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A STUDY ON GENERAL CLASS OF MEROMORPHICALLY UNIVALENT FUNCTIONS

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Abstract. In this paper, we define the general class $V_n(\beta, \alpha, \gamma)$ of certain subclasses of meromorphically univalent functions and we derive distortion theorems and modified-Hadamard products for functions belonging to the class.

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1. Introduction

Let Σ_n denote the class of meromorphic functions of the form:

$$f(z) = \frac{1}{z} + \sum_{k=n}^{\infty} a_k z^k \quad \left(a_k \geq 0; n \in \mathbb{N} = \{1, 2, ...\}\right), \tag{1.1}$$

which are regular and univalent in the punctured unit disc $U^* = \{z : z \in \mathbb{C}$ and $0 < |z| < 1\} = U \backslash \{0\}$. Let $g \in \Sigma_n$, be given by

$$g(z) = \frac{1}{z} + \sum_{k=n}^{\infty} b_k z^k, \tag{1.2}$$

then the Hadamard product (or convolution) of f and g is given by

$$(f * g)(z) = \frac{1}{z} + \sum_{k=n}^{\infty} a_k b_k z^k = (g * f)(z). \tag{1.3}$$

A function $f \in \Sigma_n$ is said to be meromorphically starlike of order α if

$$\text{Re}\left\{-\frac{zf'(z)}{f(z)}\right\} > \alpha \quad (z \in U^*;\ 0 \leq \alpha < 1). \tag{1.4}$$

The class of all meromorphically starlike functions of order α is denoted by $\Sigma S_n^*(\alpha)$. A function $f \in \Sigma_n$ is said to be meromorphically convex of order α if

$$\operatorname{Re}\left\{-(1+\frac{zf''(z)}{f'(z)})\right\} > \alpha \ (z \in U^*;\ (0 \le \alpha < 1)). \tag{1.5}$$

The class of all meromorphically convex functions of order α is denoted by $\Sigma K_n(\alpha)$. We note that

$$f(z) \in \Sigma K_n(\alpha) \Longleftrightarrow -zf'(z) \in \Sigma S_n^*(\alpha).$$

The classes $\Sigma S_n^*(\alpha)$ and $\Sigma K_n(\alpha)$ were introduced by Owa et al.[4]. Various subclasses of the class Σ_n when $n = 1$ were considered earlier by Pommerenke [5], Miller [3] and others.

For $\beta \ge 0,\ 0 \le \alpha < 1,\ 0 \le \lambda < \frac{1}{2}$ and g given by (1.2) with $b_k \ge 0$ $(k \ge n)$, Aouf et al. [2] defined the class $M(f, g; \beta, \alpha, \lambda)$ consisting of functions of the form (1.1) and satisfying the analytic criterion:

$$-\operatorname{Re}\left\{\frac{z(f*g)'(z)+\lambda z^2(f*g)''(z)}{(1-\lambda)(f*g)(z)+\lambda z(f*g)'(z)}+\alpha\right\} \ge \beta\left|\frac{z(f*g)'(z)+\lambda z^2(f*g)''(z)}{(1-\lambda)(f*g)(z)+\lambda z(f*g)'(z)}+1\right| (z \in U). \tag{1.6}$$

When we take $g(z) = \frac{1}{z(1-z)}$, in (1.6), we obtain the class $\Sigma_n(\beta, \alpha, \lambda)(\beta \ge 0, 0 \le \alpha < 1$ and $0 \le \lambda < \frac{1}{2})$, which consisting of functions of the form (1.1) and satisfying the analytic criterion:

$$-\operatorname{Re}\left\{\frac{zf'(z)+\lambda z^2 f''(z)}{(1-\lambda)f(z)+\lambda zf'(z)}+\alpha\right\} \ge \beta\left|\frac{zf'(z)+\lambda z^2 f''(z)}{(1-\lambda)f(z)+\lambda zf'(z)}+1\right| (z \in U). \tag{1.7}$$

We note that:

$\Sigma_1(0, \alpha, 0) = \Sigma^*(\alpha)$ $(0 \le \alpha < 1)$ (see Pommerenke [5]).

