[1-e^{-\theta x}\left[1+\frac{\theta x}{\alpha+1}\right]\right]^{c(a+j)-1} = \sum_{k=0}^\infty(-1)^k\binom{c(a+j)-1}{k}e^{-\theta kx}\left[1+\frac{\theta x}{\alpha+1}\right]^k$$

therefore

$$\mu_r^{'}(x) = \sum_{j,k}^\infty(-1)^{j+k}\binom{b-1}{j}\binom{c(a+j)-1}{k}\int_0^\infty s^r(\alpha+\theta x)$$
$$\times\, e^{-\theta(k+1)x}\left[1+\frac{\theta x}{\alpha+1}\right]^k dx$$

also by using binomial expansion, we get

$$\left[1+\frac{\theta x}{\alpha+1}\right]^k = \sum_{i=0}^\infty\binom{k}{i}(\frac{\theta}{\alpha+1})^i x^i$$

Now

$$\mu_r^{'}(x) = w_{i,j,k} \int_0^\infty x^{r+i}(\alpha + \theta x)e^{-\theta(k+1)x}dx$$
$$= w_{i,j,k} \left[\alpha \int_0^\infty x^{r+i} e^{-\theta(k+1)x}dx + \theta \int_0^\infty x^{r+i+1} e^{-\theta(k+1)x}dx \right]$$
$$= w_{i,j,k} \left[\frac{\alpha\Gamma(r+i+1)}{(\theta(k+1))^{r+i+1}} + \frac{\theta\Gamma(r+i+2)}{(\theta(k+1))^{r+i+2}} \right]$$

where

$$w_{i,j,k} = \sum_{i,j,k}^{\infty} (-1)^{j+k} \binom{b-1}{j} \binom{c(a+j)-1}{k} \binom{k}{i}$$

Based on the first four moments of the $McQL$ distribution, the measures of skewness $A(\Phi)$ and kurtosis $k(\Phi)$ of the $McQL$ distribution can obtained as

$$A(\Phi) = \frac{\mu_3(\theta) - 3\mu_1(\theta)\mu_2(\theta) + 2\mu_1^3(\theta)}{\left[\mu_2(\theta) - \mu_1^2(\theta)\right] \frac{3}{2}},$$

and

$$k(\Phi) = \frac{\mu_4(\theta) - 4\mu_1(\theta)\mu_3(\theta) + 6\mu_1^2(\theta)\mu_2(\theta) - 3\mu_1^4(\theta)}{\left[\mu_2(\theta) - \mu_1^2(\theta)\right]^2}.$$

3.3. Moment Generating function

In this subsection we derived the moment generating function of $McQL$ distribution.

Theorem 2. *If X has $McQL$ distribution, then the moment generating function $M_X(t)$ has the following form*

$$M_X(t) = w_{i,j,k} \left[\frac{\alpha\Gamma(i+1)}{(\theta(k+1)-t)^{i+1}} + \frac{\theta\Gamma(i+2)}{(\theta(k+1)-t)^{i+2}} \right]. \tag{16}$$

Proof. We start with the well known definition of the moment generating function given by

$$
\begin{aligned}
M_X(t) = E(e^{tX}) &= \int_0^{\infty} e^{tx} f_{McQL}(x,\phi)dx \\
&= w_{i,j,k} \int_0^{\infty} x^i(\alpha+\theta x)e^{-x(\theta(k+1)-t)}dx \\
&= w_{i,j,k}\left[\frac{\alpha\Gamma(i+1)}{(\theta(k+1)-t)^{i+1}} + \frac{\theta\Gamma(i+2)}{(\theta(k+1)-t)^{i+2}}\right] \qquad (17)
\end{aligned}
$$

which completes the proof.

4. Distribution of the order statistics

In this section, we derive closed form expressions for the pdfs of the r^{th} order statistic of the $McQL$ distribution, also, the measures of skewness and kurtosis of the distribution of the r^{th} order statistic in a sample of size n for different choices of $n; r$ are presented in this section. Let $X_1, X_2, ..., X_n$ be a simple random sample from $McQL$ distribution with pdf and cdf given by (7) and (8), respectively.

