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HARDY'S TYPE INEQUALITY FOR PSEUDO-INTEGRALS

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Abstract. In this paper, we prove Hardy's type inequality for two classes of pseudo-integrals. One of them concerns the pseudo-integrals based on a function reduces on the g-integral where pseudo-operations are defined by a monotone and continuous function g. The other one concerns the pseudo-integrals based on a semiring $([a,b], \max, \odot)$ where $\odot$ generated.

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1. Introduction and Preliminaries

Recently, some authors ([3, 10, 17, 18]) have studied some fuzzy integral inequalities. The purpose of this paper is to prove a Hardy type inequality for the pseudo-integrals.

Pseudo-analysis is a generalization of the classical analysis, where instead of the field of real numbers a semiring is taken on a real interval $[a,b] \subset [-\infty, \infty]$ endowed with pseudo-addition $\oplus$ and with pseudo-multiplication $\odot$ ([1, 2, 9, 11, 12, 19]). Based on this structure there where developed the concepts of $\oplus$-measure (pseudo-additive measure), pseudo-integral, pseudo-convolution, pseudo-Laplace transform and etc. ([4, 5, 6, 13, 15, 16, 18]).

The well-known Hardy inequality is a part of the classical mathematical analysis ([7]). The classical Hardy's integral inequality holds

$$\Big(\frac{P}{P-1}\Big)^P \int_0^\infty f^P(x)dx > \int_0^\infty \Big(\frac{F}{x}\Big)^P dx,$$

where $P > 1$ and $f : [0,\infty) \to [0,\infty)$ is an integrable function ($f \neq 0$) and $F(x) = \int_0^x f(t)dt$. Furthermore, for parameters a, b such that $0 < a < b < \infty$, the following inequality is also valid ([20]):

$$\Big(\frac{P}{P-1}\Big)^P \int_a^b f^P(x)dx > \int_a^b \Big(\frac{F}{x}\Big)^P dx,$$

where $0 < \int_0^\infty f^P(t)dt < \infty$. H. Román-Flores et al. have proved a Hardy type inequality for fuzzy integrals ([17]). The fuzzy Hardy's integral inequality holds

$$\Big(⨍_0^1 f^P(x)dx\Big)^{\frac{1}{P+1}} \geq ⨍_0^1 \Big(\frac{F}{x}\Big)^P dx \tag{1}$$

where $P \geq 1$, $f : [0,1] \to [0, \infty)$ is an integrable function and $F(x) = ⨍_0^x f(t)dt$.

In this paper, we generalize their work for pseudo-integrals. In special case, if in the pseudo-integral version of the Hardy type inequality we put $\oplus = max$ and $\odot = min$, then we get the fuzzy Hardy type inequality that has been studied in ([17]) by H. Román-Flores et al.

Let $[a, b]$ be a closed (in some cases can be considered semiclosed) subinterval of $[-\infty, \infty]$. The full order on $[a, b]$ will be denoted by $\preceq$.

The operation $\oplus$ (pseudo-addition) is a function $\oplus : [a, b] \times [a, b] \to [a, b]$ which is commutative, nondecreasing (with respect to $\preceq$), associative and with a zero (neutral) element denoted by $\mathbf{0}$, i.e., for each $x \in [a, b], \mathbf{0} \oplus x = x$ holds (usually $\mathbf{0}$ is either a or b). Let $[a, b]_+ = \{x | x \in [a, b], \mathbf{0} \preceq x\}$.

Definition 1. *The operation $\odot$ (pseudo-multiplication) is a function $\odot : [a, b] \times [a, b] \to [a, b]$ which is commutative, positively non-decreasing, i.e., $x \preceq y$ implies $x \odot z \preceq y \odot z$ for all $z \in [a, b]_+$, associative and for which there exists a unit element $\mathbf{1} \in [a, b]$, i.e., for each $x \in [a, b], \mathbf{1} \odot x = x$.*

We assume also $\mathbf{0} \odot x = \mathbf{0}$ that $\odot$ is a distributive pseudo-multiplication with respect to $\oplus$, i.e., $x \odot (y \oplus z) = (x \odot y) \oplus (x \odot z)$. The structure $([a, b], \oplus, \odot)$ is a semiring ([8, 17]). In this paper, we will consider semirings with the following continuous operations:

Case I: The pseudo-addition is idempotent operation and the pseudo-multiplication is not.

