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COEFFICIENT ESTIMATES FOR A CERTAIN SUBCLASS OF ANALYTIC AND BI-UNIVALENT FUNCTIONS

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Abstract. In the present investigation, we find estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in the function class $B_\Sigma(n,\lambda,\phi)$. The results presented in this paper improve or generalize the recent work of Porwal and Darus [8].

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1. Introduction and Definitions

Let A denote the class of analytic functions in the unit disk

$$U = \{z \in \mathbb{C} : |z| < 1\}$$

that have the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1}$$

Further, by S we shall denote the class of all functions in A which are univalent in U.

The Koebe one-quarter theorem [3] states that the image of U under every function f from S contains a disk of radius $\frac{1}{4}$. Thus every such univalent function has an inverse f^{-1} which satisfies

$$f^{-1}(f(z)) = z \ , \ \ (z \in U)$$

and

$$f\left(f^{-1}(w)\right) = w \ , \ \ \left(|w| < r_0(f) \ , \ r_0(f) \geq \frac{1}{4}\right),$$

where

$$f^{-1}(w) = w \ - a_2 w^2 + \left(2a_2^2 - a_3\right) w^3 - \left(5a_2^3 - 5a_2 a_3 + a_4\right) w^4 + \cdots .$$

A function $f(z) \in A$ is said to be bi-univalent in U if both $f(z)$ and $f^{-1}(z)$ are univalent in U.

If the functions f and g are analytic in U, then f is said to be subordinate to g, written as $f(z) \prec g(z)$, if there exists a Schwarz function w such that $f(z) = g(w(z))$.

Let Σ denote the class of bi-univalent functions defined in the unit disk U. For a brief history and interesting examples in the class Σ, (see [10]).

Lewin [5] studied the class of bi-univalent functions, obtaining the bound 1.51 for modulus of the second coefficient $|a_2|$. Subsequently, Brannan and Clunie [2] conjectured that $|a_2| \leq \sqrt{2}$ for $f \in \Sigma$. Netanyahu [6] showed that $max\,|a_2| = \frac{4}{3}$ if $f(z) \in \Sigma$.

Brannan and Taha [1] introduced certain subclasses of the bi-univalent function class Σ similar to the familiar subclasses $\delta^{\star}(\alpha)$ and $K(\alpha)$ of starlike and convex function of order α $(0 < \alpha \leq 1)$ respectively (see [6]). Thus, following Brannan and Taha [1], a function $f(z) \in A$ is the class $\delta^{\star}_{\Sigma}(\alpha)$ of strongly bi-starlike functions of order α $(0 < \alpha \leq 1)$ if each of the following conditions is satisfied:

$$f \in \Sigma, \quad \left|\arg\left(\frac{zf'(z)}{f(z)}\right)\right| < \frac{\alpha\pi}{2} \quad (0 < \alpha \leq 1,\ z \in U)$$

and

$$\left|\arg\left(\frac{wg'(w)}{g(w)}\right)\right| < \frac{\alpha\pi}{2} \quad (0 < \alpha \leq 1,\ w \in U)$$

where g is the extension of f^{-1} to U. Similarly, a function $f(z) \in A$ is the class $K_{\Sigma}(\alpha)$ of strongly bi-convex functions of order $\alpha\,(0 < \alpha \leq 1)$ if each of the following conditions is satisfied:

$$f \in \Sigma, \quad \left|\arg\left(\frac{z^2 f''(z) + zf'(z)}{zf'(z)}\right)\right| < \frac{\alpha\pi}{2} \quad (0 < \alpha \leq 1,\ z \in U)$$

and

$$\left|\arg\left(\frac{w^2 g''(w) + wg'(w)}{wg'(w)}\right)\right| < \frac{\alpha\pi}{2} \quad (0 < \alpha \leq 1,\ w \in U)$$

where g is the extension of f^{-1} to U. The classes $\delta^{\star}_{\Sigma}(\alpha)$ and $K_{\Sigma}(\alpha)$ of bi-starlike functions of order α and bi-convex functions of order α, corresponding to the function classes $\delta^{\star}(\alpha)$ and $K(\alpha)$, were also introduced analogously. For each of the function classes $\delta^{\star}_{\Sigma}(\alpha)$ and $K_{\Sigma}(\alpha)$, they found non-sharp estimates on the initial coefficients. Recently, many authors investigated bounds for various subclasses of bi-univalent functions ([4], [10], [11]). The coefficient estimate problem for each of the following Taylor-Maclaurin coefficients $|a_n|$ for $n \in \mathbb{N}\backslash\{1,2\};\ \mathbb{N} = \{1,2,3,...\}$ is presumably still an open problem.