Let $X_1, X_2, ..., X_n$ denote the order statistics obtained from this sample. We now give the probability density function of $X_{r:n}$, say $f_{r:n}(x,\phi)$ and the moments of $X_{r:n}$, $r = 1, 2, ..., n$. Therefore, the measures of skewness and kurtosis of the distribution of the $X_{r:n}$ are presented. The probability density function of $X_{r:n}$ is given by

$$
f_{r:n}(x,\Phi) = \frac{1}{B(r,n-r+1)}\left[F(x,\phi)\right]^{r-1}\left[1-F(x,\phi)\right]^{n-r} f(x,\phi) \qquad (18)
$$

where $f(x,\phi)$ and $F(x,\phi)$ are the pdf and cdf of the $McQL$ distribution given by (7) and (8), respectively, and $B(.,.)$ is the beta function, since $0 < F(x,\phi) < 1$, for $x > 0$, by using the binomial series expansion of $[1-F(x,\phi)]^{n-r}$, given by

$$
\left[1-F(x,\phi)\right]^{n-r} = \sum_{j=0}^{n-r}(-1)^j\binom{n-r}{j}\left[F(x,\phi)\right]^j, \qquad (19)
$$

we have

$$
f_{r:n}(x,\phi) = \sum_{j=0}^{n-r}(-1)^j\binom{n-r}{j}\left[F(x,\Phi)\right]^{r+j-1} f(x,\phi), \qquad (20)
$$

substituting from (7) and (8) into (20), we can express the k^{th} ordinary moment of the r^{th} order statistics $X_{r:n}$ say $E(X_{r:n}^k)$ as a liner combination of the k^{th} moments of the $McQL$ distribution with different shape parameters. Therefore, the measures of skewness and kurtosis of the distribution of $X_{r:n}$ can be calculated.

5. Estimation and Inference

In this section, we determine the maximum likelihood estimates (MLEs) of the parameters of the $McQL$ distribution from complete samples only. Let $X_1, X_2, ..., X_n$ be a random sample of size n from $McQL(\phi, x)$. The likelihood function for the vector of parameters $\phi = (\alpha, \theta, a, b, c)$ can be written as

$$\begin{aligned}
Lf(x_{(i)}, \Phi) &= \Pi_{i=1}^{n} f(x_{(i)}, \phi) \\
&= \left(\frac{c\theta}{B(a,b)(\alpha+1)}\right)^{n} \Pi_{i=1}^{n}(\alpha+\theta x_{(i)}) e^{-\theta \sum\limits_{i=1}^{n} x_{(i)}} \Pi_{i=1}^{n}\left[1-e^{-\theta x}\left[1+\frac{\theta x}{\alpha+1}\right]\right]^{ac-1} \\
&\times \Pi_{i=1}^{n}\left[1-\left\{1-e^{-\theta x}\left[1+\frac{\theta x}{\alpha+1}\right]\right\}^{c}\right]^{b-1}. \qquad (21)
\end{aligned}$$

Taking the log-likelihood function for the vector of parameters $\phi = (\alpha, \theta, a, b, c)$ we get

$$\begin{aligned}
\log L &= n\log c + n\log\theta - n\log(1+\alpha) + n\log\left[\Gamma(a+b)\right] - n\log\left[\Gamma(a)\right] - n\log\left[\Gamma(b)\right] \\
&+ \sum_{i=1}^{n}\log(\alpha+\theta x_{(i)}) - \theta\sum_{i=1}^{n} x_{(i)} + (ac-1)\sum_{i=1}^{n}\log\left[1-e^{-\theta x_{(i)}}\left[1+\frac{\theta x_{(i)}}{\alpha+1}\right]\right] \\
&+ (b-1)\sum_{i=1}^{n}\log\left[1-\left\{1-e^{-\theta x_{(i)}}\left[1+\frac{\theta x_{(i)}}{\alpha+1}\right]\right\}^{c}\right]. \qquad (22)
\end{aligned}$$

