Acta Universitatis Apulensis No. 40/2014
ISSN: 1582-5329 pp. 265-282
http://www.uab.ro/auajournal/ doi: 10.17114/j.aua.2014.40.22

SOME PROPERTIES OF CERTAIN SUBCLASSES OF MEROMORPHIC MULTIVALENT FUNCTIONS OF COMPLEX ORDER DEFINED BY CERTAIN LINEAR OPERATOR

R.M. El-Ashwah, M.K. Aouf, S.M. El-Deeb

Abstract. In this paper we introduce two subclasses $H_p^n(q,s,\alpha_1;A,B,b)$ and $H_p^{n,*}(q,s,\alpha_1;A,B,b)$ of meromorphic p-valent functions of complex order defined by certain linear operator. We study the various important properties and characteristics of these two subclasses such as, coefficients estimate, radii of starlikeness and convexity and closure theorems. We also extend the familiar concept of δ-neighborhoods of analytic functions to these subclasses.

2000 *Mathematics Subject Classification*: 30C45.

Keywords: Meromorphic functions, Hadamard product, generalized hypergeometric function, δ-neighborhoods, partial sums.

1. Introduction

Let $\sum_p$ be the class of functions of the form:

$$f(z) = z^{-p} + \sum_{k=1-p}^{\infty} a_k z^k \qquad (p \in N = \{1,2,....\}), \tag{1.1}$$

which are analytic and p-valent in the punctured unit disc $U^* = \{z : z \in C$ and $0 < |z| < 1\} = U\backslash\{0\}$. Let f and F two analytic functions in the unit disc U, we say that f is subordinate to F if there exists an analytic function $w(z)$ with $w(0) = 0$ and $|w(z)| < 1$ $(z \in U)$ such that $f = F(w(z))$. We denote by $f \prec F$ this subordination. For functions $f(z) \in \sum_p$ given by (1.1) and $g(z) \in \sum_p$ defined by

$$g(z) = z^{-p} + \sum_{k=1-p}^{\infty} b_k z^k \quad (p \in N), \tag{1.2}$$

then the Hadamard product (or convolution) of $f(z)$ and $g(z)$ is given by

$$(f*g)(z)=z^{-p}+\sum_{k=1-p}^{\infty}a_kb_kz^k=(g*f)(z). \tag{1.3}$$

For complex numbers $\alpha_1,...,\alpha_q$ and $\beta_1,...,\beta_s$ ($\beta_j\notin\mathbb{Z}_0^-=\{0,-1,-2,...\};j=1,2,...,s$), we now define the generalized hypergeometric function ${}_qF_s(\alpha_1,...,\alpha_q;\beta_1,...,\beta_s;z)$ by (see, for example, [18, p.19])

$${}_qF_s(\alpha_1,...,\alpha_q;\beta_1,...,\beta_s;z)=\sum_{k=0}^{\infty}\frac{(\alpha_1)_k...(\alpha_q)_k}{(\beta_1)_k...(\beta_s)_k}.\frac{z^k}{k!} \tag{1.4}$$

$$(q\le s+1;q,s\in\mathbb{N}_0=\mathbb{N}\cup\{0\};z\in U),$$

where $(\theta)_\nu$ is the Pochhammer symbol defined, in terms of the Gamma function Γ, by

$$(\theta)_\nu=\frac{\Gamma(\theta+\nu)}{\Gamma(\theta)}=\begin{cases}1 & (\nu=0;\theta\in\mathbb{C}^*=\mathbb{C}\backslash\{0\})\\ \theta(\theta+1)...(\theta+\nu-1) & (\nu\in\mathbb{N};\theta\in\mathbb{C}).\end{cases} \tag{1.5}$$

Corresponding to the function $h_p(\alpha_1,...,\alpha_q;\beta_1,...,\beta_s;z)$, defined by

$$h_p(\alpha_1,...,\alpha_q;\beta_1,...,\beta_s;z)=z^{-p}{}_qF_s(\alpha_1,...,\alpha_q;\beta_1,...,\beta_s;z), \tag{1.6}$$

we consider a linear operator

$$H_p(\alpha_1,...,\alpha_q;\beta_1,...,\beta_s):\sum\nolimits_p\to\sum\nolimits_p,$$

which is defined by the following Hadamard product:

$$H_p(\alpha_1,...,\alpha_q;\beta_1,...,\beta_s)f(z)=h_p(\alpha_1,...,\alpha_q;\beta_1,...,\beta_s;z)*f(z). \tag{1.7}$$

We observe that, for a function $f(z)$ of the form (1.1), we have

$$H_p(\alpha_1,...,\alpha_q;\beta_1,...,\beta_s)f(z)=z^{-p}+\sum_{k=1-p}^{\infty}\Gamma_{k+p}(\alpha_1)a_kz^k. \tag{1.8}$$

where, for convenience

$$\Gamma_m(\alpha_1)=\frac{(\alpha_1)_m...(\alpha_q)_m}{(\beta_1)_m...(\beta_s)_m}.\frac{1}{(m)!} \tag{1.9}$$

If, for convenience, we write

$$H_{p,q,s}(\alpha_1)=H_p(\alpha_1,...,\alpha_q;\beta_1,...,\beta_s), \tag{1.10}$$

then one can easily verify from the definition (1.8) that (see [16])

$$z(H_{p,q,s}(\alpha_1)f(z))'=\alpha_1 H_{p,q,s}(\alpha_1+1)f(z)-(\alpha_1+p)H_{p,q,s}(\alpha_1)f(z). \tag{1.11}$$

The linear operator $H_{p,q,s}(\alpha_1)$ was investigated recently by Liu and Srivastava [16] and Aouf [5]. Some interesting subclasses of analytic functions associated with the generalized hypergeometric function, were considered recently by (for example) Dziok and Srivastava ([7] and [8]),Gangadharan et al. [9], Liu [14].

