Acta Universitatis Apulensis No. 40/2014
ISSN: 1582-5329 pp. 305-314
http://www.uab.ro/auajournal/ doi: 10.17114/j.aua.2014.40.25

LIPSCHITZ ESTIMATE FOR VECTOR-VALUED MULTILINEAR COMMUTATOR OF FRACTIONAL AREA INTEGRAL OPERATOR

Wei-ping Kuang

Abstract. In this paper, we proved the vector-valued multilinear commutator $|S_{\psi,\delta}^{\vec{b}}|_r$, which is generated by the fractional area integral operator and functions in $Lip_\beta(R^n)$, is bounded on Triebel-Lizorkin and Lebegue spaces.

2000 *Mathematics Subject Classification*: 42B20.

Keywords: Fractional area integral operator; Vector-valued multilinear commutator; BMO space; Triebel-Lizorkin space; Lipschitz space; Lebegue space.

1. Introduction

As the development of the singular integral operators, their commutators and multilinear operators have been well studied ([1-3]). A well known result of Coifman, Rochberg and Weiss (see [4]) states that the commutator $[b,T](f)(x) = b(x)T(f)(x) - T(bf)(x)$ (where $b \in BMO$) is bounded on $L^p(R^n)$ for $1 < p < \infty$. Chanillo (see [5]) proves a similiar result when T is replaced by the fractional operators. In [6-15], the authors study these results for the Triebel-Lizorkin spaces and the case $b \in Lip_\beta$, where Lip_β is the homogeneous Lipschitz space. In this paper, we will prove the vector-valued multilinear commutator $|S_{\psi,\delta}^{\vec{b}}|_r$, which is generated by the fractional area integral operator and functions in $Lip_\beta(R^n)$, is bounded on Triebel-Lizorkin and Lebegue spaces.

2. Definitions and Results

Throughout this paper, Q will denote a cube of R^n with side parallel to the axes. Let f is locally integrable function with $f(Q) = \int_Q f(x)dx$, $f_Q = |Q|^{-1}\int_Q f(x)dx$, and

$$M^{\#}(f)(x) = \sup_{Q \ni x} \frac{1}{|Q|}\int_Q |f(y) - f_Q|dy.$$

It is well known that (see [6][7])

$$M^{\#}(f)(x) \approx \sup_{Q \ni x} \inf_{c \in C} \frac{1}{|Q|} \int_Q |f(y) - c| dy.$$

Let

$$M(f)(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)| dy.$$

For $\beta > 0$ and $p > 1$, let $\dot{F}_p^{\beta,\infty}$ be the homogeneous Tribel-Lizorkin space. The Lipschitz space $Lip_\beta(R^n)$ is the space of functions f such that

$$||f||_{Lip_\beta} = \sup_{\substack{x,y \in R^n \\ x \neq y}} \frac{|f(x) - f(y)|}{|x - y|^\beta} < \infty.$$

Given some functions b_j $(j = 1, \cdots, m)$ and a positive integer m and $1 \le j \le m$, we denote by C_j^m the family of all finite subsets $\sigma = \{\sigma(1), \cdots, \sigma(j)\}$ of $\{1, \cdots, m\}$ of j different elements. For $\sigma \in C_j^m$, set $\sigma^c = \{1, \cdots, m\} \setminus \sigma$. For $\vec{b} = (b_1, \cdots, b_m)$ and $\sigma = \{\sigma(1), \cdots, \sigma(j)\} \in C_j^m$, set $\vec{b}_\sigma = (b_{\sigma(1)}, \cdots, b_{\sigma(j)})$, $b_\sigma = b_{\sigma(1)} \cdots b_{\sigma(j)}$ and $||\vec{b}_\sigma||_{Lip_\beta(w)} = ||b_{\sigma(1)}||_{Lip_\beta(w)} \cdots ||b_{\sigma(j)}||_{Lip_\beta(w)}$.

In this paper, we will study vector-valued multilinear commutator of fractional area integral operator as follows.