In this paper, by using the method [7] different from that used by other authors, we obtain bounds for the coefficients $|a_2|$ and $|a_3|$ for the subclasses of bi-univalent functions considered Porwal and Darus and get more accurate estimates than that given in [8].

2. Coefficient Estimates

In the following, let ϕ be an analytic function with positive real part in U, with $\phi(0) = 1$ and $\phi'(0) > 0$. Also, let $\phi(U)$ be starlike with respect to 1 and symmetric with respect to the real axis. Thus, ϕ has the Taylor series expansion

$$\phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots \quad (B_1 > 0). \tag{2}$$

Suppose that $u(z)$ and $v(z)$ are analytic in the unit disk U with $u(0) = v(0) = 0$, $|u(z)| < 1$, $|v(z)| < 1$, and suppose that

$$u(z) = b_1 z + \sum_{n=2}^{\infty} b_n z^n, \quad v(z) = c_1 z + \sum_{n=2}^{\infty} c_n z^n \quad (|z| < 1). \tag{3}$$

It is well known that

$$|b_1| \le 1, \quad |b_2| \le 1 - |b_1|^2, \quad |c_1| \le 1, \quad |c_2| \le 1 - |c_1|^2. \tag{4}$$

Next, the equations (2) and (3) lead to

$$\phi(u(z)) = 1 + B_1 b_1 z + \left(B_1 b_2 + B_2 b_1^2\right) z^2 + \cdots, \quad |z| < 1 \tag{5}$$

and

$$\phi(v(w)) = 1 + B_1 c_1 w + \left(B_1 c_2 + B_2 c_1^2\right) w^2 + \cdots, \quad |w| < 1. \tag{6}$$

Definition 1. *[8] A function $f(z)$ given by (1) is said to be in the class $B_\Sigma(n, \alpha, \lambda)$ if the following conditions are satisfied:*

$$f \in \Sigma, \quad \left|\arg\left(\frac{(1-\lambda) D^n f(z) + \lambda D^{n+1} f(z)}{z}\right)\right| < \frac{\alpha\pi}{2}, \qquad (0 < \alpha \le 1, \lambda \ge 1,\ z \in U)$$

and

$$\left|\arg\left(\frac{(1-\lambda) D^n g(w) + \lambda D^{n+1} g(w)}{w}\right)\right| < \frac{\alpha\pi}{2}, \qquad (0 < \alpha \le 1, \lambda \ge 1,\ w \in U)$$

where D^n stands for Salagean derivative introduced by Salagean [9].

Definition 2. *A function $f \in \Sigma$ is said to be* $\mathrm{B}_{\Sigma}(n,\lambda,\phi)$, $n \in \mathbb{N}_0$, $0<\alpha\leq 1$ *and* $\lambda \geq 1$, *if the following subordination hold*

$$\frac{(1-\lambda)D^n f(z)+\lambda D^{n+1}f(z)}{z} \prec \phi(z)$$

and

$$\frac{(1-\lambda)D^n g(w)+\lambda D^{n+1}g(w)}{w} \prec \phi(w)$$

where $g(w)=f^{-1}(w)$.

Theorem 1. *Let the function $f(z)$ given by (1) be in the class* $\mathrm{B}_{\Sigma}(n,\lambda,\phi)$, $n \in \mathbb{N}_0$, $0<\alpha\leq 1$ *and* $\lambda\geq 1$. *Then*

$$|a_2| \leq \frac{B_1\sqrt{B_1}}{\sqrt{\left|3^n(2\lambda+1)B_1^2-4^n(1+\lambda)^2B_2\right|+4^n(1+\lambda)^2B_1}} \tag{7}$$

and

$$|a_3| \leq \begin{cases} \dfrac{B_1}{3^n(2\lambda+1)}; & if \quad B_1 \leq \dfrac{4^n(1+\lambda)^2}{3^n(2\lambda+1)} \\ \dfrac{\left|3^n(2\lambda+1)B_1^2-4^n(1+\lambda)^2B_2\right|B_1+3^n(2\lambda+1)B_1^3}{3^n(2\lambda+1)\left[\left|3^n(2\lambda+1)B_1^2-4^n(1+\lambda)^2B_2\right|+4^n(1+\lambda)^2B_1\right]}; & if \quad B_1 > \dfrac{4^n(1+\lambda)^2}{3^n(2\lambda+1)} \end{cases} \tag{8}$$