Also, we note that:

$\Sigma_n(\beta, \alpha, 0) = \Sigma S_n^*(\beta, \alpha) =$

$$-\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}+\alpha\right\} \ge \beta\left|\frac{zf'(z)}{f(z)}+1\right| (z \in U). \tag{1.8}$$

For $\beta \ge 0$ and $0 \le \alpha < 1$, we denote by $\Sigma K_n(\beta, \alpha)$ the subclass of Σ_n consisting of functions of the form (1.1) and satisfying the analytic criterion:

$$-\operatorname{Re}\left\{1+\frac{zf''(z)}{f'(z)}+\alpha\right\} \ge \beta\left|2+\frac{zf''(z)}{f'(z)}\right| \ (z \in U). \tag{1.9}$$

We note that

$\Sigma K_1(0,\alpha,1) = \Sigma_k^*(\alpha)$ $(0 \le \alpha < 1)$ (see Pommerenke [5]).

From (1.8) and (1.9) we have

$$f(z) \in \Sigma K_n(\beta,\alpha) \Longleftrightarrow -zf'(z) \in \Sigma S_n^*(\beta,\alpha). \tag{1.10}$$

2. General Classes Associated with Coefficient Bounds

In order to prove our results for functions belonging to the class $\Sigma_n(\beta,\alpha,\lambda)$, we shall need the following lemma given by Aouf et al. $\left[2, \text{with } g = \frac{1}{z(1-z)}\right]$.

Lemma 1. [2, *Theorem 1*]. *Let the function f be defined by (1.1). Then $f \in \Sigma_n(\beta,\alpha,\lambda)$ if and only if*

$$\sum_{k=n}^{\infty} [1+\lambda(k-1)][k(1+\beta)+(\beta+\alpha)]\, a_k \le (1-\alpha)(1-2\lambda). \tag{2.1}$$

Taking $\lambda = 0$ in Lemma 1, we obtain the following corollary.

Corollary 2. *Let the function f defined by (1.1). Then $f \in \Sigma S_n^*(\beta,\alpha)$ if and only if*

$$\sum_{k=n}^{\infty} [k(1+\beta)+(\beta+\alpha)]\, a_k \le (1-\alpha) \qquad (n \in \mathbb{N}). \tag{2.2}$$

By using Corollary 1 and (1.10), we can prove the following lemma.

Lemma 3. *Let the function f defined by (1.1). Then $f \in \Sigma K_n(\beta,\alpha)$ if and only if*

$$\sum_{k=n}^{\infty} k\,[k(1+\beta)+(\beta+\alpha)]\, a_k \le (1-\alpha) \quad (n \in \mathbb{N}). \tag{2.3}$$

Definition 1. *A function f defined by* (1.1) *and belonging to the class Σ_n is said to be in the class $V_n(\beta,\alpha,\gamma)$ if it also satisfies the coefficient inequality:*

$$\sum_{k=n}^{\infty} [k(1+\beta)+(\beta+\alpha)](1-\gamma+\gamma k)\, a_k \le (1-\alpha) \;\; (n \in \mathbb{N}; \gamma \ge 0;\; \beta \ge 0;\; 0 \le \alpha < 1). \tag{2.4}$$

It is easily to observe that

$$V_n(\beta,\alpha,0) = \Sigma S_n^*(\beta,\alpha) \text{ and } V_n(\beta,\alpha,1) = \Sigma K_n(\beta,\alpha). \tag{2.5}$$

3. Growth and Distortion Theorems

Unless otherwise mentioned, we assume in the reminder of this paper that $\gamma \geq 0,\ \beta \geq 0,\ 0 \leq \alpha < 1$ and $n \in \mathbb{N}$.

Theorem 4. *If a functions f defined by* (1.1) *is in the class $V_n(\beta,\alpha,\gamma)$, then*

$$\frac{1}{|z|} - \frac{(1-\alpha)}{[n(1+\beta)+(\beta+\alpha)](1-\gamma+\gamma n)}|z|^n \leq |f(z)|$$
$$\leq \frac{1}{|z|} + \frac{(1-\alpha)}{[n(1+\beta)+(\beta+\alpha)](1-\gamma+\gamma n)}|z|^n \ (z \in U^*), \tag{3.1}$$

and

$$\frac{1}{|z|^2} - \frac{n(1-\alpha)}{[n(1+\beta)+(\beta+\alpha)](1-\gamma+\gamma n)}|z|^{n-1} \leq \left|f'(z)\right|$$
$$\leq \frac{1}{|z|^2} + \frac{n(1-\alpha)}{[n(1+\beta)+(\beta+\alpha)](1-\gamma+\gamma n)}|z|^{n-1} \ (z \in U^*). \tag{3.2}$$