(a) $x \oplus y = \sup(x, y), \odot$ is arbitrary not idempotent pseudo-multiplication on the interval $[a, b]$. We have $\mathbf{0} = a$ and the idempotent operation sup induces a full order in the following way: $x \preceq y$ if and only if $\sup(x, y) = y$.

(b) $x \oplus y = \inf(x, y), \odot$ is arbitrary not idempotent pseudo-multiplication on the interval $[a, b]$. We have $\mathbf{0}= b$ and the idempotent operation *inf* induces a full order in the following way: $x \preceq y$ if and only if $\inf(x, y) = y$.

Case II: The pseudo-operations are defined by a monotone and continuous function $g : [a, b] \to [0, \infty]$, i.e., pseudo operations are given with $x \oplus y = g^{-1}(g(x)+g(x))$ and $x \odot y = g^{-1}(g(x)g(x))$. If the zero element for the pseudo-addition is a, we will consider increasing generators. Then $g(a) = 0$ and $g(b) = \infty$. If the zero element for the pseudo-addition is b, we will consider decreasing generators. Then $g(b) = 0$ and $g(a) = \infty$. If the generator g is increasing (respectively decreasing), then the operation $\oplus$ induces the usual order (respectively opposite to the usual order) on the interval $[a, b]$ in the following way: $x \preceq y$ if and only if $g(x) \leq g(y)$.

Case III: Both operations are idempotent. We have
(a) $x \oplus y = \sup(x, y), x \odot y = \inf(x, y)$, on the interval $[a, b]$. We have $\mathbf{0} = a$ and $\mathbf{1} = b$. The idempotent operation *sup* induces the usual order ($x \preceq y$ if and only if $\sup(x, y) = y$).
(b) $x \oplus y = \inf(x, y), x \odot y = \sup(x, y)$, on the interval $[a, b]$. We have $\mathbf{0} = b$ and $\mathbf{1} = a$. The idempotent operation *inf* induces an order opposite to the usual order ($x \preceq y$ if and only if $\inf(x, y) = y$).

Let X be a non-empty set. Let $\mathbb{A}$ be a σ-algebra of subsets of a set X.

We shall consider the semiring $([a, b], \oplus, \odot)$, when pseudo-operations are generated by a monotone and continuous function $g : [a, b] \to [0, \infty]$, i.e., pseudo-operations are given with $x \oplus y = g^{-1}(g(x) + g(y))$ and $x \odot y = g^{-1}(g(x)g(y))$.

Then the pseudo-integral for a function $f : [c, d] \to [a, b]$ reduces on the g-integral ([12, 14]),

$$\int_{[c,d]}^{\oplus} f(x)dx = g^{-1}\Big(\int_c^d g(f(x))dx\Big). \tag{2}$$

More on this structure as well as corresponding measures and integrals can be found in [7, 11]. The second class is when $x \oplus y = \max(x, y)$ and $x \odot y = g^{-1}(g(x)g(y))$, the pseudo-integral for a function $f : \mathbb{R} \to [a, b]$ is given by

$$\int_{\mathbb{R}}^{\oplus} f \odot dm = \sup\Big(f(x) \odot \psi(x)\Big),$$

where function ψ defines sup-measure m. Any sup-measure generated as essential supremum of a continuouse denisty can be obtained as a limit of pseudo-additive measures with respect to generated pseudo-additive [5]. For any continuouse function $f : [0, \infty] \to [0, \infty]$ the integral $\int^{\oplus} f \odot dm$ can be obtained as a limit of g-integrals, [5]. We denoted by μ the usual Lebesgue measure on $\mathbb{R}$. We have