Proof. Let $f \in B_{\Sigma}(n,\lambda,\phi)$, $\lambda \geq 1$ and $0<\alpha\leq 1$. Then there are analytic functions $u,v: U \to U$ given by (3) such that

$$\frac{(1-\lambda)D^n f(z)+\lambda D^{n+1}f(z)}{z} = \phi(u(z)) \tag{9}$$

and

$$\frac{(1-\lambda)D^n g(w)+\lambda D^{n+1}g(w)}{w} = \phi(v(w)) \tag{10}$$

where $g(w)=f^{-1}(w)$ .Since

$$\begin{aligned} &\frac{(1-\lambda)D^n f(z)+\lambda D^{n+1}f(z)}{z} \\ &= 1+\left[(1-\lambda)2^n+\lambda 2^{n+1}\right]a_2 z+\left[(1-\lambda)3^n+\lambda 3^{n+1}\right]a_3 z^2+\cdots \end{aligned}$$

and

$$\begin{aligned}&\frac{(1-\lambda)\,D^n g\,(w)+\lambda D^{n+1} g\,(w)}{w}\\ =\;&1-\left[(1-\lambda)\,2^n+\lambda 2^{n+1}\right] a_2 w+\left[(1-\lambda)\,3^n+\lambda 3^{n+1}\right]\left(2a_2^2-a_3\right) w^2+\cdots,\end{aligned}$$

it follows from (5), (6), (9) and (10) that

$$\left[(1-\lambda)\,2^n+\lambda 2^{n+1}\right] a_2=B_1 b_1, \tag{11}$$

$$\left[(1-\lambda)\,3^n+\lambda 3^{n+1}\right] a_3=B_1 b_2+B_2 b_1^2, \tag{12}$$

and

$$-\left[(1-\lambda)\,2^n+\lambda 2^{n+1}\right] a_2=B_1 c_1, \tag{13}$$

$$\left[(1-\lambda)\,3^n+\lambda 3^{n+1}\right]\left(2a_2^2-a_3\right)=B_1 c_2+B_2 c_1^2. \tag{14}$$

From (11) and (13) we obtain

$$c_1=-b_1. \tag{15}$$

By adding (14) to (12), further computations using (11) to (15) lead to

$$\left[2\left[(1-\lambda)\,3^n+\lambda 3^{n+1}\right] B_1^2-2\left[(1-\lambda)\,2^n+\lambda 2^{n+1}\right]^2 B_2\right] a_2^2=B_1^3\,(b_2+c_2)\,. \tag{16}$$

(15) and (16), together with (4), give that

$$\left|2\left[(1-\lambda)\,3^n+\lambda 3^{n+1}\right] B_1^2-2\left[(1-\lambda)\,2^n+\lambda 2^{n+1}\right]^2 B_2\right| |a_2|^2\le 2B_1^3\left(1-|b_1|^2\right). \tag{17}$$

From (11) and (17) we get

$$|a_2|\le\frac{B_1\sqrt{B_1}}{\sqrt{\left|3^n\,(2\lambda+1)\,B_1^2-4^n\,(1+\lambda)^2\,B_2\right|+4^n\,(1+\lambda)^2\,B_1}}.$$

Next, in order to find the bound on $|a_3|$, by subtracting (14) from (12), we obtain

$$2\left[(1-\lambda)\,3^n+\lambda 3^{n+1}\right] a_3-2\left[(1-\lambda)\,3^n+\lambda 3^{n+1}\right] a_2^2=B_1\,(b_2-c_2)+B_2\left(b_1^2-c_1^2\right). \tag{18}$$

Then, in view of (4) and (15) , we have

$$\left[(1-\lambda)\,3^n+\lambda 3^{n+1}\right] B_1\,|a_3|\le\left[\left[(1-\lambda)\,3^n+\lambda 3^{n+1}\right] B_1-4^n\,(1+\lambda)^2\right] |a_2|^2+B_1^2.$$

Notice that (7), we get

$$|a_3| \leq \begin{cases} \dfrac{B_1}{3^n (2\lambda+1)}; & if \quad B_1 \leq \dfrac{4^n (1+\lambda)^2}{3^n (2\lambda+1)} \\ \dfrac{\left|3^n (2\lambda+1) B_1^2 - 4^n (1+\lambda)^2 B_2\right| B_1 + 3^n (2\lambda+1) B_1^3}{3^n (2\lambda+1) \left[\left|3^n (2\lambda+1) B_1^2 - 4^n (1+\lambda)^2 B_2\right| + 4^n (1+\lambda)^2 B_1\right]}; & if \quad B_1 > \dfrac{4^n (1+\lambda)^2}{3^n (2\lambda+1)} \end{cases}$$