The bounds in (3.1) *and* (3.2) *are attained for the function f given by*

$$f(z) = \frac{1}{z} + \frac{(1-\alpha)}{[n(1+\beta)+(\beta+\alpha)](1-\gamma+\gamma n)}z^n. \tag{3.3}$$

Proof. First of all, for $f \in V_n(\beta,\alpha,\gamma)$, it follows from (2.4) that

$$\sum_{k=n}^{\infty} a_k \leq \frac{(1-\alpha)}{[n(1+\beta)+(\beta+\alpha)](1-\gamma+\gamma n)}, \tag{3.4}$$

which, in view of (1.1), yields

$$|f(z)| \geq \frac{1}{|z|} - |z|^n \sum_{k=n}^{\infty} a_k \tag{3.5}$$
$$\geq \frac{1}{|z|} - \frac{(1-\alpha)}{[n(1+\beta)+(\beta+\alpha)](1-\gamma+\gamma n)}|z|^n \ (z \in U^*),$$

and

$$|f(z)| \leq \frac{1}{|z|} + |z|^n \sum_{k=n}^{\infty} a_k \tag{3.6}$$
$$\leq \frac{1}{|z|} + \frac{(1-\alpha)}{[n(1+\beta)+(\beta+\alpha)](1-\gamma+\gamma n)}|z|^n \ (z \in U^*).$$

Next, we see from (2.4) that

$$\frac{[n(1+\beta)+(\beta+\alpha)]\,(1-\gamma+\gamma n)}{n}\sum_{k=n}^{\infty} ka_k \tag{3.7}$$
$$\leq \sum_{k=n}^{\infty}[k(1+\beta)+(\beta+\alpha)]\,(1-\gamma+\gamma k)\,a_k$$
$$\leq (1-\alpha)\,,$$

then

$$\sum_{k=n}^{\infty} ka_k \leq \frac{n\,(1-\alpha)}{[n(1+\beta)+(\beta+\alpha)]\,(1-\gamma+\gamma n)}.$$

which, again in view of (1.1) , yields

$$\left|f^{'}(z)\right| \geq \frac{1}{|z|^2} - |z|^{n-1}\sum_{k=n}^{\infty} ka_k \tag{3.8}$$
$$\geq \frac{1}{|z|^2} - \frac{n\,(1-\alpha)}{[n(1+\beta)+(\beta+\alpha)]\,(1-\gamma+\gamma n)}\,|z|^{n-1}\,(z\in U^*)\,,$$

and

$$\left|f^{'}(z)\right| \leq \frac{1}{|z|^2} + |z|^{n-1}\sum_{k=n}^{\infty} ka_k \tag{3.9}$$
$$\leq \frac{1}{|z|^2} + \frac{n\,(1-\alpha)}{[n(1+\beta)+(\beta+\alpha)]\,(1-\gamma+\gamma n)}\,|z|^{n-1}\,(z\in U^*)\,.$$

Finally, it is easy to see that the bounds in (3.1) and (3.2) are attained for the function f given by (3.3) .

Taking $\gamma = 0$ in Theorem 1, and making use of the first relationship in (2.5) , we obtain the following corollary.

Corollary 5. *If a functions f defined by* (1.1) *is in the class $\Sigma S_n^*(\beta,\alpha)$, then*

$$\frac{1}{|z|} - \frac{(1-\alpha)}{[n(1+\beta)+(\beta+\alpha)]}\,|z|^n \leq |f(z)|$$

$$\leq \frac{1}{|z|} + \frac{(1-\alpha)}{[n(1+\beta)+(\beta+\alpha)]}\,|z|^n\,(z\in U^*)\,, \tag{3.10}$$

and

$$\frac{1}{|z|^2}-\frac{n(1-\alpha)}{[n(1+\beta)+(\beta+\alpha)]}|z|^{n-1}\le\left|f'(z)\right|$$
$$\le\frac{1}{|z|^2}+\frac{n(1-\alpha)}{[n(1+\beta)+(\beta+\alpha)]}|z|^{n-1}\,(z\in U^*)\,. \tag{3.11}$$

The bounds in (3.10) *and* (3.11) *are attained for the function f given by*

$$f(z)=\frac{1}{z}+\frac{(1-\alpha)}{[n(1+\beta)+(\beta+\alpha)]}z^n. \tag{3.12}$$

Letting $\gamma=1$ in Theorem 1, and applying the second relationship in (2.5), we obtain the following corollary.