$$m(A) = ess \sup(x|x \in A) = \sup\{a|\mu(x|x \in A, x > a) > 0\}.$$

Theorem 1. *([9]). Let m be a* sup*-measure on* $([0, \infty], \mathbb{B}[0, \infty])$*, where* $\mathbb{B}([0, \infty])$ *is the Borel σ-algebra on* $[0, \infty]$*,* $m(A) = ess \sup_{\mu}(\psi(x)|x \in A)$*, and* $\psi : [0, \infty] \to [0, \infty]$ *is a continuouse density. Then for any pseudo-addition $\oplus$ with a generator g there exists a family m_λ of $\oplus_\lambda$-measure on* $([0, \infty], \mathbb{B})$*, where $\oplus_\lambda$ is a generated by g^λ (the function g of the power λ), $\lambda \in (0, \infty)$, such that* $\lim_{\lambda \to \infty} m_\lambda = m$.

Theorem 2. *([9]). Let* $([0, \infty], \sup, \odot)$ *be a semiring, when $\odot$ is a generated with g, i.e., we have $x \odot y = g^{-1}(g(x)g(y))$ for every $x, y \in (0, \infty)$. Let m be the same* *as in Theorem 2.1., Then there exists a family $\{m_\lambda\}$ of $\oplus_\lambda$ -measures, where $\oplus_\lambda$ is a generated by $g^\lambda, \lambda \in (0,\infty)$ such that for every continuous function $f : [0,\infty] \to [0,\infty]$,*

$$\int^{\sup} f \odot dm = \lim_{\lambda\to\infty} \int^{\oplus_\lambda} f \odot dm_\lambda = \lim_{\lambda\to\infty} (g^\lambda)^{-1}\Big(\int g^\lambda(f(x))dx\Big).$$

Now, we recall the following inequality which is the pseudo version of Chebyshev's inequality and appears ([1]).

Theorem 3. *(Chebyshev's inequality for pseudo-integrals). Let $f, h : [0,1] \to [0,1]$ be two measurable function and $g : [a,b] \to [0,\infty)$ be an increasing generator function for pseudo-operation. If f, h are comonotone, then the inequality*

$$\int_{[0,1]}^{\oplus} (f \odot h)dx \geq (\int_{[0,1]}^{\oplus} f dx) \odot (\int_{[0,1]}^{\oplus} hdx)$$

holds.

Theorem 4. *([12]). For any measurable function f, f_1, f_2 and $\lambda \in \mathbb{R}$, we have*
(i) $\int_{[c,d]}^{\oplus}(f_1 \oplus f_2)dx = \int_{[c,d]}^{\oplus} f_1dx \oplus \int_{[c,d]}^{\oplus} f_2dx,$
(ii) $\int_{[c,d]}^{\oplus}(\lambda \otimes f)dx = \lambda \otimes \int_{[c,d]}^{\oplus} fdx,$
(iii)$f_1 \leq f_2 \Longrightarrow \int_{[c,d]}^{\oplus} f_1dx \leq \int_{[c,d]}^{\oplus} f_2dx.$

2. Hardy's inequality for pseudo-integrals

Our purpose in this section is to prove a Hardy type inequality for pseudo-integrals. Unfortunately, the following example shows that, the Hardy's integral inequality is not valid for the pseudo-integrals.

