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SPECIAL SUBCLASSES OF HARMONIC FUNCTIONS

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Abstract. In the present paper we have introduced and studied special subclasses of harmonic function, we obtain the basic properties such as coefficients characterization and distortion theorem, extreme points and obtain convolution and convex combinations results

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1. Introduction

A continuous function $f = u + iv$ is a complex valued harmonic function in a simply connected complex domain $D \subset \mathbb{C}$ if both u and v are real harmonic in D. It was shown by Clunie and Sheil-Small [2] that such harmonic function can be represented by $f = h + \overline{g}$, where h and g are analytic in D. Also, a necessary and sufficient condition for f to be locally univalent and sense preserving in D is that $\left|h'(z)\right| > \left|g'(z)\right|$, (see also, [3, 4] and [6, 7].

Denote by S_H the class of functions f that are harmonic univalent and sense-preserving in the open unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$ for which $f(0) = h(0) = f_z'(0) - 1 = 0$. Then for $f = h + \overline{g} \in S_H$ we may express the analytic functions h and g as

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad , \quad g(z) = \sum_{n=1}^{\infty} b_n z^n \quad , |b_1| < 1. \tag{1}$$

Clunie and Shell-Small [2] investigated the class S_H as well as its geometric subclasses and obtained some coefficient bounds.

Also let $S_{\overline{H}}$ denote the subclass of S_H consisting of functions $f = h + \overline{g}$ such that the functions h and g are of the form

$$h(z) = z - \sum_{n=2}^{\infty} |a_n| z^n \quad , \quad g(z) = \sum_{n=1}^{\infty} |b_n| z^n \quad , |b_1| < 1. \tag{2}$$

In this work, we may express the analytic functions h^β and g^β as follows (see e.g.[1])

$$h(z)^\beta = z^\beta + \sum_{n=2}^{\infty} \beta a_n z^{n+\beta-1} \quad , \quad g(z)^\beta = \sum_{n=1}^{\infty} \beta b_n z^{n+\beta-1} \quad , |b_1| < 1. \tag{3}$$

Therefore,

$$f(z)^\beta = h(z)^\beta + \overline{g(z)^\beta} \tag{4}$$

We let $H(\alpha, \beta)$ denote the class of harmonic functions of the form (4) , which satisfy the condition

$$\Re\left\{(1+e^{i\theta})\frac{zf'(z)^\beta}{z'f(z)^\beta} - \beta e^{i\theta}\right\} \geq \alpha\beta \ (0 \leq \alpha < 1; \theta \in \mathbb{R}; \beta \geq 1), \tag{5}$$

where $f'(z)^\beta = \dfrac{\partial f(z)^\beta}{\partial\theta}$ *and* $z' = \dfrac{\partial z}{\partial\theta}$.

Replacing $h(z)^\beta + \overline{g(z)^\beta}$ for $f(z)^\beta$ in (5), we have:

$$\Re\left\{(1+e^{i\theta})\frac{zh'(z)^\beta - \overline{zg'(z)^\beta}}{h(z)^\beta + \overline{g(z)^\beta}} - \beta e^{i\theta}\right\} \geq \alpha\beta \ (0 \leq \alpha < 1; \theta \in \mathbb{R}; \beta \geq 1). \tag{6}$$

It is clear that the class $H(\alpha, 1) = G_H(\alpha)(0 \leq \alpha < 1)$ defined by Rosy et al. [5].

We let the class $\overline{H}(\alpha, \beta)$ consist of harmonic functions $f \in H(\alpha, \beta)$ so h^β and g^β of the form

$$h(z)^\beta = z^\beta - \sum_{n=2}^{\infty} \beta a_n z^{n+\beta-1} \quad , \quad g(z)^\beta = \sum_{n=1}^{\infty} \beta b_n z^{n+\beta-1} \quad , |b_1| < 1. \tag{7}$$

In this paper, we extend the results of the class $G_H(\alpha)$ defined by Rosy et al. [5] to the classes $H(\alpha, \beta)$ and $\overline{H}(\alpha, \beta)$, we also obtain some basic properties for the class $\overline{H}(\alpha, \beta)$.

2. Coefficient characterization and distortion theorem

Unless otherwise mentioned, we assume throughout this paper that $0 \leq \alpha < 1, \beta \geq 1$ and θ is $\Re$. We begin with a sufficient condition for functions in the class $H(\alpha, \beta)$.