Remark 1. *f let*

$$\phi(z) = \left(\frac{1+z}{1-z}\right)^{\alpha} = 1 + 2\alpha z + 2\alpha^2 z^2 + ... \qquad (0 < \alpha \leq 1),$$

then inequalities (7) and (8) become

$$|a_2| \leq \frac{2\alpha}{\sqrt{\left|2.3^n (2\lambda+1) - 4^n (1+\lambda)^2\right| \alpha + 4^n (1+\lambda)^2}} \tag{19}$$

and

$$|a_3| \leq \begin{cases} \dfrac{2\alpha}{3^n (2\lambda+1)}; & if \quad 0 < \alpha \leq \dfrac{2^{n-1} (1+\lambda)}{3^n (2\lambda+1)} \\ \dfrac{2\left[\left|2.3^n (2\lambda+1) - 4^n (1+\lambda)^2\right| + 2.3^n (2\lambda+1)\right] \alpha^2}{3^n (2\lambda+1) \left[\left|2.3^n (2\lambda+1) - 4^n (1+\lambda)^2\right| \alpha + 4^n (1+\lambda)^2\right]}; & if \quad \dfrac{2^{n-1} (1+\lambda)}{3^n (2\lambda+1)} < \alpha \leq 1. \end{cases} \tag{20}$$

The bounds on $|a_2|$ and $|a_3|$ given by (19) and (20) are more accurate than that given in Theorem 2.1 in [8].

Remark 2. *If let*

$$\phi(z) = \frac{1 + (1-2\alpha) z}{1-z} = 1 + 2(1-\alpha) z + 2(1-\alpha) z^2 + \cdots \qquad (0 < \alpha \leq 1),$$

then inequalities (7) and (8) become

$$|a_2| \leq \frac{2(1-\alpha)}{\sqrt{\left|2(1-\alpha) 3^n (2\lambda+1) - 4^n (1+\lambda)^2\right| + 4^n (1+\lambda)^2}} \tag{21}$$

and

$$|a_3| \leq \begin{cases} \dfrac{2(1-\alpha)}{3^n(2\lambda+1)}; & if \quad \dfrac{3^n(2\lambda+1)-2^{n-1}(1+\lambda)}{3^n(2\lambda+1)} \leq \alpha < 1 \\ \dfrac{2\left[\left|2(1-\alpha)3^n(2\lambda+1)-4^n(1+\lambda)^2\right|+2(1-\alpha)3^n(2\lambda+1)\right](1-\alpha)}{3^n(2\lambda+1)\left[\left|2(1-\alpha)3^n(2\lambda+1)-4^n(1+\lambda)^2\right|+4^n(1+\lambda)^2\right]} & \\ & if \quad 0 \leq \alpha < \dfrac{3^n(2\lambda+1)-2^{n-1}(1+\lambda)}{3^n(2\lambda+1)}. \end{cases} \tag{22}$$

The bounds on $|a_2|$ and $|a_3|$ given by (21) and (22) are more accurate than that given in Theorem 3.1 in [8].

References

[1] D. A. Brannan and T. S. Taha, *On some classes of bi-univalent functions*, Studia Universitatis Babeş-Bolyai. Mathematica, 31, 2 (1986), 70-77.

[2] D. A. Brannan and J. G. Clunie, *Aspects of comtemporary complex analysis*, (Proceedings of the NATO Advanced Study Instute Held at University of Durham:July 1-20, 1979). New York: Academic Press, (1980).

[3] P. L. Duren, *Univalent Functions*, Grundlehren der Mathematischen Wissenschaften, Springer, New York, NY, USA, 259 (1983).

[4] B. A. Frasin and M. K. Aouf, *New subclasses of bi-univalent functions*, Applied Mathematics Letters, 24 (2011), 1569-1573.

[5] M. Lewin, *On a coefficient problem for bi-univalent functions*, Proceeding of the American Mathematical Society, 18 (1967), 63-68.

[6] E. Netanyahu, *The minimal distance of the image boundary from the orijin and the second coefficient of a univalent function in $|z| < 1$*, Archive for Rational Mechanics and Analysis, 32 (1969), 100-112.

[7] Z. Peng and Q. Han, *On the coefficients of several classes of bi-univalent functions*, Acta Mathematica Scientia, 34B(1) (2014), 228-240.

[8] S. Porwal and M. Darus, *On a new subclass of bi-univalent functions*, Journal of the Egyptian Mathematical Society, 21, 3 (2013), 190-193.

[9] G.S. Salagean, *Subclasses of univalent functions*, in: Complex Analysis- Fifth Romanian Finish Seminar, Bucharest, 1 (1983), 362-372.

[10] H. M. Srivastava, A. K. Mishra, and P. Gochhayat, *Certain subclasses of analytic and bi-univalent functions*, Applied Mathematics Letters, 23, 10 (2010), 1188-1192.

[11] Q. H. Xu, Y. C. Gui, and H. M. Srivastava, *Coefficient estimates for a certain subclass of analytic and bi-univalent functions*, Applied Mathematics Letters, 25 (2012), 990-994.

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