The log-likelihood can be maximized either directly or by solving the nonlinear likelihood equations obtained by differentiating (5). The components of the score vector are given by

$$\frac{\partial \log L}{\partial a} = n\psi(a+b) - n\psi(a) + c\sum_{i=1}^{n}\log\left[1-e^{-\theta x_{(i)}}\left[1+\frac{\theta x_{(i)}}{\alpha+1}\right]\right], \qquad (23)$$

$$\frac{\partial \log L}{\partial b} = n\psi(a+b) - n\psi(b) + \sum_{i=1}^{n}\log\left[1-\left[1-e^{-\theta x_{(i)}}\left[1+\frac{\theta x_{(i)}}{\alpha+1}\right]\right]^{c}\right], \qquad (24)$$

$$\begin{aligned}
\frac{\partial \log\ L}{\partial c} &= \frac{n}{c} + a\sum_{i=1}^{n}\log\left[1-\left[1-e^{-\theta x_{(i)}}\left[1+\frac{\theta x_{(i)}}{\alpha+1}\right]\right]\right] \\
&- (b-1)\sum_{i=1}^{n}\frac{\left\{1-e^{-\theta x_{(i)}}\left[1+\frac{\theta x_{(i)}}{\alpha+1}\right]\right\}^{c}\log\left\{1-e^{-\theta x_{(i)}}\left[1+\frac{\theta x_{(i)}}{\alpha+1}\right]\right\}}{\left[1-\left\{1-e^{-\theta x_{(i)}}\left[1+\frac{\theta x_{(i)}}{\alpha+1}\right]\right\}^{c}\right]}, \qquad (25)
\end{aligned}$$

$$\frac{\partial \log\ L}{\partial \theta} = \frac{n}{\theta} + \sum_{i=1}^{n} \frac{x_{(i)}}{(\alpha+\theta x_{(i)})} - \sum_{i=1}^{n} x_{(i)} + (ac-1)\sum_{i=1}^{n} \frac{x_{(i)}(\alpha+\theta x_{(i)})(e^{-\theta x_{(i)}})}{\left[1-e^{-\theta x_{(i)}}\left[1+\frac{\theta x_{(i)}}{\alpha+1}\right]\right]}$$
$$+ c(b-1)\sum_{i=1}^{n} \frac{x_{(i)}(\alpha+\theta x_{(i)})(e^{-\theta x_{(i)}})\left\{1-e^{-\theta x_{(i)}}\left[1+\frac{\theta x_{(i)}}{\alpha+1}\right]\right\}^{c-1}}{\left[1-\left\{1-e^{-\theta x_{(i)}}\left[1+\frac{\theta x_{(i)}}{\alpha+1}\right]\right\}^{c}\right]}, \qquad (26)$$

and

$$\frac{\partial \log\ L}{\partial \alpha} = \frac{-n}{\alpha+1} - \sum_{i=1}^{n} \frac{1}{(\alpha+\theta x_{(i)})} + (ac-1)\sum_{i=1}^{n} \frac{\theta x_{(i)} e^{-\theta x_{(i)}}}{(\alpha+1)^2\left[1-e^{-\theta x_{(i)}}\left[1+\frac{\theta x_{(i)}}{\alpha+1}\right]\right]}$$
$$- c(b-1)\sum_{i=1}^{n} \frac{\theta x_{(i)} e^{-\theta x_{(i)}}\left\{1-e^{-\theta x}\left[1+\frac{\theta x}{\alpha+1}\right]\right\}^{c-1}}{(\alpha+1)^2\left[1-\left\{1-e^{-\theta x}\left[1+\frac{\theta x}{\alpha+1}\right]\right\}^{c}\right]}, \qquad (27)$$

and where $\psi(.)$ is the digamma function. We can find the estimates of the unknown parameters by maximum likelihood method by setting these above non-linear equations (23)- (27) to zero and solve them simultaneously. Therefore, we have to use mathematical package to get the MLE of the unknown parameters.

