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ON SPACELIKE PARALLEL P_I-EQUIDISTANT RULED SURFACES IN THE MINKOWSKI 3-SPACE R_1^3

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Abstract. In this paper, radii and curvature axes of osculator Lorentz spheres and arc lengths of indicatrix curves of base curves of spacelike parallel p_i-Equidistant ruled surfaces in the Minkowski 3-space R_1^3 are given.

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1. Introduction

I. E. Valeontis, (see [3]), defined parallel p-equidistant ruled surfaces in E^3 and gave some results related with striction curves of ruled surfaces. Then he also studied on existence theorem related with homothety of parallel p-equidistant ruled surfaces.
M. Masal, N. Kuruoğlu, (see [1]) obtained arc lengths, curvature radii, curvature axes, spherical involute and areas of real closed spherical indicatrix curves of base curves (leading curves) of parallel p-equidistant ruled surfaces in E^3.
And also, M. Masal, N. Kuruoğlu, (see [2]) defined spacelike parallel p_i-equidistant ruled surfaces in the Minkowski 3-space R_1^3 and obtained dralls, the shape operators, Gaussian curvatures, mean curvatures, shape tensor, q^{th} fundamental forms of these surfaces.
This paper is organized as follows: in Section 3 we find radii and curvature axes of osculator Lorentz spheres of spacelike parallel p_i-equidistant ruled surfaces in the Minkowski 3-space.
And later in Section 4 we give arc lengths of indicatrice curves of spacelike parallel p_i-equidistant ruled surfaces.

2. Preliminaries

Let $\alpha : I \to R_1^3$, $\alpha(t) = (\alpha_1(t), \alpha_2(t), \alpha_3(t))$ be a differentiable spacelike curve with arc-length in the Minkowski 3-space, where I is an open interval in R containing the origin. Let V_1 be the tangent vector field of α, D be the Levi-Civita connection on R_1^3 and $D_{V_1}V_1$ be a spacelike vector. If V_1 moves along α, then we obtain a spacelike ruled surface which is given by the parametrization

$$M : \varphi(t,v) = \alpha(t) + vV_1(t). \tag{1}$$

$\{V_1, V_2, V_3\}$ is an Frenet frame field along α in R_1^3, where V_1 and V_2 are spacelike vectors and V_3 is a timelike vector, (see [2]). If k_1 and k_2 are the naturel curvature and torsion of $\alpha(t)$, respectively, then the Frenet formulas are, (see [4])

$$V_1' = k_1V_2, \quad V_2' = -k_1V_1 + k_2V_3, \quad V_3' = k_2V_2. \tag{2}$$

Using $V_1 = \alpha'$ and $V_2 = \frac{\alpha''}{\|\alpha''\|}$, we have $k_1 = \|\alpha''\| > 0$, where "$'$" means derivate with respect to time t, (see [2]).

Definition 1. *The planes which are corresponding to the subspaces $Sp\{V_1, V_2\}$, $Sp\{V_2, V_3\}$ and $Sp\{V_3, V_1\}$ are called* ***asymptotic plane****,* ***polar plane*** *and* ***central plane****, respectively, (see [2]).*

Definition 2. *Let M and M^* be two spacelike ruled surfaces in R_1^3; and p_1, p_2 and p_3 be the distances between the polar planes, central planes and asymptotic planes, respectively.*
If
i) *The generator vectors of M and M^* are parallel,*
ii) *The distances p_i, $1 \leq i \leq 3$, at the corresponding points of α and α^* are constant, then the pair of ruled surfaces M and M^* are called the spacelike parallel p_i-equidistant ruled surfaces in R_1^3. If $p_i = 0$, then the pair of M and M^* are called the spacelike parallel p_i-equivalent ruled surfaces in R_1^3.*

From the definition 2, the spacelike parallel p_i-equidistant ruled surfaces have the following parametric representations, (see [2]).

$$M : \varphi(t,v) = \alpha(t) + vV_1(t), \quad (t,v) \in I \times R,$$
$$M^* : \varphi^*(t^*,v^*) = \alpha^*(t^*) + v^*V_1(t^*), \quad (t^*,v^*) \in I \times R.$$

where, t and t^* are the arc parameters of curves α and α^*, respectively.
From now on M and M^* will be assumed the spacelike parallel p_i-equidistant ruled surfaces.