In particular, for $q=2, s=1$ and $\alpha_2=1$, we obtain the linear operator

$$H_{p,2,1}(\alpha_1,1;\beta_1)f(z)=L_p(\alpha_1,\beta_1)f(z) \qquad (f(z)\in\sum\nolimits_p;\ \alpha_1>0;\ \beta_1>0),$$

which was introduced and studied by Liu and Srivastava [15]. Also we note that, for any integer $n>-p$ and $f(z)\in\sum_p$, we have

$$H_{p,2,1}(n+p,1;1)f(z)=D^{n+p-1}f(z)=\frac{1}{z^p(1-z)^{n+p}}*f(z),$$

where $D^{n+p-1}f(z)$ is the differential operator studied by Uralegaddi and Somanatha [19] and Aouf [4].

For a function $f\in\sum_p$, we define

$$I^0\left(H_{p,q,s}(\alpha_1)f(z)\right)=H_{p,q,s}(\alpha_1)f(z),$$

$$I^1\left(H_{p,q,s}(\alpha_1)f(z)\right)=z\left(H_{p,q,s}(\alpha_1)f(z)\right)'+\frac{p+1}{z^p},$$

$$I^2\left(H_{p,q,s}(\alpha_1)f(z)\right)=z\left(I^1\left(H_{p,q,s}(\alpha_1)f(z)\right)\right)'+\frac{p+1}{z^p}$$

and (in general)

$$\begin{aligned} I^n\left(H_{p,q,s}(\alpha_1)f(z)\right) &= z\left(I^{n-1}\left(H_{p,q,s}(\alpha_1)f(z)\right)\right)'+\frac{p+1}{z^p} \\ &= \frac{1}{z^p}+\sum_{k=1-p}^{\infty}k^n\Gamma_{k+p}a_kz^k\ (p\in\mathbb{N};\ n\in\mathbb{N}_0=\mathbb{N}\cup\{0\}), \end{aligned}$$

where Γ_{k+p} is given by (1.9).

We note that $I^n\left(H_{p,2,1}(\alpha,1;\beta)f(z)\right)=I^n\left(L_p(\alpha,\beta)f(z)\right)$ (see Ghanim and Darus [10]).

Making use of the operator $I^n\left(H_{p,q,s}(\alpha_1)f(z)\right)$, we say that a function $f(z) \in \sum_p$ is in the class $H_p^n(q,s,\alpha_1;A,B,b)$ if it satisfies the following inequality:

$$p - \frac{1}{b}\left\{\frac{z\left(I^n\left(H_{p,q,s}(\alpha_1)f(z)\right)\right)'}{I^n\left(H_{p,q,s}(\alpha_1)f(z)\right)} + p\right\} \prec p\frac{1+Az}{1+Bz} \tag{1.12}$$

or, equivalently, to

$$\left|\frac{\dfrac{z\left(I^n\left(H_{p,q,s}(\alpha_1)f(z)\right)\right)'}{I^n\left(H_{p,q,s}(\alpha_1)f(z)\right)} + p}{B\dfrac{z\left(I^n\left(H_{p,q,s}(\alpha_1)f(z)\right)\right)'}{I^n\left(H_{p,q,s}(\alpha_1)f(z)\right)} + p\left[(A-B)b + B\right]}\right| < 1$$

$$\begin{gathered}(-1 \le B < A \le 1; \alpha_1, ..., \alpha_q \in \mathbb{C} \text{ and } \beta_1, ..., \beta_s \in \mathbb{C}\backslash\mathbb{Z}_0^-; \\ p \in \mathbb{N}; q, s, n \in \mathbb{N}_0; q \le s+1; b \in \mathbb{C}^*; z \in U).\end{gathered} \tag{1.13}$$

Let $\sum_p^*$ denote the subclass of $\sum_p$ consisting of functions of the form:

$$f(z) = z^{-p} + \sum_{k=p}^{\infty} |a_k| z^k \quad (p \in N). \tag{1.14}$$

We now write

$$H_p^{n,*}(q,s;\alpha_1;A,B,b) = H_p^n(q,s;\alpha_1;A,B,b) \cap \sum\nolimits_p^*.$$

We note the following interesting relationship with some of the special function classes which were investigated recently:

(i) $H_p^{0,*}(q,s;\alpha_1;A,B,1) = \sum_{p,q}^*(A,B)$ (see Goyal et al. [12]) ;

(ii) $H_p^{0,*}(2,1;a,1,c;A,B,b) = H_{a,c}^+(p;A,B,b,1)$ ($a \in \mathbb{R};\ c \in \mathbb{R}\backslash\{0,-1,-2,...\}$)(see Aqlan and Kulkarni [6]).

Also, we note that:

(i) $H_p^{n,*}(q,s;\alpha_1;\beta,-\beta,b) = H_p^{n,*}(q,s;\alpha_1;\beta,b)$

$$= \left\{ f(z) \in \sum\nolimits_p^* : \left|\frac{\dfrac{z\left(I^n\left(H_{p,q,s}(\alpha_1)f(z)\right)\right)'}{I^n\left(H_{p,q,s}(\alpha_1)f(z)\right)} + p}{\dfrac{z\left(I^n\left(H_{p,q,s}(\alpha_1)f(z)\right)\right)'}{I^n\left(H_{p,q,s}(\alpha_1)f(z)\right)} + p(1-2b)}\right| < \beta \right.$$

$$\left.\begin{gathered}(0 < \beta \le 1;\ b \in \mathbb{C}^*;\ \alpha_1, ..., \alpha_q \in \mathbb{C} \text{ and } \beta_1, ..., \beta_s \in \mathbb{C}\backslash\mathbb{Z}_0^-; \\ q \le s+1; p \in \mathbb{N}\ ; q, s, n \in \mathbb{N}_0;\ z \in U)\end{gathered}\right\}; \tag{1.15}$$

(ii) $H_p^{n,*}(q,s;\alpha_1;\beta,-(2\gamma-1)\beta,b)=H_p^{n,*}(q,s;\alpha_1;\beta,\gamma,b)$

$$=\left\{f(z)\in\sum\nolimits_p^*:\left|\frac{\dfrac{z\left(I^n\left(H_{p,q,s}(\alpha_1)f(z)\right)\right)'}{I^n\left(H_{p,q,s}(\alpha_1)f(z)\right)}+p}{(2\gamma-1)\dfrac{z\left(I^n\left(H_{p,q,s}(\alpha_1)f(z)\right)\right)'}{I^n\left(H_{p,q,s}(\alpha_1)f(z)\right)}+p\left[2\gamma(1-b)-1\right]}\right|<\beta\right.$$

$$\left.\begin{array}{c}(0<\beta\leq 1;\ \frac{1}{2}\leq\gamma\leq 1;\ b\in\mathbb{C}^*;\ \alpha_1,...,\alpha_q\in\mathbb{C}\text{ and }\beta_1,...,\beta_s\in\mathbb{C}\backslash\mathbb{Z}_0^-;\\ q\leq s+1;\ p\in\mathbb{N};\ q,s,n\in\mathbb{N}_0;\ z\in U)\end{array}\right\};\quad(1.16)$$