Example 1. *Let $f(x) = k$ where $k > 1$ and $P \geq 1$. If $g : [0,1] \to [0,1]$ is defined as follows*

$$g(x) = x.$$

Then by using (2) we have

$$\begin{aligned}
\Big(\int_{[0,1]}^{\oplus} f^P(x)dx\Big)^{\frac{1}{P+1}} &= \Big(\int_{[0,1]}^{\oplus} k^P dx\Big)^{\frac{1}{P+1}} \\
&= \Big(g^{-1}\int_0^1 g(k^P)dx\Big)^{\frac{1}{P+1}} \\
&= \Big(g^{-1}\int_0^1 k^P dx\Big)^{\frac{1}{P+1}} \\
&= \Big(g^{-1}(k^P)\Big)^{\frac{1}{P+1}} \\
&= \Big(k^P\Big)^{\frac{1}{P+1}} \\
&= k^{\frac{P}{P+1}}.
\end{aligned}$$

Since

$$F(x) = \int_{[0,x]}^{\oplus} f(t)dt,$$

then by (2) we obtain that

$$\begin{aligned}
F(x) = g^{-1}\int_0^x g(f(t))dt &= g^{-1}\int_0^x g(k)dt \\
&= g^{-1}\int_0^x kdt \\
&= g^{-1}(kx) \\
&= kx.
\end{aligned}$$

It follows that

$$\frac{F(x)}{x} = k.$$

So by using (2) we have

$$\begin{aligned}
\int_{[0,1]}^{\oplus} \Big(\frac{F}{x}\Big)^P dx &= g^{-1}\int_0^1 g\Big(\frac{F}{x}\Big)^P dx \\
&= g^{-1}\int_0^1 g(k^P)dx \\
&= g^{-1}\int_0^1 k^P dx \\
&= g^{-1}(k^P) \\
&= k^P.
\end{aligned}$$

Consequently, (1) is not valid for pseudo-integrals.

In order to prove Theorem 2.4. and 2.6. we need some Lemmas.

Lemma 5. *If $f:[0,1]\to[0,1]$ is a μ-measurable function and $g:[0,1]\to[0,1]$ is a continuous and decreasing function, then*

$$\int_{[0,1]}^{\oplus} f^P d\mu \geq \Big(\int_{[0,1]}^{\oplus} f d\mu\Big)^P \tag{3}$$

holds for all $P\geq 1$.

Proof. By induction: For $P=2$, inequality (3) is valid by Theorem 1.4.
For $P-1$, we suppose that the Lemma is valid as follows

$$\int_{[0,1]}^{\oplus} f^{P-1} d\mu \geq \Big(\int_{[0,1]}^{\oplus} f d\mu\Big)^{P-1}.$$

Hence for P we have

$$\begin{aligned}\int_{[0,1]}^{\oplus} f^P d\mu &= \int_{[0,1]}^{\oplus} f\dots f d\mu\\ &\geq \int_{[0,1]}^{\oplus} (f^{P-1}) f d\mu.\end{aligned}$$

So from case $P=2$, we get

$$\int_{[0,1]}^{\oplus} f^P d\mu \geq \Big(\int_{[0,1]}^{\oplus} f d\mu\Big)^P.$$

Thereby, the Lemma is proved.

Lemma 6. *Let $f:[0,1]\to[0,1]$ be a continuouse function. If m be the same as in Theorem 2.1., and $g:[0,1]\to[0,1]$ is a continuous and decreasing function, then*

$$\int_{[0,1]}^{\sup} f^P dm \geq \Big(\int_{[0,1]}^{\sup} f dm\Big)^P$$

holds for all $P\geq 1$.

Proof. Using the same arguments in Lemma 2.2. proof is easy.

Theorem 7. *(Pseudo Hardy's inequality). Let* $f:[0,1]\to[0,1]$ *be a* μ*-measurable and* $g:[0,1]\to[0,1]$ *be a continuous and decreasing function. If*

$$F(x)=\int_{[0,x]}^{\oplus} f(t)dt$$

where $x\in[0,1]$*, then the inequality*

$$\Big(\frac{P}{P-1}\Big)^{P}\int_{[0,1]}^{\oplus} f^{P}(x)dx > \int_{[0,1]}^{\oplus}\Big(\frac{F}{x}\Big)^{P}dx \tag{4}$$

holds for all $P>1$.