Theorem 1. *Let $f^\beta = h^\beta + \overline{g^\beta}$ be such that h^β and g^β are given by (3). Furthermore, let*

$$\sum_{n=2}^{\infty}\left(\tfrac{2\beta(n+\beta-1)-\beta(1+\alpha)}{\beta(1-\alpha)+1)-|\beta(1-\alpha)-1|}\right)|a_n| + \sum_{n=1}^{\infty}\left(\tfrac{2\beta(n+\beta-1)+\beta(1+\alpha)}{\beta(1-\alpha)+1)-|\beta(1-\alpha)-1|}\right)|b_n| \leq \frac{1}{2}. \tag{8}$$

Then $f \in H(\alpha, \beta)$.

Proof. Now, suppose that the condition (8) holds. It suffices to show that $Re\{A(z)/B(z)\} > 0$, where

$$\begin{aligned} A(z) &= \left(1+e^{i\theta}\right)\left(zh'(z)^{\beta}+\overline{zg'(z)^{\beta}}\right)-\left(\alpha\beta+\beta e^{i\theta}\right)\left(h(z)^{\beta}+\overline{g(z)^{\beta}}\right) \\ B(z) &= h(z)^{\beta}+\overline{g(z)^{\beta}} \end{aligned} \tag{9}$$

Substituting for h^{β} and g^{β} in (9) and using the fact that $\Re(w) \geq 0$ if and only if $|w+1| > |w-1|$ in U, it suffices to show that $|A(z)+B(z)| - |A(z)-B(z)| \geq 0$. Substituting for $A(z)$ and $B(z)$ gives

$$\begin{aligned} & |A(z)+B(z)| - |A(z)-B(z)| \\ = & \left|\beta(1-\alpha)+1+\sum_{n=2}^{\infty}\beta(n+\beta-1)\left(1+e^{i\theta}\right)-\left(\alpha\beta+\beta e^{i\theta}-\beta\right)a_n z^{n-1}\right. \\ & \left.-\left(\frac{\overline{z}}{z}\right)^{\beta}\sum_{n=1}^{\infty}\beta(n+\beta-1)(1+e^{-i\theta})+\left(\alpha\beta+\beta e^{i\theta}-\beta\right)b_n\overline{z}^{n-1}\right| \\ & -\left|\beta(1-\alpha)-1+\sum_{n=2}^{\infty}\left(1+e^{i\theta}\right)(n+\beta-1)-\left(\alpha\beta+\beta e^{i\theta}+\beta\right)a_n z^{n-1}\right. \\ & \left.-\left(\frac{\overline{z}}{z}\right)^{\beta}\sum_{n=1}^{\infty}(1+e^{-i\theta})(n+\beta-1)+\left(\alpha\beta+\beta e^{i\theta}+\beta\right)b_n\overline{z}^{n-1}\right| \end{aligned}$$

$$\begin{aligned} \geq & \ (\beta(1-\alpha)+1)-|\beta(1-\alpha)-1|-\sum_{n=2}^{\infty}\left(4\beta(n+\beta-1)-2\beta(\alpha+1)\right)|a_n|\,|z|^{n-1} \\ & -\sum_{n=1}^{\infty}\left(4\beta(n+\beta-1)+2\beta(1+\alpha)\right)|b_n|\,|\overline{z}|^{n-1} \\ \geq & \ \left((\beta(1-\alpha)+1)-|\beta(1-\alpha)-1|\right)\left\{1-2\sum_{n=2}^{\infty}\left(\tfrac{2\beta(n+\beta-1)-\beta(1+\alpha)}{\beta(1-\alpha)+1)-|\beta(1-\alpha)-1|}\right)|a_n|\right. \\ & \left.-2\sum_{n=1}^{\infty}\left(\tfrac{2\beta(n+\beta-1)+\beta(1+\alpha)}{\beta(1-\alpha)+1)-|\beta(1-\alpha)-1|}\right)|b_n|\right\} \\ \geq & \ 0, \quad \text{by } (8) \end{aligned}$$