Corollary 6. *If a functions f defined by* (1.1) *is in the class $\Sigma K_n(\beta,\alpha)$, then*

$$\frac{1}{|z|}-\frac{(1-\alpha)}{n\,[n(1+\beta)+(\beta+\alpha)]}|z|^n\le|f(z)|$$
$$\le\frac{1}{|z|}+\frac{(1-\alpha)}{n\,[n(1+\beta)+(\beta+\alpha)]}|z|^n\,(z\in U^*)\,, \tag{3.13}$$

and

$$\frac{1}{|z|^2}-\frac{(1-\alpha)}{[n(1+\beta)+(\beta+\alpha)]}|z|^{n-1}\le\left|f'(z)\right|$$
$$\le\frac{1}{|z|^2}+\frac{(1-\alpha)}{[n(1+\beta)+(\beta+\alpha)]}|z|^{n-1}\,(z\in U^*)\,. \tag{3.14}$$

The bounds in (3.13) *and* (3.14) *are attained for the function f given by*

$$f(z)=\frac{1}{z}+\frac{(1-\alpha)}{n\,[n(1+\beta)+(\beta+\alpha)]}z^n. \tag{3.15}$$

4. Modified Hadamard Product

Let each of the functions f_1 and f_1 defined by

$$f_j(z)=\frac{1}{z}+\sum_{k=n}^{\infty}a_{k,j}z^k \qquad (j=1,2) \tag{4.1}$$

belong to the class Σ_n. We denote by $(f_1 * f_2)$ the modified Hadamard product (or convolution) of the functions f_1 and f_2, that is,

$$(f_1 * f_2)(z) = \frac{1}{z} + \sum_{k=n}^{\infty} a_{k,1} a_{k,2} z^k. \tag{4.2}$$

Now we derive the following modified Hadamard product of the general class $V_n(\beta, \alpha, \gamma)$:

Theorem 7. *Let each of the functions $f_j\,(j = 1, 2)$ defined by* (4.1) *be in the class $V_n(\beta, \alpha, \gamma)$. Then*

$$(f_1 * f_2)(z) \in V_n(\beta, \eta, \gamma),$$

where

$$\eta = 1 - \frac{(1-\alpha)^2 (1+\beta)(n+1)}{[n(1+\beta) + (\beta+\alpha)]^2 (1-\gamma+\gamma n) + (1-\alpha)^2}. \tag{4.3}$$

The result is sharp for the functions $f_j\ (j = 1, 2)$ given by

$$f_j(z) = \frac{1}{z} + \frac{(1-\alpha)}{[n(1+\beta) + (\beta+\alpha)](1-\gamma+\gamma n)} z^n \quad (j = 1, 2). \tag{4.4}$$

Proof. In order to prove the main assertion of Theorem 2, we must find the largest η such that

$$\sum_{k=n}^{\infty} \frac{[k(1+\beta) + (\beta+\eta)](1-\gamma+\gamma k)}{(1-\eta)} a_{k,1} a_{k,2} \leq 1 \tag{4.5}$$

for $f_j \in V_n(\beta, \alpha, \gamma)\,(j = 1, 2)$. Indeed, since each of the functions $f_j\ (j = 1, 2)$ does belongs to the class $V_n(\beta, \alpha, \gamma)$, then

$$\sum_{k=n}^{\infty} \frac{[k(1+\beta) + (\beta+\alpha)](1-\gamma+\gamma k)}{(1-\alpha)} a_{k,j} \leq 1 \quad (j = 1, 2). \tag{4.6}$$

Now, by the Cauchy-Schwarz inequality, we find from (4.6) that

$$\sum_{k=n}^{\infty} \frac{[k(1+\beta) + (\beta+\alpha)](1-\gamma+\gamma k)}{(1-\alpha)} \sqrt{a_{k,1} a_{k,2}} \leq 1. \tag{4.7}$$

Equation (4.7) implies that we need only to show that

$$\frac{[k(1+\beta) + (\beta+\eta)]}{(1-\eta)} a_{k,1} a_{k,2} \leq \frac{[k(1+\beta) + (\beta+\alpha)]}{(1-\alpha)} \sqrt{a_{k,1} a_{k,2}} \qquad (k \geq n), \tag{4.8}$$

that is, that

$$\sqrt{a_{k,1}a_{k,2}} \leq \frac{[k(1+\beta)+(\beta+\alpha)]\,(1-\eta)}{[k(1+\beta)+(\beta+\eta)]\,(1-\alpha)} \quad (k \geq n)\,. \tag{4.9}$$