Proof. By using Lemma 2.2. we have

$$\begin{aligned}\int_{[0,1]}^{\oplus}\Big(\frac{F}{x}\Big)^{P}dx &= \int_{[0,1]}^{\oplus}\Big(\frac{\int_{[0,x]}^{\oplus} f(t)dt}{x}\Big)^{P}dx\\ &= \int_{[0,1]}^{\oplus}\frac{\Big(\int_{[0,x]}^{\oplus} f(t)dt\Big)^{P}}{x^{P}}dx\\ &\leq \int_{[0,1]}^{\oplus}\frac{\int_{[0,x]}^{\oplus} f^{P}(t)dt}{x^{P}}dx.\end{aligned}$$

Thus, by (2), we have

$$\begin{aligned}\int_{[0,1]}^{\oplus}\Big(\frac{F}{x}\Big)^{P}dx &\leq \int_{[0,1]}^{\oplus}\frac{\int_{[0,x]}^{\oplus} f^{P}(t)dt}{x^{P}}dx\\ &= \int_{[0,1]}^{\oplus}\int_{[0,x]}^{\oplus}\Big(\frac{f^{P}(t)}{x^{P}}\Big)dtdx\\ &= g^{-1}\int_{0}^{1} g\int_{[0,x]}^{\oplus}\Big(\frac{f^{P}(t)}{x^{P}}\Big)dtdx\\ &= g^{-1}\int_{0}^{1} g(g^{-1}\int_{0}^{x} g\Big(\frac{f^{P}(t)}{x^{P}}\Big)dt)dx\\ &= g^{-1}\int_{0}^{1}\int_{0}^{x} g(\frac{f(t)}{x})^{P}dtdx\\ &= g^{-1}\int_{0}^{1}\int_{0}^{x} g(f(t))^{P}g(\frac{1}{x^{P}})dtdx\\ &= g^{-1}\Big(\int_{0}^{1} g(\frac{1}{x^{P}})dx\Big)\Big(\int_{0}^{x} g(f(t))^{P}dt\Big).\end{aligned}$$

Since $\frac{1}{x^P} > 1$ and g is a decreasing function, we have $g(\frac{1}{x^P}) < g(1)$, It follows that

$$\begin{aligned}
\int_{[0,1]}^{\oplus} \Big(\frac{F}{x}\Big)^P dx &\leq g^{-1}\Big(\int_0^1 g(\frac{1}{x^P})dx\Big)\Big(\int_0^x g(f(t))^P dt\Big) \\
&< g^{-1}\Big(\int_0^1 g(1)dx\Big)\Big(\int_0^x g(f(t))^P dt\Big) \\
&= g^{-1}\Big(gg^{-1}\int_0^1 g(1)dx\Big)\Big(gg^{-1}\int_0^x g(f(t))^P dt\Big) \\
&= \Big(\int_{[0,1]}^{\oplus} 1dx\Big) \odot \Big(\int_{[0,x]}^{\oplus} g(f(t))^P dt\Big).
\end{aligned}$$

By using Theorem(1.5.(ii)), we have

$$\begin{aligned}
\int_{[0,1]}^{\oplus} \Big(\frac{F}{x}\Big)^P dx &< \Big(\int_{[0,1]}^{\oplus} 1dx\Big) \odot \Big(\int_{[0,x]}^{\oplus} g(f(t))^P dt\Big) \\
&< \Big(\int_{[0,x]}^{\oplus} g(f(t))^P dt\Big) \\
&< \Big(\int_{[0,1]}^{\oplus} g(f(x))^P dx\Big) \\
&< \Big(\frac{P}{P-1}\Big)^P \int_{[0,1]}^{\oplus} f^P(x)dx.
\end{aligned}$$

Which complete the proof.