The harmonic functions

$$f(z) = z^{\beta} + \sum_{n=2}^{\infty} \tfrac{\beta(1-\alpha)+1)-|\beta(1-\alpha)-1|}{2[2(n+\beta-1)-\beta(1+\alpha)]} x_{n+\beta-1} z^{n+\beta-1}$$
$$+ \sum_{n=1}^{\infty} \tfrac{\beta(1-\alpha)+1)-|\beta(1-\alpha)-1|}{2[2(n+\beta-1)+\beta(1+\alpha)]} \overline{y}_{n+\beta-1} \overline{z}^{n+\beta-1}, \tag{10}$$

where $\sum_{n=2}^{\infty} |x_{n+\beta-1}| + \sum_{n=1}^{\infty} \left|\overline{y}_{n+\beta-1}\right| = 1$, show that the coefficient bound given by (8) is sharp. This completes the proof of Theorem 1.

Corollary 2. *Let $f^{\beta} = h^{\beta} + \overline{g^{\beta}}$ be such that h^{β} and g^{β} are given by* (3)*. Furthermore, let $\beta \geq \frac{1}{1-\alpha}$. If the condition*

$$\sum_{n=2}^{\infty} (2\beta(n+\beta-1) - \beta(1+\alpha))|a_n| + \sum_{n=1}^{\infty} (2\beta(n+\beta-1) + \beta(1+\alpha))|b_n| \leq 1, \tag{11}$$

is satisfied, then $f \in H(\alpha, \beta)$.

Corollary 3. *Let $f^{\beta} = h^{\beta} + \overline{g^{\beta}}$ be such that h^{β} and g^{β} are given by* (3)*. Furthermore, let $1 \leq \beta \leq \frac{1}{1-\alpha}$. If the condition*

$$\sum_{n=2}^{\infty} (2(n+\beta-1) - (1+\alpha))|a_n| + \sum_{n=1}^{\infty} (2(n+\beta-1) + (1+\alpha))|b_n| \leq (1-\alpha), \tag{12}$$

is satisfied, then $f \in H(\alpha, \beta)$

In the following theorem, it is shown that the condition (11) and (12) is also necessary for functions $f^{\beta} = h^{\beta} + \overline{g^{\beta}}$ *be such that h^{β} and g^{β} are given by* (7), belong to the class $\overline{H}(\alpha, \beta)$.

Theorem 4. *Let $f^{\beta} = h^{\beta} + \overline{g^{\beta}}$ be such that h^{β} and g^{β} are given by* (7)*. Furthermore,*

(i) if *$1 \leq \beta \leq \frac{1}{1-\alpha}$, then $f \in \overline{H}(\alpha, \beta)$ if and only if*

$$\sum_{n=2}^{\infty} \left(\tfrac{2(n+\beta-1)-(1+\alpha)}{(1-\alpha)}\right)|a_n| + \sum_{n=1}^{\infty} \left(\tfrac{2(n+\beta-1)+(1+\alpha)}{(1-\alpha)}\right)|b_n| \leq 1, \tag{13}$$

(ii) if $\beta(1-\alpha) \geq 1$, *then $f \in \overline{H}(\alpha, \beta)$ if and only if*

$$\sum_{n=2}^{\infty}\left(2\beta(n+\beta-1)-\beta\left(1+\alpha\right)\right)\left|a_n\right|+\sum_{n=1}^{\infty}\left(2\beta(n+\beta-1)+\beta\left(1+\alpha\right)\right)\left|b_n\right|\leq 1, \tag{14}$$

Proof. In view of Corollary 1 and Corollary 2, it is suffices to show that $f\in\overline{H}(\alpha,\beta)$ if the condition (13) and (14) does hold. We note that a necessary and sufficient condition for functions $f^{\beta}=h^{\beta}+\overline{g^{\beta}}$ *be such that* h^{β} *and* g^{β} *are given by* (7), to be in $\overline{H}(\alpha,\beta)$, is that the condition (5) to be satisfied. Equivalent we must have

$$\Re\left\{\frac{\left(1+e^{i\theta}\right)\left(h'(z)^{\beta}+\overline{g'(z)^{\beta}}\right)-\left(\alpha\beta+\beta e^{i\theta}\right)\left(h(z)^{\beta}+\overline{g(z)^{\beta}}\right)}{h(z)^{\beta}+\overline{g(z)^{\beta}}}\right\}\geq 0,$$