Hence, by the inequality (4.7) it is sufficient to prove that

$$\frac{(1-\alpha)}{[k(1+\beta)+(\beta+\alpha)]\,(1-\gamma+\gamma k)} \leq \frac{[k(1+\beta)+(\beta+\alpha)]\,(1-\eta)}{[k(1+\beta)+(\beta+\eta)]\,((1-\alpha))} \quad (k \geq n)\,. \tag{4.10}$$

It follows from (4.10) that

$$\eta \leq 1 - \frac{(1-\alpha)^2\,(1+\beta)\,(k+1)}{[k(1+\beta)+(\beta+\alpha)]^2\,(1-\gamma+\gamma k)+(1-\alpha)^2} \quad (k \geq n)\,. \tag{4.11}$$

Defining the function $\Phi\,(k)$ by

$$\Phi\,(k) = 1 - \frac{(1-\alpha)^2\,(1+\beta)\,(k+1)}{[k(1+\beta)+(\beta+\alpha)]^2\,(1-\gamma+\gamma k)+(1-\alpha)^2} \quad (k \geq n)\,, \tag{4.12}$$

we see that $\Phi\,(k)$ is an increasing function of $k\,(k \geq n)$. Therefore, we conclude from (4.11) that

$$\eta \leq \Phi\,(n) = 1 - \frac{(1-\alpha)^2\,(1+\beta)\,(n+1)}{[n(1+\beta)+(\beta+\alpha)]^2\,(1-\gamma+\gamma n)+(1-\alpha)^2}, \tag{4.13}$$

which completes the proof of the main assertion of Theorem 2.

Setting $\gamma = 0$ in Theorem 2, and making use of first relationship in (2.5), we obtain the following corollary.

Corollary 8. *Let each of the functions $f_j\,(j=1,2)$ defined by (4.1) be in the class $\Sigma S_n^*\,(\beta,\alpha)$. Then*

$$(f_1 * f_2)\,(z) \in \Sigma S_n^*\,(\beta,\mu)\,,$$

where

$$\mu = 1 - \frac{(1-\alpha)^2\,(1+\beta)\,(n+1)}{[n(1+\beta)+(\beta+\alpha)]^2+(1-\alpha)^2}. \tag{4.14}$$

The result is sharp for the functions $f_j\,(j=1,2)$ given by

$$f_j\,(z) = \frac{1}{z} + \frac{(1-\alpha)}{[n(1+\beta)+(\beta+\alpha)]}z^n \quad (j=1,2)\,. \tag{4.15}$$

Putting $\gamma = 1$ in Theorem 2, and applying the second relationship in (2.5), we obtain the following corollary.

Corollary 9. *Let each of the functions $f_j\,(j = 1, 2)$ defined by (4.1) be in the class $\Sigma K_n\,(\beta, \alpha)$. Then*

$$(f_1 * f_2)\,(z) \in \Sigma K_n\,(\beta, \nu)\,,$$

where

$$\nu = 1 - \frac{(1-\alpha)^2\,(1+\beta)\,(n+1)}{n\,[n(1+\beta)+(\beta+\alpha)]^2 + (1-\alpha)^2}. \tag{4.16}$$

The result is sharp for the functions $f_j\,(j = 1, 2)$ given by

$$f_j\,(z) = \frac{1}{z} + \frac{(1-\alpha)}{n\,[n(1+\beta)+(\beta+\alpha)]} z^n \quad (j = 1, 2)\,. \tag{4.17}$$

Theorem 10. *Let each of the functions $f_j\,(j = 1, 2)$ defined by (4.1) be in the class $V_n\,(\beta, \alpha, \gamma)$. Then the function $h(z)$ defined by*

$$h\,(z) = \frac{1}{z} + \sum_{k=n}^{\infty} \left(a_{k,1}^2 + a_{k,2}^2\right) z^k \tag{4.18}$$

belongs to the class $V_n\,(\beta, \xi, \gamma)$, where

$$\xi = 1 - \frac{2\,(1-\alpha)^2\,(1+\beta)\,(n+1)}{[n(1+\beta)+(\beta+\alpha)]^2\,(1-\gamma+\gamma n) + 2\,(1-\alpha)^2}. \tag{4.19}$$

The result is sharp for the functions $f_j\,(j = 1, 2)$ given by (4.4).