Definition 1. *Let $0 < \delta < n$. Suppose functions ψ satisfies the following properties:*

(1) $\int_{R^n} \psi(x) dx = 0$;

(2) $|\psi(x)| \le C(1 + |x|)^{-(n+1-\delta)}$;

(3) $|\psi(x + y) - \psi(x)| \le C|y|^\varepsilon (1 + |x|)^{-(n+2-\delta)}$, *where* $2|y| < |x|$.

Set $1 < r < \infty$, $b_j (j = 1, \cdots, m)$ are fixed local integratiable function on R^n. Let $\Gamma(x) = \{(y, t) \in R_+^{n+1} : |x - y| \le t\}$, its characteristic function is $\chi_{\Gamma(x)}$, vector-valued multilinear commutator of fractional area integral operator is defined as follows:

$$|S_{\psi,\delta}^{\vec{b}}(f)(x)|_r = \left(\sum_{i=1}^{\infty} (S_{\psi,\delta}^{\vec{b}}(f_i)(x))^r \right)^{1/r},$$

where

$$S_{\psi,\delta}^{\vec{b}}(f)(x) = \left(\int_{\Gamma(x)} |F_t^{\vec{b}}(f)(x, y)|^2 \frac{dydt}{t^{n+1}} \right)^{1/2}$$

and

$$F_t^{\tilde{b}}(f)(x) = \int_{R^n} \left[\prod_{j=1}^{m} (b_j(x) - b_j(z)) \right] \psi_t(y - z) f(z) dz.$$

Now we state our theorems as following.

Theorem 1. *Let $1 < r < \infty$, $0 < \beta < 1/2m$, $1 < p < \infty$, $\vec{b} = (b_1, \cdots, b_m)$, with $b_j \in Lip_\beta(R^n)$ for $1 \leq j \leq m$, then $1 < p < n/\delta$,$1/p - 1/q = \delta/n$, $|S^{\vec{b}}_{\psi,\delta}|_r$ is bounded from $L^p(R^n)$ to $\dot{F}^{m\beta,\infty}_q(R^n)$.*

Theorem 2. *Let $1 < r < \infty$, $0 < \beta < 1/2m$, $1 < p < \infty$, $\vec{b} = (b_1, \cdots, b_m)$, with $b_j \in Lip_\beta(R^n)$, $1 \leq j \leq m$, then $1/p - 1/q = (m\beta + \delta)/n$, $1/p > (m\beta + \delta)/n$, $|S^{\vec{b}}_{\psi,\delta}|_r$is bounded from $L^p(R^n)$ to $L^q(R^n)$.*

3. Proof of Theorems

The proofs of our main results are based the following lemmas.

Lemma 3. *(see [9]) For $0 < \beta < 1$, $1 < p < \infty$, we have*

$$\begin{aligned} ||f||_{\dot{F}^{\beta,\infty}_p} &\approx \left\|\sup_Q \frac{1}{|Q|^{1+\frac{\beta}{n}}} \int_Q |f(x) - f_Q| dx\right\|_{L^p} \\ &\approx \left\|\sup_{\cdot \in Q} \inf_c \frac{1}{|Q|^{1+\frac{\beta}{n}}} \int_Q |f(x) - c| dx\right\|_{L^p}. \end{aligned}$$

Lemma 4. *(see [9]) For $0 < \beta < 1$, $1 \leq p \leq \infty$, we have*

$$\begin{aligned} ||f||_{Lip_\beta} &\approx \sup_Q \frac{1}{|Q|^{1+\frac{\beta}{n}}} \int_Q |f(x) - f_Q| dx \\ &\approx \sup_Q \frac{1}{|Q|^{\frac{\beta}{n}}} \left(\frac{1}{|Q|} \int_Q |f(x) - f_Q|^p dx\right)^{1/p}. \end{aligned}$$