Example 2. *Let $f(x) = \frac{1}{2}$, and $g : [0,1] \to [0,\infty]$ define as follows $g(x) = \frac{1}{x^2}$. By using (2) we have*

$$\begin{aligned}
\int_{[0,1]}^{\oplus} f^P(x)dx &= g^{-1}\int_0^1 g(f^P(x))dx \\
&= g^{-1}\int_0^1 g((\frac{1}{2})^P)dx \\
&= g^{-1}\int_0^1 \frac{1}{(\frac{1}{2^P})^2}dx \\
&= g^{-1}(2^{2P}) \\
&= \frac{1}{\sqrt{2^{2p}}} \\
&= \frac{1}{2^P}.
\end{aligned}$$

Then a straightforward calcules shows that

$$\begin{aligned}
F(x) &= \int_{[0,x]}^{\oplus} f(t)dt \\
&= g^{-1}\int_0^x g(\frac{1}{2})dt \\
&= g^{-1}\int_0^x 4dt \\
&= g^{-1}(4x) \\
&= \frac{1}{\sqrt{4x}} \\
&= \frac{1}{2\sqrt{x}}.
\end{aligned}$$

It follows that,

$$\frac{F(x)}{x} = \frac{1}{2x\sqrt{x}} = \frac{1}{2}x^{-\frac{3}{2}}$$

On the other hand,

$$\begin{aligned}
\int_{[0,1]}^{\oplus}\Big(\frac{F}{x}\Big)^p dx &= g^{-1}\int_0^1 g(\frac{F}{x})^P dx \\
&= g^{-1}\int_0^1 g\Big(\frac{1}{2}x^{-\frac{3}{2}}\Big)^P dx \\
&= g^{-1}\int_0^1 \frac{2^{2P}}{x^{-3P}}dx \\
&= g^{-1}\Big(\frac{2^{2P}}{3P+1}\Big) \\
&= \frac{1}{\sqrt{\frac{2^{2P}}{3P+1}}}.
\end{aligned}$$

This shows that the Hardy's inequality is valid for pseudo-integral.

Now, we generalize the Hardy type inequality by the semiring $([a,b],\max,\odot)$, where $\odot$ is generated.

Theorem 8. *Let $f:[0,1]\to[0,1]$ be a μ-measurable, $g:[0,1]\to[0,1]$ be a continuous and decreasing function and m be the same as in Theorem 2.1. If $\odot$ is represented by a decreasing multiplicative generator g and*

$$F(x) = \int_{[0,x]}^{\sup} f dm$$

where $x \in [0,1]$, then the inequality

$$\Big(\frac{P}{P-1}\Big)^P \int_{[0,1]}^{\sup} f^P dm > \int_{[0,1]}^{\sup} \Big(\frac{F}{x}\Big)^P dm \tag{5}$$

holds for all $P > 1$.

Proof. By using Lemma 2.3. and Theorem 1.3. we have

$$\begin{aligned}
\int_{[0,1]}^{\sup} \Big(\frac{F}{x}\Big)^P dm &= \lim_{\lambda\to\infty} \int_{[0,1]}^{\oplus_\lambda} \Big(\frac{F}{x}\Big)^P dm_\lambda \\
&= \lim_{\lambda\to\infty} (g^\lambda)^{-1} \int_0^1 g^\lambda \Big(\frac{F}{x}\Big)^P dx \\
&= \lim_{\lambda\to\infty} (g^\lambda)^{-1} \int_0^1 g^\lambda \Big(F(x)\Big)^P g^\lambda \Big(\frac{1}{x^p}\Big) dx \\
&= \lim_{\lambda\to\infty} (g^\lambda)^{-1} \int_0^1 g^\lambda \Big(\int_{[0,x]}^{\sup} f(t) dm\Big)^P g^\lambda \Big(\frac{1}{x^p}\Big) dx \\
&= \lim_{\lambda\to\infty} (g^\lambda)^{-1} \int_0^1 g^\lambda \Big(\lim_{\lambda\to\infty} \int_{[0,x]}^{\oplus_\lambda} f(t) dm_\lambda\Big)^P g^\lambda \Big(\frac{1}{x^P}\Big) dx \\
&\le \lim_{\lambda\to\infty} (g^\lambda)^{-1} \int_0^1 g^\lambda \Big(\lim_{\lambda\to\infty} \int_{[0,x]}^{\oplus_\lambda} f^P(t) dm_\lambda\Big) g^\lambda \Big(\frac{1}{x^P}\Big) dx \\
&= \lim_{\lambda\to\infty} (g^\lambda)^{-1} \int_0^1 g^\lambda \Big(\lim_{\lambda\to\infty} (g^\lambda)^{-1} \int_0^x g^\lambda (f^P(t)) dt\Big) g^\lambda \Big(\frac{1}{x^P}\Big) dx \\
&= \lim_{\lambda\to\infty} \lim_{\lambda\to\infty} (g^\lambda)^{-1} \int_0^1 \int_0^x g^\lambda (g^\lambda)^{-1} g^\lambda (f^P(t)) g^\lambda (\frac{1}{x^P}) dt dx.
\end{aligned}$$