(iii) $H_p^{n,*}(2,1;\delta+p,1,1;A,B,b)=D_p^{n,*}(\delta;A,B,b)$

$$=\left\{f(z)\in\sum\nolimits_p^*:\left|\frac{\dfrac{z\left(I^n\left(D^{\delta+p-1}f(z)\right)\right)'}{I^n\left(D^{\delta+p-1}f(z)\right)}+p}{B\dfrac{z\left(I^n\left(D^{\delta+p-1}f(z)\right)\right)'}{I^n\left(D^{\delta+p-1}f(z)\right)}+p\left[(A-B)b+B\right]}\right|<1\right.$$

$$\left.(-1\leq B<A\leq 1;\ \delta>-p;\ p\in\mathbb{N};\ n\in\mathbb{N}_0;b\in\mathbb{C}^*;z\in U)\right\};\quad(1.17)$$

(iv) $H_p^{n,*}(2,1;a,1,c;A,B,b)=L_p^{n,*}(a,c;A,B,b)$

$$=\left\{f(z)\in\sum\nolimits_p^*:\left|\frac{\dfrac{z\left(I^n\left(L_p(a,c)f(z)\right)\right)'}{I^n\left(L_p(a,c)f(z)\right)}+p}{B\dfrac{z\left(I^n\left(L_p(a,c)f(z)\right)\right)'}{I^n\left(L_p(a,c)f(z)\right)}+p\left[(A-B)b+B\right]}\right|<1\right.$$

$$\left.\begin{array}{c}(-1\leq B<A\leq 1;\ a\in\mathbb{R};\ c\in\mathbb{R}\backslash\{0,-1,-2,...\};\\ p\in\mathbb{N};\ n\in\mathbb{N}_0;b\in\mathbb{C}^*;z\in U)\end{array}\right\}.\quad(1.18)$$

2. Some basic properties of the class $H_p^{n,*}(q,s,\alpha_1;A,B,b)$

We begin by proving the necessary and sufficient condition (invovling coefficient bounds) for the class $H_p^{n,*}(q,s,\alpha_1;A,B,b)$.

Theorem 1. *Let the function $f(z)$ defined by (1.14) be in the class $\sum_p^*$. Then the function $f(z)$ belongs to the class $H_p^{n,*}(q,s,\alpha_1;A,B,b)$ if and only if*

$$\sum_{k=p}^{\infty} k^n \left[(k+p)(1-B) - p\,|b|\,(A-B)\right] \Gamma_{k+p}\,|a_k| \leq p\,|b|\,(A-B). \tag{2.1}$$

Proof. Assuming that the inequality (2.1) holds true then from (2.1), we find that

$$\left| \frac{z\left(I^n\left(H_{p,q,s}(\alpha_1)f(z)\right)\right)' + pI^n\left(H_{p,q,s}(\alpha_1)f(z)\right)}{Bz\left(I^n\left(H_{p,q,s}(\alpha_1)f(z)\right)\right)' + \left[Bp(1-b)+Apb\right] I^n\left(H_{p,q,s}(\alpha_1)f(z)\right)} \right|$$
$$= \left| \frac{\sum\limits_{k=p}^{\infty} k^n (k+p)\,\Gamma_{k+p}\,|a_k|\, z^{k+p}}{pb(A-B) + \sum\limits_{k=p}^{\infty} k^n \left[B(k+p) + pb(A-B)\right] \Gamma_{k+p}\,|a_k|\, z^{k+p}} \right| < 1 \quad (z \in U^*), \tag{2.2}$$

$z \in \partial U = \{z : z \in \mathbb{C} \text{ and } |z| = 1\}$. Hence, by the maximum modulus theorem, we have $f(z) \in H_p^{n,*}(q,s,\alpha_1;A,B,b)$.

Conversely, suppose that $f(z)$ is in the class $H_p^{n,*}(q,s,\alpha_1;A,B,b)$ with $f(z)$ of the form (1.14), then we find from (1.13), that

$$\left| \frac{z\left(I^n\left(H_{p,q,s}(\alpha_1)f(z)\right)\right)' + pI^n\left(H_{p,q,s}(\alpha_1)f(z)\right)}{Bz\left(I^n\left(H_{p,q,s}(\alpha_1)f(z)\right)\right)' + \left[Bp(1-b)+Apb\right] I^n\left(H_{p,q,s}(\alpha_1)f(z)\right)} \right|$$
$$= \left| \frac{\sum\limits_{k=p}^{\infty} k^n (k+p)\,\Gamma_{k+p}\,|a_k|\, z^{k+p}}{pb(A-B) + \sum\limits_{k=p}^{\infty} k^n \left[B(k+p) + pb(A-B)\right] \Gamma_{k+p}\,|a_k|\, z^{k+p}} \right| < 1. \tag{2.3}$$

If we choose z to be real and $z \to 1^-$, we get

$$\sum_{k=p}^{\infty} k^n \left[(k+p)(1-B) - p\,|b|\,(A-B)\right] \Gamma_{k+p}\,|a_k| \leq p\,|b|\,(A-B), \tag{2.4}$$

which is precisely the assertion (2.1) of Theorem 1.

Corollary 2. *Let the function $f(z)$ be defined by (1.14) be in the class $H_p^{n,*}(q,s,\alpha_1;A,B,b)$ Then*

$$|a_k| \leq \frac{p\,|b|\,(A-B)}{k^n \left[(k+p)(1-B) - p\,|b|\,(A-B)\right] \Gamma_{k+p}} \quad (k \geq p). \tag{2.5}$$

The result is sharp for the function $f(z)$ given by

$$f(z) = z^{-p} + \frac{p\,|b|\,(A-B)}{k^n\left[(k+p)\,(1-B) - p\,|b|\,(A-B)\right]\Gamma_{k+p}} z^k \quad (k \geq p). \tag{2.6}$$

Next we prove the following growth and distortion properties for the class $H_p^{n,*}(q, s, \alpha_1; A, B, b)$.