Theorem 1. ***i)*** *The Frenet frames $\{V_1, V_2, V_3\}$ and $\{V_1^*, V_2^*, V_3^*\}$ are equivalent at the corresponding points in M and M^*, respectively. (For $\frac{dt^*}{dt} > 0$.)*
ii) *If k_1 and k_1^* are the naturel curvatures and k_2, k_2^* are the torsions of base curves of M and M^*, respectively, then we have, (see [2]).*

$$k_i^* = k_i \frac{dt}{dt^*}, \quad 1 \leq i \leq 2.$$

3. The Osculator Lorentz Spheres of Spacelike Parallel p_i-equidistant Ruled Surfaces

In this Section, we will investigate radii and curvature axes of osculator Lorentz spheres of the spacelike parallel p_i-equidistant ruled surfaces M and M^*.
We compute the locus of center of the osculator sphere S_1^2 which is fourth order contact with the base curve α of M. Let us consider the function f defined by

$$\begin{aligned} f &: I \to R \\ t &\to f(t) = \langle \alpha(t) - a, \alpha(t) - a \rangle - R^2, \end{aligned} \tag{3}$$

where a and R are the center and radius of S_1^2 , respectively. Since S_1^2 is fourth order contact with the curve α, we get

$$f(t) = f'(t) = f''(t) = f'''(t) = 0.$$

From $f(t) = 0$ we have

$$\langle \alpha(t) - a, \alpha(t) - a \rangle = R^2, \tag{4}$$

Then $f'(t) = 0$, we obtain

$$\langle V_1(t), \alpha(t) - a \rangle = 0, \tag{5}$$

Using $f''(t) = 0$ and equation (2) we get

$$\langle V_2(t), \alpha(t) - a \rangle = -\frac{1}{k_1(t)}. \tag{6}$$

For the vector $\alpha(t) - a$, we can write

$$\alpha(t) - a = m_1(t)V_1(t) + m_2(t)V_2(t) + m_3(t)V_3(t), \quad m_i(t) \in R, \tag{7}$$

where $\{V_1, V_2, V_3\}$ is the Frenet frame field of M. From equation (7), we obtain

$$\langle \alpha(t) - a, V_1(t) \rangle = m_1(t), \langle \alpha(t) - a, V_2(t) \rangle = m_2(t), \langle \alpha(t) - a, V_3(t) \rangle = -m_3(t). \tag{8}$$

From equations (5) and (6), we get

$$m_1(t) = 0, \quad m_2(t) = -\frac{1}{k_1(t)}. \tag{9}$$

Using equations (4), (7) and (9) we find

$$R = \sqrt{m_2^2 - m_3^2} \tag{10}$$

or

$$m_3 = \pm\sqrt{m_2^2 - R^2}. \tag{11}$$

Substituting equation (9) to equation (7), we have the center a of S_1^2 as follows

$$a = \alpha(t) + \frac{1}{k_1}V_2(t) - \lambda V_3(t), \quad \lambda = m_3(t) \in R. \tag{12}$$