which implies that

$$\Re\left\{\frac{\beta\left(1-\alpha\right)z^{\beta}-\sum\limits_{n=2}^{\infty}\beta\left[(n+\beta-1)\left(1+e^{i\theta}\right)-\left(\alpha+e^{i\theta}\right)\right]\left|a_n\right|z^{n+\beta-1}}{z^{\beta}-\sum_{n=2}^{\infty}\beta\left|a_n\right|z^{n}+\sum_{n=1}^{\infty}\beta\left|b_n\right|\overline{z}^{n+\beta-1}}\right.$$

$$\left.-\frac{\sum\limits_{n=1}^{\infty}\beta\left[(n+\beta-1)\left(1+e^{i\theta}\right)+\left(\alpha+e^{i\theta}\right)\right]\left|b_n\right|z^{n+\beta-1}}{z^{\beta}-\sum_{n=2}^{\infty}\beta\left|a_n\right|z^{n+\beta-1}+\sum_{n=1}^{\infty}\beta\left|b_n\right|\overline{z}^{n+\beta-1}}\right\}$$

$$=\Re\left\{\frac{(1-\alpha)-\sum\limits_{n=2}^{\infty}\left[\left(1+e^{i\theta}\right)(n+\beta-1)-\left(\alpha+e^{i\theta}\right)\right]\left|a_n\right|z^{n-1}}{1-\sum_{n=2}^{\infty}\beta\left|a_n\right|z^{n-1}+\left(\frac{\overline{z}}{z}\right)^{\beta}\sum_{n=1}^{\infty}\beta\left|b_n\right|\overline{z}^{n-1}}\right.$$

$$\left.-\frac{\left(\frac{\overline{z}}{z}\right)^{\beta}\sum\limits_{n=1}^{\infty}\left[\left(1+e^{i\theta}\right)(n+\beta-1)+\left(\alpha+e^{i\theta}\right)\right]\left|b_n\right|\overline{z}^{n-1}}{1-\sum_{n=2}^{\infty}\beta\left|a_n\right|z^{n}+\left(\frac{\overline{z}}{z}\right)^{\beta}\sum_{n=1}^{\infty}\beta\left|b_n\right|\overline{z}^{n-1}}\right\}>0. \tag{15}$$

Upon choosing the value of z on the positive real axis and using $\Re\left(e^{i\theta}\right)\leq\left|e^{i\theta}\right|=1$, the required condition that (15) is equivalent to

$$\left\{\frac{1-\sum\limits_{n=2}^{\infty}\frac{2(n+\beta-1)-(1+\alpha)}{(1-\alpha)}\left|a_n\right|r^{n-1}}{1-\sum_{n=2}^{\infty}\beta\left|a_n\right|r^{n-1}+\sum_{n=1}^{\infty}\beta\left|b_n\right|r^{n-1}}\right.$$

$$
\left.-\frac{\sum\limits_{n=1}^{\infty}\dfrac{2(n+\beta-1)+(1+\alpha)}{(1-\alpha)}\left|b_n\right|r^{n-1}}{1-\sum_{n=2}^{\infty}\beta\left|a_n\right|r^{n-1}+\sum_{n=1}^{\infty}\beta\left|b_n\right|r^{n-1}}\right\}\geq 0. \tag{16}
$$

If the condition (15) does not hold, then the numerator in (16) is negative for $z=r$ sufficiently close to 1^{-}. Hence there exists $z_0=r_0$ in $(0,1)$ for which the quotient in (16) is negative. This contradicts the required condition for $f\in\overline{H}(\alpha,\beta)$, and so the proof of Theorem 2 is completed.

Theorem 5. *Let $f\in\overline{H}(\alpha,\beta)$. Then for $|z|=r<1$ we have*

$$
|f(z)|\leq\begin{cases}(1+|b_1|)r^{\beta}+\left\{\frac{(1-\alpha)}{2\beta+1-\alpha}-\dfrac{2\beta+1+\alpha}{2\beta+1-\alpha}\left|b_1\right|\right\}r^{1+\beta}, \\ \qquad \beta(1-\alpha)\leq 1 \text{ and } |b_1|\leq\frac{(1-\alpha)}{2\beta+1+\alpha} \\ (1+|b_1|)r^{\beta}+\left\{\frac{1}{\beta(2\beta+1-\alpha)}-\dfrac{2\beta+1+\alpha}{2\beta+1-\alpha}\left|b_1\right|\right\}r^{1+\beta}, \\ \qquad \beta(1-\alpha)\geq 1 \text{ and } |b_1|\leq\frac{1}{\beta(2\beta+1+\alpha)}\end{cases}
$$