Theorem 3. *If a function $f(z)$ defined by (1.14) is in the class $H_p^{n,*}(q, s, \alpha_1; A, B, b)$, then for $|z| = r < 1$, we have*

$$\left\{\frac{(p+m-1)!}{(p-1)!} - \frac{p!\,|b|\,(A-B)}{\left[2(1-B) - |b|\,(A-B)\right]p^n(p-m)!\Gamma_{2p}} r^{2p}\right\} r^{-(p+m)}$$
$$\leq \left|f^{(m)}(z)\right| \leq \left\{\frac{(p+m-1)!}{(p-1)!} - \frac{p!\,|b|\,(A-B)}{\left[2(1-B) - |b|\,(A-B)\right]p^n(p-m)!\Gamma_{2p}} r^{2p}\right\} r^{-(p+m)}. \tag{2.7}$$

The result is sharp for the function $f(z)$ given by

$$f(z) = z^{-p} + \frac{|b|\,(A-B)}{p^n\left[2(1-B) - |b|\,(A-B)\right]\Gamma_{2p}} z^p. \tag{2.8}$$

Proof. Let $f(z) \in H_p^{n,*}(q, s, \alpha_1; A, B, b)$. Then we find from Theorem 1 that

$$\frac{p\left[2(1-B) - |b|\,(A-B)\right]p^n(p-m)!\Gamma_{2p}}{p!} \sum_{k=p}^{\infty} \frac{k!}{(k-m)!}|a_k|$$
$$\leq \sum_{k=p}^{\infty}\left[(k+p)\,(1-B) - p\,|b|\,(A-B)\right]k^n\Gamma_{k+p}.\,|a_k| \leq p\,|b|\,(A-B),$$

which yields

$$\sum_{k=p}^{\infty} \frac{k!}{(k-m)!}|a_k| \leq \frac{|b|\,(A-B)}{p^n\left[2(1-B) - |b|\,(A-B)\right]\Gamma_{2p}} \frac{p!}{(p-m)!}. \tag{2.9}$$

Now, by differentiating both sides of (1.14) m times with respect to z, we have

$$f^{(m)}(z) = (-1)^m \frac{(p+m-1)!}{(p-1)!} z^{-(p+m)} + \sum_{k=p}^{\infty} \frac{k!}{(k-m)!}|a_k|\, z^{k-m} \quad (p \in N; 0 \leq m < p) \tag{2.10}$$

and Theorem 3 follows easily from (2.9) and (2.10), respectively.

Finally, it is easy to see that the bounds in (2.7) are attained for the function $f(z)$ given by (2.8).

Next, we determine the radii of meromorphically p-valent starlikeness and convexity of order φ $(0 \leq \varphi < p)$ for functions in the class $H_p^{n,*}(q, s, \alpha_1; A, B, b)$.

Theorem 4. *Let the function $f(z)$ defined by (1.14) be in the class $H_p^{n,*}(q, s, \alpha_1; A, B, b)$, then we have:*

(i) $f(z)$ is meromorphically p-valent starlike of order φ $(0 \leq \varphi < p)$ in the disc $|z| < r_1$, that is,

$$\Re\left\{-\frac{zf'(z)}{f(z)}\right\} > \varphi \quad (|z| < r_1;\ 0 \leq \varphi < p;\ p \in \mathbb{N}), \tag{2.11}$$

where

$$r_1 = \inf_{k \geq p}\left\{\frac{k^n\left[(k+p)(1-B) - p|b|(A-B)\right]\Gamma_{k+p}}{p|b|(A-B)}\frac{(p-\varphi)}{(k+\varphi)}\right\}^{\dfrac{1}{k+p}}, \tag{2.12}$$

(ii) $f(z)$ is meromorphically p-valent convex of order φ $(0 \leq \varphi < p)$ in the disc $|z| < r_2$, that is,

$$\Re\left\{-\left(1 + \frac{zf''(z)}{f'(z)}\right)\right\} > \varphi \quad (|z| < r_2;\ 0 \leq \varphi < p;\ p \in \mathbb{N}), \tag{2.13}$$

where

$$r_2 = \inf_{k \geq p}\left\{\frac{k^{n-1}\left[(k+p)(1-B) - p|b|(A-B)\right]\Gamma_{k+p}}{p|b|(A-B)}\frac{p(p-\varphi)}{(k+\varphi)}\right\}^{\dfrac{1}{k+p}}. \tag{2.14}$$

Each of these results is sharp for the function $f(z)$ given by (2.6).

Proof. (i) From the definition (1.14), we easily get

$$\left|\frac{\dfrac{zf'(z)}{f(z)} + p}{\dfrac{zf'(z)}{f(z)} - p + 2\varphi}\right| \leq \frac{\sum\limits_{k=p}^{\infty}(k+p)|a_k||z|^{k+p}}{2(p-\varphi) - \sum\limits_{k=p}^{\infty}(k-p+2\varphi)|a_k||z|^{k+p}}. \tag{2.15}$$

Thus, we have the desired inequality

$$\left|\frac{\dfrac{zf'(z)}{f(z)} + p}{\dfrac{zf'(z)}{f(z)} - p + 2\varphi}\right| \leq 1 \quad (0 \leq \varphi < p;\ p \in \mathbb{N}), \tag{2.16}$$

if

$$\sum_{k=p}^{\infty}\left(\frac{k+\varphi}{p-\varphi}\right)|a_k|\,|z|^{k+p}\leq 1. \tag{2.17}$$

Hence, by Theorem 1, (2.17) will be true

$$\left(\frac{k+\varphi}{p-\varphi}\right)|z|^{k+p}\leq \frac{k^n\left[(k+p)(1-B)-p\,|b|\,(A-B)\right]\Gamma_{k+p}}{p\,|b|\,(A-B)}$$

$$|z|\leq\left\{\frac{k^n\left[(k+p)(1-B)-p\,|b|\,(A-B)\right]\Gamma_{k+p}}{p\,|b|\,(A-B)}\frac{(p-\varphi)}{(k+\varphi)}\right\}^{\frac{1}{k+p}}. \tag{2.18}$$

The last inequality (2.18) leads us immediately to the disc $|z|<r_1$, where r_1 is given by (2.12).