Lemma 5. *(see [15]) Let $1 < r < \infty$, $0 < \delta < n$, $1 < p < n/\delta$, $1/q = 1/p - \delta/n$, $w \in A_1$. then $|S_{\psi,\delta}|_r$ is bounded from $L^p(w)$ to $L^q(w)$.*

Lemma 6. *(see [5]) Let $1 \leq l < \infty$, $1 < r < \infty$, $\beta > 0$, with*

$$M_{l,\beta}(f)(x) = \sup_{x \in Q} \left(\frac{1}{|Q|^{1-\frac{\beta l}{n}}} \int_Q |f(y)|^l dy\right)^{1/l},$$

if $l < p < n/\beta$, $1/q = 1/p - \beta/n$,then

$$||M_{l,\beta}(|f|_r)||_{L^q} \leq C|||f|_r||_{L^p}.$$

Lemma 7. *(see [9]) Let* $Q_1 \subset Q_2$*,then*

$$|f_{Q_1} - f_{Q_2}| \leq C||f||_{Lip_\beta}|Q_2|^{\beta/n}.$$

Proof of Theorem 1. Fix a cube $Q = (x_0, l)$ and $\tilde{x} \in Q$. We first consider the case $m \geq 2$. Set $\vec{b}_Q = ((b_1)_Q, \cdots, (b_m)_Q)$ with $(b_j)_Q = |Q|^{-1}\int_Q b_j(y)dy$, $1 \leq j \leq m$. Let f decompose into $f = g + h = \{g_i\} + \{h_i\}$, with $g_i = f_i\chi_Q$, $h_i = f_i\chi_{(Q)^c}$. We have

$$\begin{aligned}
&F_t^{\vec{b}}(f_i)(x,y)\\
= &\int_{R^n}\left[\prod_{j=1}^m(b_j(x) - b_j(z))\right]\psi_t(y-z)f_i(z)dz\\
= &\int_{R^n}\prod_{j=1}^m[(b_j(x)-(b_j)_Q) - (b_j(z)-(b_j)_Q)]\psi_t(y-z)f_i(z)dz\\
= &\sum_{j=o}^m\sum_{\sigma\in C_j^m}(-1)^{m-j}(b(x)-b_Q)_\sigma\int_{R^n}(b_j(z)-b_Q)_{\sigma^c}\psi_t(y-z)f_i(z)dz\\
= &(b_1(x)-(b_1)_Q)\cdots(b_m(x)-(b_m)_Q)F_t(f_i)(y)\\
&+(-1)^mF_t((b_1-(b_1)_Q)\cdots(b_m-(b_m)_Q)f_i)(y)\\
&+\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}(-1)^{m-j}(b(x)-b_Q)_\sigma\int_{R^n}(b(z)-b_Q)_{\sigma^c}\psi_t(y-z)f_i(z)dz\\
= &(b_1(x)-(b_1)_Q)\cdots(b_m(x)-(b_m)_Q)F_t(f_i)(y)\\
&+(-1)^mF_t((b_1-(b_1)_Q)\cdots(b_m-(b_m)_Q)g_i)(y)\\
&+(-1)^mF_t((b_1-(b_1)_Q)\cdots(b_m-(b_m)_Q)h_i)(y)\\
&+\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}(-1)^{m-j}(b(x)-b_Q)_\sigma F_t((b-b_Q)_{\sigma^c}f_i)(x,y),
\end{aligned}$$