Thus, we conclude

$$\begin{aligned}
\int_{[0,1]}^{\sup} \Big(\frac{F}{x}\Big)^P dm &\le \lim_{\lambda\to\infty} \lim_{\lambda\to\infty} (g^\lambda)^{-1} \int_0^1 \int_0^x g^\lambda (f^P(t)) g^\lambda (\frac{1}{x^P}) dt dx \\
&= \Big(\lim_{\lambda\to\infty} (g^\lambda)^{-1} \int_0^x g^\lambda (f^P(t)) dt\Big)\Big(\lim_{\lambda\to\infty} (g^\lambda)^{-1} \int_0^1 g^\lambda (\frac{1}{x^P}) dx\Big).
\end{aligned}$$

Since $\frac{1}{x^P} > 1, g$ is a decreasing function and $\lambda \in (0, \infty)$, so we have

$$g^\lambda(\frac{1}{x^P}) < g^\lambda(1),$$

then

$$\begin{aligned}
\int_{[0,1]}^{\sup}\Big(\frac{F}{x}\Big)^{P} dm &\leq \Big(\lim_{\lambda\to\infty}(g^{\lambda})^{-1}\int_0^x g^{\lambda}(f^P(t))dt\Big)\Big(\lim_{\lambda\to\infty}(g^{\lambda})^{-1}\int_0^1 g^{\lambda}(\frac{1}{x^P})dx\Big)\\
&< \Big(\lim_{\lambda\to\infty}(g^{\lambda})^{-1}\int_0^x g^{\lambda}(f^P(t))dt\Big)\Big(\lim_{\lambda\to\infty}(g^{\lambda})^{-1}\int_0^1 g^{\lambda}(1)dx\Big)\\
&< \Big(\lim_{\lambda\to\infty}\int_{[0,x]}^{\oplus_\lambda}(f^P(t))dm\Big)\Big(\lim_{\lambda\to\infty}\int_{[0,1]}^{\oplus_\lambda}(1)dm\Big)\\
&< \Big(\int_{[0,x]}^{\sup} f^P(t)dm\Big)\\
&< \Big(\int_{[0,1]}^{\sup} f^P(x)dm\Big)\\
&< \Big(\frac{P}{P-1}\Big)^P\int_{[0,1]}^{\sup} f^P(x)dm.
\end{aligned}$$

Which complete the proof.

Example 3. *Let $f:[0,1]\to[0,1]$ be a μ-measurable, and $g^{\lambda}(x)=x^{-\lambda}$. So*

$$x\oplus y=(x^{-\lambda}+y^{-\lambda})^{-\lambda} \quad \text{and} \quad x\odot y=xy.$$

Therefore Relation (5) reduces on the following inequality:

$$\sup\Big((\frac{F}{x})^P+\psi(x)\Big)<(\frac{P}{P-1})^P\sup\Big(f^p(x)+\psi(x)\Big).$$

where ψ is from Theorem 2.1.

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