and

$$
|f(z)|\geq\begin{cases}(1-|b_1|)r^{\beta}-\left\{\frac{(1-\alpha)}{2\beta+1-\alpha}-\dfrac{2\beta+1+\alpha}{2\beta+1-\alpha}\left|b_1\right|\right\}r^{1+\beta}, \\ \qquad \beta(1-\alpha)\leq 1 \text{ and } |b_1|\leq\frac{(1-\alpha)}{2\beta+1+\alpha} \\ (1-|b_1|)r^{\beta}-\left\{\frac{1}{\beta(2\beta+1-\alpha)}-\dfrac{2\beta+1+\alpha}{2\beta+1-\alpha}\left|b_1\right|\right\}r^{1+\beta}, \\ \qquad \beta(1-\alpha)\geq 1 \text{ and } |b_1|\leq\frac{1}{\beta(2\beta+1+\alpha)}\end{cases}.
$$

The results are sharp.

Proof. We prove the first left hand side inequality for $|f|$. The proof for the rest inequalities can be done by using similar arguments. Let $f\in\overline{H}(\alpha,\beta)$, then we have

$$
\left|f(z)^{\beta}\right|=\left|z^{\beta}-\sum_{n=2}^{\infty}\beta\left|a_n\right|z^{n+\beta-1}+\sum_{n=1}^{\infty}\beta\left|b_n\right|z^{n+\beta-1}\right|
$$

$$
\geq r^{\beta}-\left|b_1\right|r^{\beta}-\sum_{n=2}^{\infty}\left(\left|a_n\right|+\left|b_n\right|\right)r^{1+\beta}
$$

$$
\geq r^{\beta}-\left|b_1\right|r^{\beta}-
$$

$$
\frac{(1-\alpha)}{2\beta+1-\alpha}\sum_{n=2}^{\infty}\frac{2(n+\beta-1)-(1+\alpha)}{(1-\alpha)}(|a_n|+|b_n|)r^{1+\beta}
$$

$$\geq r^{\beta} - |b_1|\, r^{\beta} -$$

$$\frac{(1-\alpha)}{2\beta+1-\alpha} \sum_{n=2}^{\infty} \left\{ \frac{2(n+\beta-1)-(1+\alpha)}{(1-\alpha)} |a_n| + \right.$$

$$\left. \frac{(2(n+\beta-1)+(1+\alpha))}{(1-\alpha)} |b_n| \right\} r^{1+\beta}$$

$$\geq (1-|b_1|) r^{\beta} - \frac{(1-\alpha)}{2\beta+1-\alpha} \left\{ 1 - \tfrac{2\beta+1+\alpha}{(1-\alpha)} |b_1| \right\} r^{1+\beta}$$

$$\geq (1-|b_1|) r^{\beta} - \left\{ \frac{(1-\alpha)}{2\beta+1-\alpha} - \frac{2\beta+1+\alpha}{2\beta+1-\alpha} |b_1| \right\} r^{1+\beta}.$$

The upper and lower bounds given in Theorem 3 are respectively attained for the following functions:

$$f(z) = \begin{cases} z^{\beta} - |b_1|\, \overline{z}^{\beta} - \left(\tfrac{(1-\alpha)}{2\beta+1-\alpha} - \dfrac{2\beta+1+\alpha}{2\beta+1-\alpha} |b_1| \right) \overline{z}^{1+\beta} \\ (1-|b_1|) z^{\beta} - \left\{ \tfrac{1}{\beta(2\beta+1-\alpha)} - \dfrac{2\beta+1+\alpha}{2\beta+1-\alpha} |b_1| \right\} z^{1+\beta} \end{cases}$$

and $f(z) = \begin{cases} (1-|b_1|)\, \overline{z}^{\beta} - \left(\tfrac{(1-\alpha)}{2\beta+1-\alpha} - \dfrac{2\beta+1+\alpha}{2\beta+1-\alpha} |b_1| \right) \overline{z}^{1+\beta} \\ (1-|b_1|) z^{\beta} - \left\{ \tfrac{1}{\beta(2\beta+1-\alpha)} - \dfrac{2\beta+1+\alpha}{2\beta+1-\alpha} |b_1| \right\} z^{1+\beta} \end{cases}.$

The following covering result follows from the left side inequality in Theorem 3.