by Minkowski's inequality, we obtain

$$\begin{aligned}
&||S^{\vec{b}}_{\psi,\delta}(f)(x)|_r - |S_{\psi,\delta}((((b_1)_{2Q}-b_1)\cdots((b_m)_{2Q}-b_m))h)(x_0)|_r| \\
\leq\ & \left(\sum_{i=1}^{\infty}\left\|(b_1(x)-(b_1)_{2Q})\cdots(b_m(x)-(b_m)_{2Q})\chi_{\Gamma(x)}F_t(f_i)(y)\right\|^r\right)^{1/r} \\
&+\left(\sum_{i=1}^{\infty}\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\left\|(b(x)-(b)_{2Q})_\sigma\chi_{\Gamma(x)}F_t^{\vec{b}_{\sigma^c}}(f_i)(x,y)\right\|^r\right)^{1/r} \\
&+\left(\sum_{i=1}^{\infty}\left\|\chi_{\Gamma(x)}F_t((b_1-(b_1)_{2Q})\cdots(b_m-(b_m)_{2Q})g_i)(y)\right\|^r\right)^{1/r} \\
&+\left\|\chi_{\Gamma(x)}F_t(\prod_{j=1}^m(b_j-(b_j)_{2Q})h)(y)-\chi_{\Gamma(x_0)}F_t(\prod_{j=1}^m(b_j-(b_j)_{2Q})h)(y)\right\|_r \\
=\ & L_1(x)+L_2(x)+L_3(x)+L_4(x),
\end{aligned}$$

so

$$\begin{aligned}
&\frac{1}{|Q|^{1+m\beta/n}}\int_Q ||S^{\vec{b}}_{\psi,\delta}(f)(x)|_r - |S_{\psi,\delta}((((b_1)_{2Q}-b_1)\cdots((b_m)_{2Q}-b_m))h)(x_0)|_r|dx \\
\leq\ & \frac{1}{|Q|^{1+m\beta/n}}\int_Q L_1(x)dx+\frac{1}{|Q|^{1+m\beta/n}}\int_Q L_2(x)dx \\
&+\frac{1}{|Q|^{1+m\beta/n}}\int_Q L_3(x)dx+\frac{1}{|Q|^{1+m\beta/n}}\int_Q L_4(x)dx \\
=\ & L_1+L_2+L_3+L_4.
\end{aligned}$$

For L_1, by using Lemma 6, we have

$$\begin{aligned}
L_1 &= \frac{1}{|Q|^{1+m\beta/n}}\int_Q ||\chi_{\Gamma(x)}(b_1(x)-(b_1)_Q)\cdots(b_m(x)-(b_m)_Q)F_t(f)(y)||_r dx \\
&= \frac{1}{|Q|^{1+m\beta/n}}\int_Q\left(\int\int_{R^{n+1}_+}|\chi_{\Gamma(x)}(b_1(x)-(b_1)_Q)\cdots(b_m(x)-(b_m)_Q|||F_t(f)(y))||_r^2\frac{dydt}{t^{n+1}}\right)^{1/2}dx \\
&= \frac{1}{|Q|^{1+m\beta/n}}\int_Q|(b_1(x)-(b_1)_Q)\cdots(b_m(x)-(b_m)_Q)||S_{\psi,\delta}(f)(x)|_r dx \\
&\leq \frac{1}{|Q|^{1+m\beta/n}}\sup_{x\in Q}|b_1(x)-(b_1)_Q|\cdots|b_m(x)-(b_m)_Q|\int_Q|S_{\psi,\delta}(f)(x)|_r dx \\
&\leq C||\vec{b}||_{Lip_\beta}\frac{1}{|Q|^{1+m\beta/n}}|Q|^{m\beta/n}\int_Q|S_{\psi,\delta}(f)(x)|_r dx \\
&\leq C||\vec{b}||_{Lip_\beta}M(|S_{\psi,\delta}(f)|_r)(\tilde{x}) \\
&\leq C||\vec{b}||_{Lip_\beta}M_l(|S_{\psi,\delta}(f)|_r)(\tilde{x}).
\end{aligned}$$