(ii) In order to prove the second assertion of Theorem 3, we find from the definition (1.14) that

$$\left|\frac{1+\frac{zf''(z)}{f'(z)}+p}{1+\frac{zf''(z)}{f'(z)}-p+2\varphi}\right|\leq\frac{\sum\limits_{k=p}^{\infty}k(k+p)\,|a_k|\,|z|^{k+p}}{2p(p-\varphi)-\sum\limits_{k=p}^{\infty}k(k-p+2\varphi)\,|a_k|\,|z|^{k+p}}. \tag{2.19}$$

Thus, we have the desired inequality

$$\left|\frac{1+\frac{zf''(z)}{f'(z)}+p}{1+\frac{zf''(z)}{f'(z)}-p+2\varphi}\right|\leq 1\quad(0\leq\varphi<p;p\in\mathbb{N}), \tag{2.20}$$

if

$$\sum_{k=p}^{\infty}\frac{k}{p}\left(\frac{k+\varphi}{p-\varphi}\right)|a_k|\,|z|^{k+p}\leq 1. \tag{2.21}$$

Hence, by Theorem 1, (2.21) will be true if

$$\frac{k}{p}\left(\frac{k+\varphi}{p-\varphi}\right)|z|^{k+p}\leq\frac{k^n\left[(k+p)(1-B)-p\,|b|\,(A-B)\right]\Gamma_{k+p}}{p\,|b|\,(A-B)}. \tag{2.22}$$

The last inequality (2.22) readily yields the disc $|z|<r_2$, where r_2 defined by (2.14), and the proof of Theorem 4 is completed by merely verifying that each assertion is sharp for the function $f(z)$ given by (2.6).

3. Closure theorems

In this section we first prove:

Theorem 5. *The class $H_p^{n,*}(q,s,\alpha_1;A,B,b)$ is closed under convex linear combinations.*

Proof. Let each of the functions

$$f_j(z) = z^{-p} + \sum_{k=p}^{\infty} |a_{k,j}|\, z^k \qquad (j=1,2;\ p\in\mathbb{N}) \tag{3.1}$$

be in the class $H_p^{n,*}(q,s,\alpha_1;A,B,b)$. It is sufficient to show that the function $F(z)$ defined by

$$F(z) = (1-t)f_1(z) + tf_2(z)\ (0\le t\le 1) \tag{3.2}$$

is also in the class $H_p^{n,*}(q,s,\alpha_1;A,B,b)$. Since

$$F(z) = z^{-p} + \sum_{k=p}^{\infty} \left[(1-t)\,|a_{k,1}| + t\,|a_{k,2}|\right] z^k \quad (0\le t\le 1). \tag{3.3}$$

With the aid of Theorem 1, we have

$$\begin{aligned}
&\sum_{k=p}^{\infty} k^n \left[(k+p)(1-B) - p\,|b|\,(A-B)\right]\Gamma_{k+p}.\left\{(1-t)\,|a_{k,1}| + t\,|a_{k,2}|\right\}\\
= \quad & (1-t)\sum_{k=p}^{\infty} k^n \left[(k+p)(1-B) - p\,|b|\,(A-B)\right]\Gamma_{k+p}\,|a_{k,1}| +\\
& t\sum_{k=p}^{\infty} k^n \left[(k+p)(1-B) - p\,|b|\,(A-B)\right]\Gamma_{k+p}\,|a_{k,2}|\\
\le \quad & (1-t)p\,|b|\,(A-B) + tp\,|b|\,(A-B) = p\,|b|\,(A-B),
\end{aligned}$$

which shows that $F(z)\in H_p^{n,*}(q,s,\alpha_1;A,B,b)$.

Theorem 6. *Let* $f_{p-1}(z) = \dfrac{1}{z^p}$ *and*

$$f_k(z) = \frac{1}{z^p} + \frac{p\,|b|\,(A-B)}{k^n\left[(k+p)(1-B) - p\,|b|\,(A-B)\right]\Gamma_{k+p}} z^k \quad (k\ge p;\ p\in\mathbb{N}). \tag{3.4}$$

Then $f(z)\in H_p^{n,*}(q,s,\alpha_1;A,B,b)$ if and only if it can be expressed in the form:

$$f(z) = \sum_{k=p-1}^{\infty} \mu_k f_k(z), \tag{3.5}$$

where $\mu_k \geq 0$ $(k \geq p-1;\ p \in \mathbb{N})$ *and* $\sum\limits_{k=p-1}^{\infty} \mu_k = 1.$

Proof. Let the function $f(z)$ expressed in the form given by (3.5), then

$$f(z) = z^{-p} + \sum_{k=p}^{\infty} \mu_k \frac{p\,|b|\,(A-B)}{k^n\,[(k+p)\,(1-B) - p\,|b|\,(A-B)]\,\Gamma_{k+p}} z^k \tag{3.6}$$

and for this function, we have

$$\sum_{k=p}^{\infty} k^n\,[(k+p)\,(1-B) - p\,|b|\,(A-B)]\,\Gamma_{k+p}\mu_k \frac{p\,|b|\,(A-B)}{k^n\,[(k+p)\,(1-B) - p\,|b|\,(A-B)]\,\Gamma_{k+p}}$$

$$= \sum_{k=p}^{\infty} \mu_k p\,|b|\,(A-B) = p\,|b|\,(A-B)(1-\mu_{p-1}) \leq p\,|b|\,(A-B), \tag{3.7}$$

which shows that $f(z) \in H_p^{n,*}(q,s,\alpha_1;A,B,b)$ by Theorem 1.