Proof. Noting that

$$\sum_{k=n}^{\infty} \frac{[k(1+\beta)+(\beta+\alpha)]^2\,(1-\gamma+\gamma k)^2}{(1-\alpha)^2} a_{k,j}^2 \tag{4.20}$$

$$\leq \left[\sum_{k=n}^{\infty} \frac{[k(1+\beta)+(\beta+\alpha)]\,(1-\gamma+\gamma k)}{(1-\alpha)} a_{k,j}\right]^2 \leq 1,$$

for $f_j \in V_n\,(\beta, \alpha, \gamma)\;\; (j = 1, 2)$, we have

$$\sum_{k=n}^{\infty} \frac{[k(1+\beta)+(\beta+\alpha)]^2\,(1-\gamma+\gamma k)^2}{2\,(1-\alpha)^2} \left(a_{k,1}^2 + a_{k,2}^2\right) \leq 1. \tag{4.21}$$

Thus we need to find the largest ξ such that

$$\frac{[k(1+\beta)+(\beta+\xi)]}{(1-\xi)} \leq \frac{[k(1+\beta)+(\beta+\alpha)]^2(1-\gamma+\gamma k)}{2(1-\alpha)^2} \quad (k \geq n), \tag{4.22}$$

that is, that

$$\xi \leq 1-\frac{2(1-\alpha)^2(1+\beta)(k+1)}{[k(1+\beta)+(\beta+\alpha)]^2(1-\gamma+\gamma k)+2(1-\alpha)^2} \quad (k \geq n). \tag{4.23}$$

Defining the function $\Theta(k)$ by

$$\Theta(k)=1-\frac{2(1-\alpha)^2(1+\beta)(k+1)}{[k(1+\beta)+(\beta+\alpha)]^2(1-\gamma+\gamma k)+2(1-\alpha)^2} \quad (k \geq n), \tag{4.24}$$

we observe that $\Theta(k)$ is an increasing function of $k \geq n$. Therefore, we conclude from (4.23) that

$$\xi \leq \Theta(n)=1-\frac{2(1-\alpha)^2(1+\beta)(n+1)}{[n(1+\beta)+(\beta+\alpha)]^2(1-\gamma+\gamma n)+2(1-\alpha)^2}, \tag{4.25}$$

which completes the proof of Theorem 3.

In its special case when $\gamma=0$, Theorem 3 yields

Corollary 11. *Let each of the functions $f_j\,(j=1,2)$ defined by (4.1) be in the class $\Sigma S_n^*(\beta,\alpha)$. Then the function $h(z)$ defined by (4.18) belongs to the class $\Sigma S_n^*(\beta,\sigma)$, where*

$$\sigma=1-\frac{2(1-\alpha)^2(1+\beta)(n+1)}{[n(1+\beta)+(\beta+\alpha)]^2+2(1-\alpha)^2}. \tag{4.26}$$

The result is sharp for the functions f_1 and f_2 given by (4.15).

Setting $\gamma=1$ in Theorem 3, we obtain the following corollary.

Corollary 12. *Let each of the functions $f_j\,(j=1,2)$ defined by (4.1) be in the class $\Sigma K_n(\beta,\alpha)$. Then the function $h(z)$ defined by (4.18) belongs to the class $\Sigma K_n(\beta,\rho)$, where*

$$\rho=1-\frac{2(1-\alpha)^2(1+\beta)(n+1)}{n[n(1+\beta)+(\beta+\alpha)]^2+2(1-\alpha)^2}. \tag{4.27}$$

The result is sharp for the functions f_1 and f_2 given by (4.17).

Remark 1. *Putting $\beta=0$ in Theorems 1, 2 and 3, respectively, we obtain the results obtained by Aouf et al. [1, Theorems 1, 2 and 3, respectively with $\beta=B=1$ and $A=-1$].*

References

[1] M. K. Aouf, H. E. Darwish, and H. M. Srivastava, *A general class of meromorphically univalent functions*, PanAmer. Math. J., 7 (1997) , no. 3, 69-84.

[2] M. K. Aouf, R. M. EL-Ashwah and H. M. Zayed, *Subclass of meromorphic funcions with positive coefficients defined by convolution*, Studia Univ. Babes-Bolai Math. (To appear).

[3] J. E. Miller, *Convex meromorphic mapping and related functions*, Proc. Amer. Math. Soc., 25 (1970) , 220-228.

[4] S. Owa, M. Nunokawa and H. Saitoh,*Generalization of certain subclasses of meromorphic univalent functions*, J. Fac. Sci. Tech. Kinki Univ., 25 (1989) , 21-27.

[5] Ch. Pommerenke, *On meromorphic starlike functions*, Pacific J. Math., 13 (1963) , 221-235.

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