Corollary 6. *Let $f \in \overline{H}(\alpha,\beta)$, then the set*

$$\left\{ w : |w| < \begin{cases} 1 - \tfrac{(1-\alpha)}{2\beta+1-\alpha} + \left\{ \dfrac{2\alpha}{2\beta+1-\alpha} \right\} |b_1|, \\ \beta(1-\alpha) \leq 1 \text{ and } |b_1| \leq \tfrac{(1-\alpha)}{2\beta+1+\alpha} \\ 1 - \tfrac{1}{\beta(2\beta+1-\alpha)} + \left\{ \dfrac{2\alpha}{2\beta+1-\alpha} \right\} |b_1|, \\ \beta(1-\alpha) \geq 1 \text{ and } |b_1| \leq \tfrac{1}{\beta(2\beta+1+\alpha)} \end{cases} \right\}$$

is included in $f(U)$.

3. Extreme points

Our next theorem is on the extreme points of convex hulls of the class $\overline{H}(\alpha,\beta)$, denoted by $clco\overline{H}(\alpha,\beta)$.

Theorem 7. *Let $f^{\beta}=h^{\beta}+\overline{g^{\beta}}$ be such that h^{β} and g^{β} are given by (3). Then $f\in clco\overline{H}(\alpha,\beta)$ if and only if f can be expressed as*

$$f(z)^{\beta}=\sum_{n=1}^{\infty}\left[X_n h_n(z)^{\beta}+Y_n g_n(z)^{\beta}\right], \tag{17}$$

where

$$\begin{aligned} h_1(z)^{\beta} &= z^{\beta}, \\ h_n(z)^{\beta} &= \begin{cases} z^{\beta}-\frac{(1-\alpha)}{2(n+\beta-1)-(1+\alpha)}z^{n+\beta-1}, \beta(1-\alpha)\leq 1 \\ z^{\beta}-\frac{1}{2\beta(n+\beta-1)-\beta(1+\alpha)}z^{n+\beta-1}, \beta(1-\alpha)\geq 1 \end{cases} \end{aligned}$$

$$g_n(z)^{\beta}=\begin{cases} z^{\beta}+\frac{(1-\alpha)}{2(n+\beta-1)+(1+\alpha)}\overline{z}^{n+\beta-1}, \beta(1-\alpha)\leq 1 \\ z^{\beta}+\frac{1}{2\beta(n+\beta-1)+\beta(1+\alpha)}\overline{z}^{n+\beta-1}, \beta(1-\alpha)\geq 1 \end{cases}$$

$$X_n\geq 0, Y_n\geq 0, \quad \sum_{n=1}^{\infty}[X_n+Y_n]=1.$$

In particular, the extreme points of the class $\overline{H}(\alpha,\beta)$ are $\left\{h_n^{\beta}\right\}$ and $\left\{g_n^{\beta}\right\}$ respectively.

Proof. For functions $f(z)^{\beta}$ of the form (17), for $\beta(1-\alpha)\leq 1$, we have

$$\begin{aligned} f(z)^{\beta} &= \sum_{n=1}^{\infty}[X_n+Y_n]\,z^{\beta}-\sum_{n=2}^{\infty}\frac{(1-\alpha)}{2(n+\beta-1)-(1+\alpha)}X_n z^{n+\beta-1} \\ &\quad +\sum_{n=1}^{\infty}\frac{(1-\alpha)}{2(n+\beta-1)+(1+\alpha)}Y_n\overline{z}^{n+\beta-1}. \end{aligned}$$