Conversely, suppose that the function $f(z)$ defined by (1.14) belongs to the class $H_p^{n,*}(q,s,\alpha_1;A,B,b)$. Since

$$|a_k| \leq \frac{p\,|b|\,(A-B)}{k^n\,[(k+p)\,(1-B) - p\,|b|\,(A-B)]\,\Gamma_{k+p}} \quad (k \geq p;\ p \in \mathbb{N}), \tag{3.8}$$

by Corollary 2, setting

$$\mu_k = \frac{k^n\,[(k+p)\,(1-B) - p\,|b|\,(A-B)]\,\Gamma_{k+p}}{p\,|b|\,(A-B)} |a_k| \quad (k \geq p;\ p \in \mathbb{N}) \tag{3.9}$$

and

$$\mu_{p-1} = 1 - \sum_{k=p}^{\infty} \mu_k,$$

it follows that

$$f(z) = \sum_{k=p-1}^{\infty} \mu_k f_k(z).$$

This completes the proof of Theorem 6.

4. Neighborhoods and partial sums

Following the earlier works (based upon the familiar concept of neighborhoods of analytic functions) by Goodman [11] and Ruscheweyh [17] and (more recently) by

Altintas et al. ([1], [2] and [3]), Liu [13] and Liu and Srivastava [15], we begin by introducing here the δ-neighborhoods of a function $f(z) \in \Sigma_p$ of the form (1.1) by means of the definition given below:

$$N_\delta(f) = \left\{ g : \; g \in \Sigma_p, \; g(z) = z^{-p} + \sum_{k=1-p}^{\infty} b_k z^k \text{ and} \right.$$
$$\sum_{k=1-p}^{\infty} \frac{k^n \left[(k+p)\left(1+|B|\right) - p\,|b|\,(A-B)\right]}{p\,|b|\,(A-B)} \Gamma_{k+p} \left|b_k - a_k\right| \leq \delta$$
$$\left. \begin{array}{c} (\delta > 0; -1 \leq B < A \leq 1; \alpha_1, ..., \alpha_q \in \mathbb{C} \text{ and } \beta_1, ..., \beta_s \in \mathbb{C} \backslash \mathbb{Z}_0^-; \\ p \in \mathbb{N}; \; q, s, n \in \mathbb{N}_0; \; q \leq s+1; b \in \mathbb{C}^*) \end{array} \right\}. \tag{4.1}$$

Making use of the definition (4.1), we now prove:

Theorem 7. *Let the function $f(z)$ defined by (1.1) be in the class $H_p^n(q, s, \alpha_1; A, B, b)$. If $f(z)$ satisfies the condition:*

$$\frac{f(z) + \epsilon z^{-p}}{1+\epsilon} \in H_p^n(q, s, \alpha_1; A, B, b) \quad (\epsilon \in \mathbb{C}; \; |\epsilon| < \delta; \; \delta > 0) \tag{4.2}$$

then

$$N_\delta(f) \subset H_p^n(q, s, \alpha_1; A, B, b). \tag{4.3}$$

Proof. It is easily from (1.13) that $g(z) \in H_p^n(q, s, \alpha_1; A, B, b)$ if and only if, for any complex number σ with $|\sigma| = 1$, we have

$$\frac{z\left(I^n\left(H_{p,q,s}(\alpha_1) f(z)\right)\right)' + pI^n\left(H_{p,q,s}(\alpha_1) f(z)\right)}{Bz\left(I^n\left(H_{p,q,s}(\alpha_1) f(z)\right)\right)' + \left[Bp\left(1-b\right) + Apb\right] I^n\left(H_{p,q,s}(\alpha_1) f(z)\right)} \neq \sigma \quad (z \in U), \tag{4.4}$$

which is equivalent to

$$\frac{(g * h)(z)}{z^{-p}} \neq 0 \quad (z \in U), \tag{4.5}$$

where, for convenience,

$$\begin{aligned} h(z) &= z^{-p} + \sum_{k=1-p}^{\infty} c_k z^k \\ &= z^{-p} + \sum_{k=1-p}^{\infty} \frac{k^n \left[(k+p)\left(1 - B\sigma\right) - pb\sigma(A-B)\right]}{pb\sigma(A-B)} \Gamma_{k+p} z^k. \end{aligned} \tag{4.6}$$

From (4.6), we have

$$
\begin{aligned}
|c_k| &= \left| \frac{k^n\left[(k+p)\left(1-B\sigma\right)-pb\sigma(A-B)\right]}{pb\sigma(A-B)}\Gamma_{k+p} \right| \\
&\leq \frac{k^n\left[(k+p)\left(1+|B|\right)-p\left|b\right|\left(A-B\right)\right]}{p\left|b\right|\left(A-B\right)}\Gamma_{k+p} \quad (k \geq p; p \in \mathbb{N}).
\end{aligned} \tag{4.7}
$$

If $f(z) = z^{-p} + \sum\limits_{k=1-p}^{\infty} a_k z^k \in \Sigma_p$ satisfies the condition (4.2), then (4.5) yields

$$
\left| \frac{(f*h)(z)}{z^{-p}} \right| > \delta \quad (z \in U^*; \delta > 0). \tag{4.8}
$$

Next, if we suppose that

$$
\Phi(z) = z^{-p} + \sum_{k=1-p}^{\infty} d_k z^k \in N_\delta(f) \tag{4.9}
$$

we easily see that

$$
\begin{aligned}
\left| \frac{[\Phi(z)-f(z)]*h(z)}{z^{-p}} \right| &= \left| \sum_{k=1-p}^{\infty} (d_k - a_k) c_k z^{k+p} \right| \\
&\leq |z| \sum_{k=1-p}^{\infty} \frac{k^n\left[(k+p)\left(1+|B|\right)-p\left|b\right|\left(A-B\right)\right]}{p\left|b\right|\left(A-B\right)}\Gamma_{k+p}\left|d_k - a_k\right| \\
&< \delta \quad (z \in U;\ \delta > 0).
\end{aligned} \tag{4.10}
$$

Thus we have (4.5), and hence also (4.4) for any $\sigma \in \mathbb{C}$ such that $|\sigma| = 1$, which implies that $\Phi(z) \in H_p^n(q, s, \alpha_1; A, B, b)$. This evidently proves the assertion (4.3) of Theorem 7.