Using $f'''(t) = 0$

$$k_1' \langle V_2(t), \alpha(t) - a\rangle + k_1 \left\langle V_2'(t), \alpha(t) - a\right\rangle + k_1 \langle V_2(t), V_1(t)\rangle = 0$$

is obtained. Hence, from (2), (8) and (9) we get

$$m_3 = -\frac{k_1'}{k_1^2 k_2} = -\frac{m_2'}{k_2}. \tag{13}$$

Similarly, we find the locus of center of osculator sphere S_1^{*2} which is fourth order contact with the base curve α^* of M^*. Let us consider the function f^* defined by

$$\begin{aligned} &f^* : I \to R \\ &\quad t^* \to f^*(\,t^*) = \langle \alpha^*(t^*) - a^*, \alpha^*(t^*) - a^*\rangle - R^{*2}, \end{aligned} \tag{14}$$

where a^* and R^* are the center and the radius of S_1^{*2}. Since S_1^{*2} is fourth order contact with the curve α^*,we can write

$$f^*(t^*) = f^{*'}(t^*) = f^{*''}(t^*) = f^{*'''}(t^*) = 0.$$

From $f^*(t^*) = f^{*'}(t^*) = f^{*''}(t^*) = 0$ and (2), we get

$$\langle \alpha^*(t^*) - a^*, \alpha^*(t^*) - a^*\rangle = R^{*2}, \tag{15}$$

$$\langle V_1^*(t^*), \alpha^*(t^*) - a^*\rangle = 0, \tag{16}$$

$$\langle V_2^*(t^*), \alpha^*(t^*) - a^*\rangle = -\frac{1}{k_1^*(t^*)}. \tag{17}$$

Furthermore, for the vector $\alpha^*(t^*) - a^*$,

$$\alpha^*(t^*) - a^* = m_1^*(t^*)V_1^*(t^*) + m_2^*(t^*)V_2^*(t^*) + m_3^*(t^*)V_3^*(t^*), \quad m_i^*(t^*) \in R, \tag{18}$$

can be written, where $\{V_1^*, V_2^*, V_3^*\}$ is Frenet frame field of M^*. Using (18), we find

$$\begin{array}{l} \langle \alpha^*(t^*) - a^*, V_1^*(t^*) \rangle = m_1^*(t^*), \\ \langle \alpha^*(t^*) - a^*, V_2^*(t^*) \rangle = m_2^*(t^*), \\ \langle \alpha^*(t^*) - a^*, V_3^*(t^*) \rangle = -m_3^*(t^*). \end{array} \tag{19}$$

Considering equations (16) and (17), we have

$$m_1^*(t^*) = 0, \quad m_2^*(t^*) = -\frac{1}{k_1^*(t^*)}. \tag{20}$$

From (15), (18) and (20), we get

$$R^* = \sqrt{m_2^{*2} - m_3^{*2}} \tag{21}$$

or

$$m_3^* = \pm\sqrt{m_2^{*2} - R^{*2}}. \tag{22}$$

Using (18), for the center a^* of S_1^{*2}, we can write

$$a^* = \alpha^*(t^*) + \frac{1}{k_1^*} V_2^*(t^*) - \lambda^* V_3^*(t^*), \quad \lambda^* = m_3^*(t^*) \in R. \tag{23}$$

Then $f^{*'''}(t^*) = 0$ we find

$$k_1^{*'} \langle V_2^*(t^*), \alpha^*(t^*) - a^* \rangle + k_1^* \left\langle V_2^{*'}(t^*), \alpha^*(t^*) - a^* \right\rangle + k_1^* \langle V_2^*(t^*), V_1^*(t^*) \rangle = 0.$$

Thus from (2), (19) and (20), we have

$$m_3^* = \frac{-k_1^{*\prime}}{k_1^{*2} k_2^*} = -\frac{m_2^{*\prime}}{k_2^*}. \tag{24}$$