Then

$$\begin{aligned} &\sum_{n=2}^{\infty}\frac{2(n+\beta-1)-(1+\alpha)}{(1-\alpha)}\left(\frac{(1-\alpha)}{2(n+\beta-1)-(1+\alpha)}\right)X_n \\ &+\sum_{n=1}^{\infty}\frac{2(n+\beta-1)+(1+\alpha)}{(1-\alpha)}\left(\frac{(1-\alpha)}{2(n+\beta-1)+(1+\alpha)}\right)Y_n \\ &=\sum_{n=2}^{\infty}X_n+\sum_{n=1}^{\infty}Y_n=1-X_1\leq 1, \end{aligned}$$

and so $f(z) \in clco\overline{H}(\alpha,\beta)$. Conversely, suppose that $f(z) \in clco\overline{H}(\alpha,\beta)$. Set

$$X_n = \frac{2(n+\beta-1)-(1+\alpha)}{(1-\alpha)}\,|a_n| \;\; (n \geq 1)$$

and

$$Y_n = \frac{2(n+\beta-1)+(1+\alpha)}{(1-\alpha)}\,|b_n| \;\; (n \geq 1)\,,$$

then note that by Theorem 2, $0 \leq X_n \leq 1 \;\; (n \geq 1)$ and $0 \leq Y_n \leq 1 \;\; (n \geq 1)$. Consequently, we obtain

$$f(z)^{\beta} = \sum_{n=1}^{\infty} \left[X_n h_n(z)^{\beta} + Y_n g_n(z)^{\beta}\right].$$

Using Theorem 2 it is easily seen that the class $\overline{H}(\alpha,\beta)$ is convex and closed and so $clco\overline{H}(\alpha,\beta) = \overline{H}(\alpha,\beta)$.

4. Convolution and convex combinations result

For harmonic functions of the form:

$$f(z) = z^{\beta} - \sum_{n=2}^{\infty} |a_n|\, z^{n+\beta-1} + \sum_{n=1}^{\infty} |b_n|\, \overline{z}^{n+\beta-1} \tag{18}$$

and

$$G(z) = z^{\beta} - \sum_{n=2}^{\infty} A_n z^{n+\beta-1} + \sum_{n=1}^{\infty} B_n \overline{z}^{n+\beta-1} \;\; (A_n, B_n \geq 0), \tag{19}$$

we define the convolution of two harmonic functions f and G as

$$(f * G)(z) = f(z) * G(z)$$

$$= z^{\beta} - \sum_{n=2}^{\infty} a_n A_n z^{n+\beta-1} + \sum_{n=1}^{\infty} b_n B_n \overline{z}^{n+\beta-1}.$$

Using this definition, we show that the class $\overline{H}(\alpha,\beta)$ is closed under convolution.

Theorem 8. *For $0 \leq \alpha < 1$, let $f(z) \in \overline{H}(\alpha,\beta)$ and $G(z) \in \overline{H}(\alpha,\beta)$. Then $f(z) * G(z) \in \overline{H}(\alpha,\beta)$.*

Proof. Let the functions $f(z)$ defined by (18) be in the class $\overline{H}(\alpha,\beta)$ and let the functions $G(z)$ defined by (19) be in the class $\overline{H}(\alpha,\beta)$. Obviously, the coefficients of f and G must satisfy condition similar to the inequality (8) and $A_n \leq 1, B_n \leq 1$. So for the coefficients of $f * G$ we can write

$$\begin{aligned}
&\sum_{n=2}^{\infty}\left(\tfrac{2\beta(n+\beta-1)-\beta(1+\alpha)}{\beta(1-\alpha)+1)-|\beta(1-\alpha)-1|}\right)|a_n|\,A_n+\sum_{n=1}^{\infty}\left(\tfrac{2\beta(n+\beta-1)+\beta(1+\alpha)}{\beta(1-\alpha)+1)-|\beta(1-\alpha)-1|}\right)|b_n|\,B_n \\
\leq\ &\sum_{n=2}^{\infty}\left[\left(\tfrac{2\beta(n+\beta-1)-\beta(1+\alpha)}{\beta(1-\alpha)+1)-|\beta(1-\alpha)-1|}\right)|a_n|+\sum_{n=1}^{\infty}\left(\tfrac{2\beta(n+\beta-1)+\beta(1+\alpha)}{\beta(1-\alpha)+1)-|\beta(1-\alpha)-1|}\right)|b_n|\right],
\end{aligned}$$

the right hand side of this inequality is bounded by $\frac{1}{2}$ because $f \in \overline{H}(\alpha,\beta)$.

Then, $f(z) * G(z) \in \overline{H}(\alpha,\beta)$.

Finally, we show that $\overline{H}(\alpha,\beta)$ is closed under convex combinations of its members.