We now define the δ-neighborhoods of a function $f(z) \in \Sigma_p^*$ of the form (1.14) as follows:

$$N_{\delta}^{+}(f)=\left\{ g\in\Sigma_{p}^{*},\ g(z)=z^{-p}+\sum_{k=p}^{\infty}|b_{k}|\,z^{k}\ \text{and}\right.$$

$$\sum_{k=p}^{\infty}\frac{k^{n}\left[(k+p)\,(1+|B|)-p\,|b|\,(A-B)\right]}{p\,|b|\,(A-B)}\Gamma_{k+p}\,||b_{k}|-|a_{k}||\leq\delta$$

$$\left.\begin{array}{c}(\delta>0;;-1\leq B<A\leq1;\alpha_{1},...,\alpha_{q}\in\mathbb{C}\ \text{and}\ \beta_{1},...,\beta_{s}\in\mathbb{C}\backslash\mathbb{Z}_{0}^{-};\\ p\in\mathbb{N};\ q,s,n\in\mathbb{N}_{0};\ q\leq s+1;\ b\in\mathbb{C}^{*})\end{array}\right\}. \tag{4.11}$$

Theorem 8. *Let $-1\leq B\leq0$. If the function $f(z)$ given by (1.14) is in the class $H_{p}^{n,*}(q,s,\alpha_{1}+1;A,B,b)$, then*

$$N_{\delta}^{+}(f)\subset H_{p}^{n,*}(q,s,\alpha_{1};A,B,b)\quad(\delta=\frac{2p}{\alpha_{1}+2p}). \tag{4.12}$$

The result is sharp in the sense that δ cannot be increased.

Proof. Making use of the same method as in the proof of Theorem 7, we can show that [cf. Equation (4.6)]

$$\begin{aligned}h(z)&=z^{-p}+\sum_{k=p}^{\infty}c_{k}z^{k}\\&=z^{-p}+\sum_{k=p}^{\infty}\frac{k^{n}\left[(k+p)\,(1-B\sigma)-pb\sigma(A-B)\right]}{pb\sigma(A-B)}\Gamma_{k+p}z^{k}.\end{aligned} \tag{4.13}$$

Thus, under the hypothesis $-1\leq B<A\leq1$; $-1\leq B\leq0$; $q\leq s+1$; $p\in\mathbb{N}$; $q,s,n\in\mathbb{N}_{0}$; $b\in\mathbb{C}^{*}$, if $f(z)\in H_{p}^{n,*}(q,s,\alpha_{1}+1;A,B,b)$ is given by (1.14), we obtain

$$\begin{aligned}\left|\frac{(f*h)(z)}{z^{-p}}\right|&=\left|1+\sum_{k=p}^{\infty}c_{k}\,|a_{k}|\,z^{k+p}\right|\\&\geq1-\sum_{k=p}^{\infty}k^{n}\frac{\left[(k+p)\,(1+|B|)-p\,|b|\,(A-B)\right]}{p\,|b|\,(A-B)}\Gamma_{k+p}\,|a_{k}|\\&\geq1-\frac{\alpha_{1}}{\alpha_{1}+2p}\sum_{k=p}^{\infty}\frac{k^{n}\left[(k+p)\,(1+|B|)-p\,|b|\,(A-B)\right]}{p\,|b|\,(A-B)}\Gamma_{k+p}\,|a_{k}|\\&\geq1-\frac{\alpha_{1}}{\alpha_{1}+2p}=\frac{2p}{\alpha_{1}+2p}=\delta.\end{aligned}$$

The remaining part of the proof of Theorem 8 is similar to that of Theorem 8 and we skip the details involved.

Theorem 9. *Let $f(z) \in \Sigma_p$ defined by (1.1) and $-1 \leq B \leq 0$, we define the partial sums $S_1(z)$ and $S_q(z)$ as follows*

$$S_1(z) = z^{-p} \quad \text{and} \quad S_q(z) = z^{-p} + \sum_{k=1-p}^{q-1} a_k z^k \quad (q \in \mathbb{N}\backslash\{1\}), \tag{4.14}$$

it beging understood that an empty sum is (as usual) nil. Suppose also that

$$\sum_{k=1-p}^{\infty} c_{k+p} |a_k| \leq 1 \quad \left(c_{k+p} = \frac{k^n \left[(k+p)(1+|B|) - p|b|(A-B)\right]}{p|b|(A-B)} \Gamma_{k+p} \right). \tag{4.15}$$

Then

(i) $f(z) \in H_p^n(q, s, \alpha_1; A, B, b)$

$$\text{(ii) } \Re\left\{ \frac{f(z)}{S_q(z)} \right\} > 1 - \frac{1}{c_q} \qquad (z \in U; q \in \mathbb{N}) \tag{4.16}$$

and

$$\text{(iii) } \Re\left\{ \frac{S_q(z)}{f(z)} \right\} > \frac{c_q}{1+c_q} \qquad (z \in U; q \in \mathbb{N}). \tag{4.17}$$

The estimates in (4.16) and (4.17) are sharp.

Proof. Since $\dfrac{z^{-p} + \varepsilon z^{-p}}{1+\varepsilon} = z^{-p} \in H_p^n(q, s, \alpha_1; A, B, b)$, $|\varepsilon| < 1$, then by Theorem 7, we have $N_\delta(f) \subset H_p^n(q, s, \alpha_1; A, B, b)$, $p \in \mathbb{N}$ ($N_1(z^{-p})$ denoting the 1-neighbourhood). Now since

$$\sum_{k=1-p}^{\infty} c_k |a_k| \leq 1, \tag{4.18}$$

then $f \in N_1(z^{-p})$ and $f \in H_p^n(q, s, \alpha_1; A, B, b)$.(ii) Since $\{c_k\}$ is an increasing sequence, we obtain

$$\sum_{k=1-p}^{q-p-1} |a_k| + c_q \sum_{k=q-p}^{\infty} |a_k| \leq \sum_{k=1-p}^{\infty} c_{k+p} |a_k| \leq 1, \tag{4.19}$$

where we have used the hypothesis (4.15) again. By seeting

$$h_1(z) = c_q \left\{ \frac{f(z)}{S_q(z)} - (1 - \frac{1}{c_q}) \right\} = 1 + \frac{c_q \sum\limits_{k=q-p}^{\infty} |a_k| \, z^{k+p}}{1 + \sum\limits_{k=1-p}^{q-p-1} |a_k| \, z^{k+p}}$$