For L_2, with $1 < l < \infty$, by $H\ddot{o}lder$ inequality, we obtain

$$
\begin{aligned}
L_2 &\le \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} \frac{1}{|Q|^{1+m\beta/n}} \int_Q |(\vec{b}(x) - \vec{b}_Q)_\sigma| |S_{\psi,\delta}((\vec{b} - \vec{b}_Q)_{\sigma^c} f)(x)|_r dx \\
&\le C \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} \frac{1}{|Q|^{m\beta/n}} \sup_{x \in Q} |(\vec{b}(x) - \vec{b}_Q)_\sigma| \left(\frac{1}{|Q|} \int_Q |S_{\psi,\delta}((\vec{b} - \vec{b}_Q)_{\sigma^c} f)(x)|_r^l dx \right)^{1/l} \\
&\le C \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} \frac{1}{|Q|^{m\beta/n}} ||\vec{b}_\sigma||_{Lip_\beta} |Q|^{|\sigma|\beta/n} \sup_{x \in Q} |(\vec{b}(x) - \vec{b}_Q)_{\sigma^c}| \left(\frac{1}{|Q|} \int_Q |S_{\psi,\delta}(f)(x)|_r^l dx \right)^{1/l} \\
&\le C \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} \frac{1}{|Q|^{m\beta/n}} ||\vec{b}_\sigma||_{Lip_\beta} |Q|^{|\sigma|\beta/n} ||\vec{b}_\sigma||_{Lip_\beta} |Q|^{|\sigma|^c\beta/n} \left(\frac{1}{|Q|} \int_Q |S_{\psi,\delta}(f)(x)|_r^l dx \right)^{1/l} \\
&\le C \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} \frac{1}{|Q|^{m\beta/n}} ||\vec{b}_\sigma||_{Lip_\beta} |Q|^{|\sigma|\beta/n} ||\vec{b}_{\sigma^c}||_{Lip_\beta} |Q|^{|\sigma^c|\beta/n} M_l(|S_{\psi,\delta}(f)|_r)(\tilde{x}) \\
&\le C||\vec{b}||_{Lip_\beta} M_l(|S_{\psi,\delta}(f)|_r)(\tilde{x}).
\end{aligned}
$$

For L_3, we set $1 < l < q < n/\delta$, $1/q = 1/l - \delta/n$, by $|g_{\mu,\delta}|_r$ bounded on (L^l, L^q) and $H\ddot{o}lder$ inequality, we obtain

$$
\begin{aligned}
L_3 &= \frac{1}{|Q|^{1+m\beta/n}} \int_Q |S_{\psi,\delta}((b_1 - (b_1)_Q) \cdots (b_m - (b_m)_Q) f \chi_Q)(x)|_r dx \\
&\le \frac{C}{|Q|^{m\beta/n}} \left(\frac{1}{|Q|} \int_{R^n} |S_{\psi,\delta}(\prod_{j=1}^m (b_j - (b_j)_Q) f \chi_Q)(x)|_r^q dx \right)^{1/q} \\
&\le \frac{C}{|Q|^{m\beta/n}} \frac{1}{|Q|^{1/q}} \prod_{j=1}^m \sup_{x \in Q} |b_j(x) - (b_j)_Q| \left(\int_{R^n} |f(x)|_r^l \chi_Q(x) dx \right)^{1/l} \\
&\le \frac{C}{|Q|^{1/q}} \prod_{j=1}^m ||b_j||_{Lip_\beta} |Q|^{1/l-\delta/n} \left(\frac{1}{|Q|^{1-\delta l/n}} \int_Q |f(x)|_r^l dx \right)^{1/l} \\
&\le C||\vec{b}||_{Lip_\beta} M_{l,\delta}(|f|_r)(\tilde{x}).
\end{aligned}
$$

For L_4, if $y \in (2Q)^c$, then $|x_0 - y| \approx |x - y|$, by the properties of ψ, we obtain