Theorem 9. *The class $\overline{H}(\alpha,\beta)$ is closed under convex linear combination.*

Proof. For $i = 1,2,3,....$, let $f_i \in \overline{H}(\alpha,\beta)$, where the functions f_i^{β} are given by

$$f_i^{\beta}(z) = z^{\beta} - \sum_{n=2}^{\infty}|a_{n,i}|\,z^{n+\beta-1} + \sum_{n=1}^{\infty}|b_{n,i}|\,z^{n+\beta-1}.$$

For $\sum\limits_{i=1}^{\infty} t_i = 1; 0 \leq t_i \leq 1$, the convex linear combination of f_i may be written as

$$\sum_{i=1}^{\infty} t_i f_i(z)^{\beta} = z^{\beta} - \sum_{n=2}^{\infty}\left(\sum_{i=1}^{\infty} t_i\,|a_{n,i}|\right)z^{n+\beta-1} + \sum_{n=1}^{\infty}\left(\sum_{i=1}^{\infty} t_i\,|b_{n,i}|\right)z^{n+\beta-1},$$

Since

$$\begin{aligned}
&\sum_{n=2}^{\infty}\left[(2\beta(n+\beta-1)-\beta\,(1+\alpha))\,|a_{n,i}|+\sum_{n=1}^{\infty}(2\beta(n+\beta-1)+\beta\,(1+\alpha))\,|b_{n,i}|\right] \\
\leq\ &\begin{cases}\beta(1-\alpha), & \beta(1-\alpha)\leq 1\\ 1, & \beta(1-\alpha)\geq 1\end{cases}
\end{aligned}$$

it follows from the above equation

$$\begin{aligned} & \sum_{n=2}^{\infty} (2\beta(n+\beta-1) - \beta(1+\alpha)) \sum_{i=1}^{\infty} t_i \left|a_{n,i}\right| + \sum_{n=1}^{\infty} (2\beta(n+\beta-1) + \beta(1+\alpha)) \sum_{i=1}^{\infty} t_i \left|b_{n,i}\right| \\ = & \sum_{i=1}^{\infty} t_i \left\{ \sum_{n=2}^{\infty} \left[(2\beta(n+\beta-1) - \beta(1+\alpha)) \left|a_{n,i}\right| + \sum_{n=1}^{\infty} (2\beta(n+\beta-1) + \beta(1+\alpha)) \left|b_{n,i}\right| \right] \right\} \\ \leq & \begin{cases} \beta(1-\alpha) \sum\limits_{i=1}^{\infty} t_i = \beta(1-\alpha), \beta(1-\alpha) \leq 1 \\ \sum\limits_{i=1}^{\infty} t_i = 1, \qquad \beta(1-\alpha) \geq 1. \end{cases} \end{aligned}$$

This conditions required by (13), (14) and so $\sum\limits_{i=1}^{\infty} t_i f_i(z) \in \overline{H}(\alpha, \beta)$. This complete the proof of Theorem 6.

Remark 1. *(i) Putting $\beta = 1$ in our results we obtain the results obtained by Rosy et al. [5].*

References

[1] Khalifa Al-Shaqsi, Maslina Darus and Olubunmi Abidemi Fadipe-Joseph, *A New subclass of Salagean-Type harmonic univalent functions*, Abstract Appl. Anal., (2010), Art. ID 821531, 1-12.

[2] J. Clunie and T. Sheil-Small, *Harmonic univalent functions*, Ann. Acad. Sci. Fenn. Ser. A. I. Math., 9(1984), 3-25.

[3] J. M. Jahangiri, *Coefficient bounds and univalent criteria for harmonic functions with negative coefficients*, Ann. Univ. Marie-Curie Sklodowska Sect. A, 52(1998), 57-66.

[4] J. M. Jahangiri, *Harmonic functions starlike in the unit disc*, J. Math. Anal. Appl., 235(1999), 470-477.

[5] T. Rosy, B. Adolf Stephen, K.G. Subramanian and J. M. Jahangiri, *Goodman-Ronning-type harmonic univalent functions*, Kyungpook Math. J., 41(2001), 45-54.

[6] H. Silverman, *Harmonic univalent function with negative coefficients*, J. Math. Anal. Appl., 220(1998), 283-289.

[7] H. Silverman and E. M. Silvia, *Subclasses of harmonic univalent functions*, New Zealand J. Math., 28(1999), 275-284.

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