and applying (4.19), we find that

$$\left| \frac{h_1(z) - 1}{h_1(z) + 1} \right| \leq \frac{c_q \sum\limits_{k=q-p}^{\infty} |a_k|}{2 - 2 \sum\limits_{k=1-p}^{q-p-1} |a_k| - c_q \sum\limits_{k=q-p}^{\infty} |a_k|} \leq 1 \quad (z \in U), \tag{4.20}$$

which readily yields the assertion (4.16) of Theorem 9. If we take

$$f(z) = z^{-p} - \frac{z^q}{c_q}, \tag{4.21}$$

then

$$\frac{f(z)}{S_q(z)} = 1 - \frac{z^{p+q}}{c_q} \to 1 - \frac{1}{c_q}, \quad \text{as } z \to 1^-,$$

which shows that the bound in (4.16) is the best possible for each $q \in N$. (iii) Just as in part (ii) above, if we put

$$h_2(z) = (1 + c_q) \left\{ \frac{S_q(z)}{f(z)} - \frac{c_q}{1 + c_q} \right\} = 1 - \frac{(1 + c_q) \sum\limits_{k=q-p}^{\infty} |a_k| \, z^{k+p}}{1 + \sum\limits_{k=1-p}^{\infty} |a_k| \, z^{k+p}}, \tag{4.22}$$

and make use of (4.19), we can deduce that

$$\left| \frac{h_2(z) - 1}{h_2(z) + 1} \right| \leq \frac{(1 + c_q) \sum\limits_{k=q-p}^{\infty} |a_k|}{2 - 2 \sum\limits_{k=1-p}^{q-p-1} |a_k| - (1 - c_q) \sum\limits_{k=q-p}^{\infty} |a_k|} \leq 1,$$

which leads us immediately to the assertion (4.17) of Theorem 9.

The bound in (4.17) is sharp for each $n \in N$, with the extremal function $f(z)$ given by (4.21). The proof of Theorem 9 is thus completed.

Remark 1. *Taking* $n = 0$, $q = 2$, $s = 1$, $\alpha_1 = a$ $(a > 0)$, $\alpha_2 = 1$ *and* $\beta_1 = c$ $(c > 0)$, *in all our results, we obtain the results obtained by Aqlan and Kulkarni [6].*

References

[1] O. Altintas and S. Owa, *Neighborhoods of certain analytic functions with negative coefficients*, Internat. J. Math. Math. Sci. 19(1996), 797-800.

[2] O. Altintas, Ö. Özkan and H. M. Srivastava, *Neighborhoods of a class of analytic functions with negative coefficients*, Appl. Math. Letters, 13(2000), no. 3, 63-67.

[3] O. Altintas, Ö. Özkan and H. M. Srivastava, *Neighborhoods of a certain family of multivalent functions with negative coefficient*, Comput. Math. Appl. 47(2004), 1667-1672.

[4] M. K. Aouf, *New criteria for multivalent meromorphic starlike function of order alpha*, Proc. Japen. Acad. Ser. A 69(1993), no. 3, 66-70.

[5] M. K. Aouf, *Certain subclasses of meromorphically multivalent functions associated with generalized hypergeometric function*, Comput. Math. Appl. 55(2008), 494-509.

[6] E. Aqlan and S. R. Kulkarni, *Linear operator on p-valent function of complex order*, Math. (Cluj) 45(68)(2003), no. 1, 207-219.

[7] J. Dziok and H. M. Srivastava, *Classes of analytic functions associated with the generalized hypergeometric function*, Appl. Math.Comput. 103(1999), 1-13.

[8] J. Dziok and H. M. Srivastava, *Certain subclasses of analytic functions associated with the generalized hypergeometric function*, Integral Transforms Spec. Funct. 14(2003), 7-18.

[9] A. Gangadharan, T. N. Shanmugan and H. M. Srivastava, *Generalized hypergeometric functions with k-uniformly convex functions*, Comput. Math. Appl. 44(2002), no. 12, 1515-1526.

[10] F. Ghanim and M. Darus, *Some properties of certain subclass of meromorphically multivalent functions defined by linear operator*, J. Math. and Statistics, 6(2010), no. 1, 34-41.

[11] A. W. Goodman, *Univalent functions and nonanalytic curves*, Proc. Amer. Math. Soc. 8(1957),898-901.

[12] S. P. Goyal, M. Bhagtani and P. Vijaywargiya, *Partial sums of certain meromorphic multivalent functions*, Int. J. Contemp. Math. Sci. 3(2008), no. 26,

[13] J.-L. Liu, *Properties of some families of meromorphic p-valent functions*, Math. Japon. 52(2000), no. 3, 425-434.

[14] J.-L. Liu, *Strongly starlike functions associated with the Dziok-Srivastava operator*, Tamkang J. Math. 35(2004), no. 1, 37-42.

[15] J.-L. Liu and H. M. Srivastava, *A linear operator and associated families of meromorphically multivalent functions*, J. Math. Anal. Appl. 259(2000), 566-581.

[16] J.-L. Liu and H. M. Srivastava, *Classes of meromorphically multivalent associated with the generalized hypergeometric function*, Math. Comput. Modelling 39(2004), 21-34.

[17] St. Ruscheweyh, *Neighborhoods of univalent functions*, Proc. Amer. Math. Soc. 81(1981), 521–527.

[18] H. M. Srivastava and P. W. Karlsson, *Multiple Gaussian Hypergeometric Series*, Halsted Press, Ellis Horwood Limited, Chichester, John Wiley and Sons, New York, Chichester, Brisbane, Toronto, 1985.

[19] B. A. Uralegaddi and C. Somanatha, *Certain classes of meromorphic multivalent functions*, Tamkang J. Math. 23(1992), 223-231.

R. M. El-Ashwah
Department of Mathematics
Faculty of Science at Damietta
University of Mansoura
New Damietta 34517, Egypt
email: *r_elashwah@yahoo.com*

M. K. Aouf
Department of Mathematics
Faculty of Science
University of Mansoura
Mansoura 33516, Egypt
email: *mkaouf127@yahoo.com*

S. M. El-Deeb
Department of Mathematics
Faculty of Science at Damietta
University of Mansoura
New Damietta 34517, Egypt
email: *shezaeldeeb@yahoo.com*