Approximate two sided $100(1-\alpha)\%$ confidence intervals for θ, α, a, b and for c are, respectively, given by

$$\hat{\theta} \pm z_{\alpha/2}\sqrt{I_{11}^{-1}(\hat{\theta})}, \hat{\alpha} \pm z_{\alpha/2}\sqrt{I_{22}^{-1}(\hat{\alpha})}, \hat{a} \pm z_{\alpha/2}\sqrt{I_{33}^{-1}(\hat{a}},$$
$$\hat{b} \pm z_{\alpha/2}\sqrt{I_{44}^{-1}(\hat{b})}, \qquad \text{and} \qquad \hat{c} \pm z_{\alpha/2}\sqrt{I_{55}^{-1}(\hat{c})},$$

where z_α is the upper αth quantile of the standard normal distribution. Using `R` we can easily compute the Hessian matrix and its inverse and hence the standard errors and asymptotic confidence intervals.

We can compute the maximized unrestricted and restricted log-likelihood functions to construct the likelihood ratio (LR) test statistic for testing on some McQL sub-models. For example, we can use the LR test statistic to check whether the McQL distribution for a given data set is statistically *superior* to the quasi Lindley distribution. In any case, hypothesis tests of the type $H_0 : \phi = \phi_0$ versus $H_0 : \phi \neq \phi_0$ can be performed using a LR test. In this case, the LR test statistic for testing H_0 versus H_1 is $\omega = 2(\ell(\hat{\phi}; x) - \ell(\hat{\phi}_0; x))$, where $\hat{\phi}$ and $\hat{\phi}_0$ are the MLEs under H_1 and H_0, respectively. The statistic ω is asymptotically (as $n \to \infty$) distributed as χ_k^2, where k is the length of the parameter vector ϕ of interest. The LR test rejects H_0 if $\omega > \chi^2_{k;\gamma}$, where $\chi^2_{k;\gamma}$ denotes the upper $100\gamma\%$ quantile of the χ_k^2 distribution.

6. Application

Now we use a real data set to show that the McDonald Quasi Lindley distribution can be a better model than the Quasi Lindley distribution and Lindley distribution.

The real data set corresponds to an uncensored data set from Nichols and Padgett (2006) on breaking stress of carbon fibres (in Gba): 3.70, 2.74, 2.73, 2.50, 3.60, 3.11, 3.27, 2.87, 1.47, 3.11, 4.42, 2.41, 3.19,3.22, 1.69, 3.28, 3.09, 1.87, 3.15, 4.90, 3.75, 2.43, 2.95, 2.97, 3.39, 2.96, 2.53, 2.67, 2.93, 3.22, 3.39,2.81, 4.20, 3.33, 2.55, 3.31, 3.31, 2.85, 2.56, 3.56, 3.15, 2.35, 2.55, 2.59, 2.38, 2.81, 2.77, 2.17, 2.83,1.92, 1.41, 3.68, 2.97, 1.36, 0.98, 2.76, 4.91, 3.68, 1.84, 1.59, 3.19, 1.57, 0.81, 5.56, 1.73, 1.59, 2.00,1.22, 1.12, 1.71, 2.17, 1.17, 5.08, 2.48, 1.18, 3.51, 2.17, 1.69, 1.25, 4.38, 1.84, 0.39, 3.68, 2.48, 0.85, 1.61, 2.79, 4.70, 2.03, 1.80, 1.57, 1.08, 2.03, 1.61, 2.12, 1.89, 2.88, 2.82, 2.05, 3.65.

Table 1: Estimated parameters of the Lindley, Quasi Lindley and McQL distribution for the breaking stress of carbon fibres (in Gba).

Model	Parameter Estimate	Standard Error	$-\ell(\cdot;x)$
Lindley	$\hat{\theta}=0.617$	0.045	181.753
Quasi-Lindley	$\hat{\alpha}=-0.378$	0.033	149.155
	$\hat{\theta}=0.995$	0.079	
McDonald	$\hat{\alpha}=0.445$	3.322	141.251
Quasi-Lindley	$\hat{\theta}=0.666$	1.324	
	$\hat{a}=0.414$	1.576	
	$\hat{b}=5.213$	15.910	
	$\hat{c}=6.890$	34.615	

The variance covariance matrix of the MLEs under the McDonald Quasi Lindley distribution is computed as

$$I(\hat{\theta})^{-1}=\begin{pmatrix} 11.042 & 1.194 & -1.607 & -15.54 & 65.167 \\ 1.194 & 1.753 & -2.058 & -20.853 & 43.134 \\ -1.607 & -2.058 & 2.486 & 24.042 & -52.225 \\ -15.545 & -20.853 & 24.042 & 253.142 & -510.843 \\ 65.167 & 43.134 & -52.225 & -510.843 & 1198.209 \end{pmatrix}.$$