$$
\begin{aligned}
L_4 &= ||\chi_{\Gamma(x)} F_t((b_1 - (b_1)_Q) \cdots (b_m - (b_m)_Q)h)(y) - \chi_{\Gamma(x_0)} F_t((b_1 - (b_1)_Q) \cdots (b_m - (b_m)_Q)h)(y)||_r \\
&\le \left[\int\int_{R_+^{n+1}} (\int_{(2Q)^c} |\chi_{\Gamma(x)} - \chi_{\Gamma(x_0)}||f(z)|_r \prod_{j=1}^m |b_j(z) - (b_j)_Q| dz)^2 \frac{dydt}{t^{n+1}} \right]^{1/2} \\
&\le C \int_{(2Q)^c} |f(z)|_r \prod_{j=1}^m |b_j(z) - (b_j)_Q| \\
&\quad \times \left| \int\int_{|x-y|\le t} \frac{t^{1-n} dydt}{(t + |y - z|)^{2n+2-2\delta}} - \int\int_{|x_0 - y| \le t} \frac{t^{1-n} dydt}{(t + |y - z|)^{2n+2-2\delta}} \right|^{1/2} dz \\
&\le C \int_{(2Q)^c} |f(z)|_r \prod_{j=1}^m |b_j(z) - (b_j)_Q| \left(\int\int_{|y|\le t, |x+y-z|\le t} \frac{|x - x_0| t^{1-n}}{(t + |x + y - z|)^{2n+3-2\delta}} dydt \right)^{1/2} dz.
\end{aligned}
$$

Because if $|y| \le t$, that $2t + |x + y - z| \ge 2t + |x - z| - |y| \ge t + |x - z|$ and

$$
\int_0^\infty \frac{tdt}{(t + |x - z|)^{2n+3-2\delta}} = C|x - z|^{-2n-1+2\delta},
$$

therefore, if $x \in Q$, then we have

$$
\begin{aligned}
L_4 &\le C \int_{(2Q)^c} |f(z)|_r \prod_{j=1}^m |b_j(z) - (b_j)_Q| \left(\int\int_{|y|\le t} \frac{|x_0 - x| t^{1-n} 2^{2n+3-2\delta} dydt}{(2t + 2|x + y - z|)^{2n+3-2\delta}} \right)^{1/2} dz \\
&\le C \int_{(2Q)^c} |f(z)|_r \prod_{j=1}^m |b_j(z) - (b_j)_Q| |x - x_0|^{1/2} \left(\int\int_{|y|\le t} \frac{t^{1-n} dydt}{(t + |x - z|)^{2n+3-2\delta}} \right)^{1/2} dz \\
&\le C \int_{(2Q)^c} |f(z)|_r \prod_{j=1}^m |b_j(z) - (b_j)_Q| |x - x_0|^{1/2} \left(\int_0^\infty \frac{tdt}{(t + |x - z|)^{2n+3-2\delta}} \right)^{1/2} dz \\
&\le C \int_{(2Q)^c} |f(z)|_r \prod_{j=1}^m |b_j(z) - (b_j)_Q| \frac{|x - x_0|^{1/2}}{|x_0 - z|^{n+1/2-\delta}} dz \\
&\le C \sum_{k=1}^\infty \int_{2^{k+1}Q \setminus 2^k Q} |x_0 - x|^{1/2} |x_0 - z|^{-(n+1/2-\delta)} |f(z)|_r \prod_{j=1}^m |b_j(z) - (b_j)_Q| dz
\end{aligned}
$$

$$
\begin{aligned}
&\leq C\sum_{k=1}^{\infty}2^{-k/2}|2^{k+1}Q|^{-1+\delta/n}\int_{2^{k+1}Q}|f(z)|_r\prod_{j=1}^{m}(|b_j(z)-(b_j)_{2^{k+1}Q}|+|(b_j)_{2^{k+1}Q}-(b_j)_Q|)dz\\
&\leq C\sum_{k=1}^{\infty}2^{-k/2}\left(\frac{1}{|2^kQ|^{1-\delta/n}}\int_{2^kQ}|f(y)|_r^l dy\right)^{1/l}\left(\frac{1}{|2^kQ|^{1-\delta l/n}}\int_{2^kQ}\left|\prod_{j=1}^{m}(b_j(y)-(b_j)_Q)\right|^{l'}dy\right)^{1/l'}\\
&\leq C\sum_{k=1}^{\infty}2^{k(m\beta-1/2)}||\vec{b}||_{Lip_\beta}|Q|^{m\beta/n}M_{l,\delta}(|f|_r)(\tilde{x})\\
&\leq C||\vec{b}||_{Lip_\beta}|Q|^{m\beta/n}M_{l,\delta}(|f|_r)(\tilde{x}),
\end{aligned}
$$