Now, let us find the relations between the radii of osculator Lorentz spheres and curvature axes of the base curves of M and M^*:
Using **Theorem 1**, **(ii)** equations (9) and (20) we obtain

$$m_1^*(t^*) = m_1(t) = 0, \quad m_2^*(t^*) = \frac{dt^*}{dt} m_2(t). \tag{25}$$

If $\frac{dt}{dt^*}$ is constant, then considering the **Theorem 1**, **(ii)**, we get

$$k_1^{*'} = k_1{}' \left(\frac{dt}{dt^*} \right)^2. \tag{26}$$

So, from equations (24), (26) and (13), we have

$$m_3^* = \frac{dt^*}{dt} m_3. \tag{27}$$

Combining (7), (18),(25), (27) and **Theorem 1,(ii)**, we find

$$\alpha^* - a^* = \frac{dt^*}{dt}\left(\alpha - a\right). \tag{28}$$

Similarly, thinking (10), (21), (25) and (27), we obtain

$$R^{*2} = \left(\frac{dt^*}{dt}\right)^2 R^2$$

or

$$R^* = \left|\frac{dt^*}{dt}\right| R. \tag{29}$$

Hence, we can give the following theorem without proof:

Theorem 2. ***i)*** *If q_α and q_{α^*} are the curvature axes (the locus of center of osculator Lorentz spheres) of the base curves α and α^* of M and M^*, then we have*

$$q_{\alpha^*} - \alpha^* = \frac{dt^*}{dt}\left(q_\alpha - \alpha\right).$$

ii) *If R and R^* are the radii of osculator Lorentz spheres of base curves α and α^* of M and M^*, then we get*

$$R^* = \left|\frac{dt^*}{dt}\right| R.$$

4. Arc Lengths of Indicatrix Curves of Spacelike Parallel p_i -Equidistant Ruled Surfaces

In this section, we will investigate arc lengths of indicatrix curves of base curves of the spacelike parallel p_i-equidistant ruled surfaces M and M^*.

Since V_1 and V_2 are spacelike vectors, the curves (V_1) and (V_2) generated by the spacelike vectors V_1 and V_2 on the pseudosphere S_1^2, are called the pseudo-spherical indicatrix curves. Since V_3 is a timelike vector, the curve (V_3) generated by the vector V_3 on the pseudohyperbolic space H_1^2 is called indicatrix curve.

Let S_{V_i} and $S_{V_i^*}$ denote the arc lengths of indicatrix curves (V_i) and (V_i^*) generated by the vector fields V_i and V_i^*, respectively. So, we can write

$S_{V_i} = \int \|V_i'\| dt$ and $S_{V_i^*} = \int \left\|V_i^{*\prime}\right\| dt^*, \quad 1 \leq i \leq 3.$
From the Frenet formulas and **Theorem 1,(ii)**, we obtain

$$S_{V_1^*} = \int k_1 dt = S_{V_1}, \quad S_{V_2^*} = \int \sqrt{\left|k_1^2 - k_2^2\right|} dt = S_{V_2}, \quad S_{V_3^*} = \int |k_2|\, dt = S_{V_3},$$

where $\frac{dt}{dt^*} > 0$.
Similarly, for the arc lengths S_α and S_{α^*} of the indicatrix curves (α) and (α^*) generated by the spacelike curves α and α^* on the pseudosphere S_1^2, we find $S_\alpha = \int \|\alpha'\|\, dt = \int dt$ and $S_{\alpha^*} = \int \left\|\alpha^{*\prime}\right\| dt^* = \int dt^*$, respectively. If $\frac{k_1}{k_1^*}$ is constant, using **Theorem 1,(ii)**, we get

$$S_{\alpha^*} = \frac{k_1}{k_1^*} S_\alpha.$$

Thus, we can give the following theorems without proofs:

Theorem 3. *If S_{V_i} and $S_{V_i^*}$, $1 \leq i \leq 3$, are the arc lengths of indicatrix curves of Frenet vectors V_i and V_i^* of base curves α and α^* of M and M^*, respectively, then we have*

$$S_{V_i^*} = S_{V_i},\ 1 \leq i \leq 3.$$

Theorem 4. *Let S_α and S_{α^*} be the arc lengths of indicatrix curves of base curves α and α^* of M and M^*, respectively. If $\frac{k_1}{k_1^*}$ is constant, then we get $S_{\alpha^*} = \frac{k_1}{k_1^*} S_\alpha$.*

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