Thus, the variances of the MLE of θ,α,a,b and c is $var(\hat{\theta})=11.042, var(\hat{\alpha})=1.753, var(\hat{a})=2.486, var(\hat{b})=253.142$ and $var(\hat{c})=1198.209$. Therefore, 95% confidence intervals for θ,α,a,b and c are $[0,6.958],[-1,3.261],[0,3.505],[0,36.398]$ and $[0,74.735]$ respectively. The LR test statistic to test the hypotheses $H_0:a=$

$b = c = 1$ versus $H_1 : a \neq 1 \vee b \neq 1 \vee c \neq 1$ is $\omega = 15.808 > 7.815 = \chi^2_{3;0.05}$, so we reject the null hypothesis.

Table 2: Criteria for comparison.

Model	-2ℓ	AIC	AICC
Lindley	363.506	365.506	365.546
Quasi Lindley	298.31	302.31	302.433
McQL	282.502	292.502	293.102

In order to compare the two distribution models, we consider criteria like -2ℓ, AIC (Akaike information criterion)and AICC (corrected Akaike information criterion) for the data set. The better distribution corresponds to smaller -2ℓ, AIC and AICC values:‘

$$\mathrm{AIC} = 2k - 2\ell\,, \quad \mathrm{and AICC} = \mathrm{AIC} + \frac{2k(k+1)}{n-k-1}\,,$$

where k is the number of parameters in the statistical model, n the sample size and ℓ is the maximized value of the log-likelihood function under the considered model. Also, here for calculating the values of KS we use the sample estimates of θ, α, a, b and c. Table 1 shows the MLEs under both distributions, Table 2 shows the values of -2ℓ, AIC and AICC values. The values in Table 2 indicate that the McQL distribution leads to a better fit than the quasi Lindley distribution.

A density plot compares the fitted densities of the models with the empirical histogram of the observed data (Fig. 4). The fitted density for the McQL model is closer to the empirical histogram than the fits of the QL and Lindley sub-models.

7. Conclusion

Here we propose a new model, the so-called the McDonald Quasi Lindley distribution which extends the quasi Lindley distribution in the analysis of data with real support. An obvious reason for generalizing a standard distribution is because the generalized form provides larger flexibility in modeling real data. We derive expansions for the mean, variance, moments and for the moment generating function. The estimation of parameters is approached by the method of maximum likelihood, also the information matrix is derived. We consider the likelihood ratio statistic to compare the model with its baseline model. An application of the McQL distribution to real data show that the new distribution can be used quite effectively to provide better fits than Lindley and quasi Lindley distribution.

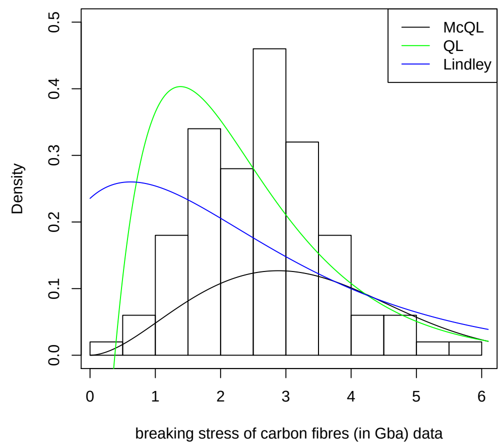

Figure 4: Estimated densities of the models for breaking stress of carbon fibres (in Gba).

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Faton Merovci
University of Mitrovica "Isa Boletini", Mitrovic,40000
Kosovo
email: *fmerovci@yahoo.com*

Ibrahim Elbatal
Institute of Statistical Studies and Research,

Department of Mathematical Statistics,
Cairo University
email: *i_elbatal @staff. cu. edu.eg*

Llukan Puka
Department of Mathematics,
Faculty of Natural Science, University of Tirana,
Albania
email: *lpuka2001@yahoo.co.uk*