we can obtain that

$$L_4\leq C||\vec{b}||_{Lip_\beta}M_{l,\delta}(|f|_r)(\tilde{x}).$$

Comprehensive above estimates, let $1<l<p$, take supremum of all cube on $\tilde{x}\in Q$ and by using Lemma 5, we obtain

$$
\begin{aligned}
|||S_{\psi,\delta}^{\vec{b}}(f)|_r||_{\dot{F}_q^{m\beta,\infty}} &\leq C||\vec{b}||_{Lip_\beta}(||M_l(|S_{\psi,\delta}(f)|_r)+M_{l,\delta}(|f|_r)||_{L^q})\\
&\leq C||\vec{b}||_{Lip_\beta}(||M_l(|S_{\psi,\delta}(f)|_r)||_{L^q}+||M_{l,\delta}(|f|_r)||_{L^q})\\
&\leq C||\vec{b}||_{Lip_\beta}(|||S_{\psi,\delta}(f)|_r||_{L^q}+||M_{l,\delta}(|f|_r)||_{L^q})\\
&\leq C||\vec{b}||_{Lip_\beta}|||f|_r||_{L^p}.
\end{aligned}
$$

This complete the proof of Theorem 1.

Proof of Theorem 2. Similar to (L_1), we have

$$
\begin{aligned}
&\frac{1}{|Q|}\int_Q||S_{\psi,\delta}^{\vec{b}}(f)(x)|_r-|S_{\psi,\delta}(((b_1)_Q-b_1)\cdots((b_m)_Q-b_m)h)(x_0)|_r|dx\\
&\leq C||\vec{b}||_{Lip_\beta}(M_{l,m\beta}(|S_{\psi,\delta}(f)|_r)(\tilde{x})+M_{l,m\beta+\delta}(|f|_r)(\tilde{x})).
\end{aligned}
$$

So

$$(|S_{\psi,\delta}^{\vec{b}}(f)|_r)^{\#}\leq C||\vec{b}||_{Lip_\beta}(M_{l,m\beta}(|S_{\psi,\delta}(f)|_r)(\tilde{x})+M_{l,m\beta+\delta}(|f|_r)(\tilde{x})).$$

Let $1<l<p$, suppose $1/s=1/p-\delta/n$, then $1/q=1/s-m\beta/n$, by using Lemma 7 and the boundedness of $|S_{\psi,\delta}|_r$, we obtain

$$
\begin{aligned}
|||S_{\psi,\delta}^{\vec{b}}(f)|_r||_{L^q} &\leq C||(|S_{\psi,\delta}^{\vec{b}}(f)|_r)^{\#}||_{L^q}\\
&\leq C||\vec{b}||_{Lip_\beta}||(M_{l,m\beta}(|S_{\psi,\delta}(f)|_r)+M_{l,m\beta+\delta}(|f|_r))||_{L^q}\\
&\leq C||\vec{b}||_{Lip_\beta}||M_{l,m\beta}(|S_{\psi,\delta}(f)|_r)||_{L^q}+|||f|_r||_{L^p}\\
&\leq C||\vec{b}||_{Lip_\beta}|||S_{\psi,\delta}(f)|_r||_{L^s}+|||f|_r||_{L^q}\\
&\leq C||\vec{b}||_{Lip_\beta}|||f|_r||_{L^p}.
\end{aligned}
$$

This complete the proof of Theorem 2.

Acknowledgements. The present investigation was supported by the *Scientific Research Fund of Huaihua University* under Grant HHUY2010-01 of the People's Republic of China.

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Wei-Ping Kuang
Department of Mathematics,
Huaihua University,
Huaihua 418008, Hunan, People's Republic of China
email: *kuangweipingppp@163.com*