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SUBCLASSES OF ANALYTIC FUNCTIONS DEFINED BY A NEW DIFFERENTIAL OPERATOR

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Abstract. In this paper we introduce and study two new subclasses $\mathcal{TS}_m^*(\alpha,\lambda)$ and $\mathcal{TC}_m(\alpha,\lambda)$ of analytic functions which are defined by means of a new differantial operator. Some results connected to coefficient estimates, distortion theorems and radii of starlikeness and convexity related these subclasses are obtained. Also, extreme points for these subclasses are determined.

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1. Introduction

Let $\mathcal{A}$ denote the class of functions f of the form

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n \tag{1}$$

which are the analytic in the open unit disk $\mathcal{U}=\{z\in\mathbb{C}:|z|<1\}$. Suppose that $\mathcal{S}$ denote the subclass of $\mathcal{A}$ consisting of functions that are the univalent in $\mathcal{U}$. A function $f\in\mathcal{A}$ is said to be starlike of order α $(0\le\alpha<1)$ if and only if

$$\Re\left(\frac{zf'(z)}{f(z)}\right)>\alpha,\quad (z\in\mathcal{U}).$$

Here, the class of all such functions is denote by $\mathcal{S}^*(\alpha)$. On the other hand, a function $f\in\mathcal{A}$ is said to be convex of order α $(0\le\alpha<1)$ if and only if

$$\Re\left(1+\frac{zf''(z)}{f'(z)}\right)>\alpha,\quad (z\in\mathcal{U}).$$

We denote by $\mathcal{C}(\alpha)$ the class of all such functions. Note that $\mathcal{S}^*(0)=\mathcal{S}^*$ and $\mathcal{C}^*(0)=\mathcal{C}$ are the usual classes of starlike and convex functions in $\mathcal{U}$, respectivelly.

Further $\mathcal{T}$ denote subclass of $\mathcal{A}$ consisting of functions f of the form

$$f(z) = z - \sum_{n=2}^{\infty} |a_n|\, z^n. \tag{2}$$

A function $f \in \mathcal{T}$ is called a function with negative coefficient and the class $\mathcal{T}$ introduced and studied by Silverman [8]. Recently, some subclasses of $\mathcal{T}$ have investigated by many mathematicians (see [1]-[5]).

For a functions f in $\mathcal{A}$, we define the a new differential operator D_λ^m as follows:

Definition 1. *Let $f \in \mathcal{A}$. For the parametres $\lambda \geq 0$ and $m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ define the differential operator D_λ^m on $\mathcal{A}$ as follows;*

$$D_\lambda^0 f(z) = f(z)$$

$$D_\lambda^1 f(z) = D_\lambda f(z) = \lambda z^3 f'''(z) + (2\lambda + 1) z^2 f''(z) + z f'(z)$$

$$D_\lambda^m f(z) = D_\lambda(D_\lambda^{m-1} f(z))$$

for $z \in \mathcal{U}$.

For a function f in $\mathcal{A}$, from definition of the differential operator D_λ^m, we can easily see that

$$D_\lambda^m f(z) = z + \sum_{n=2}^{\infty} n^{2m} \left(\lambda\,(n-1) + 1\right)^m a_n z^n.$$

Also, $D_\lambda^m f(z) \in \mathcal{A}$. For $f \in \mathcal{A}$ given by (1) and $g(z)$ given by

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n$$

their convolution (or Hadamard Product), denoted by $(f * g)$, is defined as

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n = (g * f)(z), \quad (z \in \mathcal{U}).$$

Special cases of this operator include the Sălăgean derivative operator S^m [7] as follows:

$$D_0^m f(z) = S^m f(z) * S^m f(z) = S^{2m} f(z)$$

and

$$D_1^m f(z) = S^m f(z) * S^m f(z) * S^m f(z) = S^{3m} f(z).$$

We now define subclasses related with the differential operator D_λ^m,

$$\mathcal{S}_m^*(\alpha,\lambda)=\left\{f\in\mathcal{A}:\ \Re\left(\frac{z\left(D_\lambda^m f(z)\right)'}{D_\lambda^m f(z)}\right)>\alpha,\quad \lambda\geq 0,\quad (0\leq\alpha<1)\right\}$$

and

$$\mathcal{C}_m(\alpha,\lambda)=\left\{f\in\mathcal{A}:\ \Re\left(1+\frac{z\left(D_\lambda^m f(z)\right)''}{\left(D_\lambda^m f(z)\right)'}\right)>\alpha,\quad \lambda\geq 0,\quad (0\leq\alpha<1)\right\}.$$

Further, we define the classes $\mathcal{TS}_m^*(\alpha,\lambda)$ and $\mathcal{TC}_m(\alpha,\lambda)$, respectivelly, by

$$\mathcal{TS}_m^*(\alpha,\lambda)=\mathcal{S}_m^*(\alpha,\lambda)\cap\mathcal{T}$$

and

$$\mathcal{TC}_m(\alpha,\lambda)=\mathcal{C}_m(\alpha,\lambda)\cap\mathcal{T}$$

for $\lambda\geq 0$, $0\leq\alpha<1$ and $m\in\mathbb{N}_0$. Silverman [8] proved some results for the subclasses $\mathcal{S}^*(\alpha)$, $\mathcal{C}(\alpha)$, $\mathcal{TS}_0^*(\alpha,\lambda)$ and $\mathcal{TC}_0^*(\alpha,\lambda)$.

2. Main Results

2.1. Coefficients inequalities

In this section, we provide a necessary and sufficient condition for a function f analytic in $\mathcal{U}$ to be in $\mathcal{S}_m^*(\alpha,\lambda)$, $\mathcal{C}_m(\alpha,\lambda)$, $\mathcal{TS}_m^*(\alpha,\lambda)$ and $\mathcal{TC}_m(\alpha,\lambda)$.

Theorem 1. *For $\lambda\geq 0$ and $0\leq\alpha<1$, let $f\in\mathcal{A}$ be defined by (1). If*

$$\sum_{n=2}^{\infty} n^{2m}(n-\alpha)\left(\lambda(n-1)+1\right)^m|a_n|\leq 1-\alpha, \tag{3}$$

then $f\in\mathcal{S}_m^*(\alpha,\lambda)$ where $m\in\mathbb{N}_0$.

Proof. It sufficies to show that values for $z\left(D_\lambda^m f(z)\right)'\diagdown D_\lambda^m f\ (z)$ lie in a circle centered at $w=1$ whose radius is $1-\alpha$. So, we have that

$$\begin{aligned}
&\left|\frac{z\left(D_\lambda^m f(z)\right)'}{D_\lambda^m f(z)}-1\right| \qquad (4)\\
=\ &\left|\frac{\sum_{n=2}^{\infty} n^{2m}(n-1)\left(\lambda(n-1)+1\right)^m a_n z^n}{z+\sum_{n=2}^{\infty} n^{2m}\left(\lambda(n-1)+1\right)^m a_n z^n}\right|\\
\leq\ &\frac{\sum_{n=2}^{\infty} n^{2m}(n-1)\left(\lambda(n-1)+1\right)^m|a_n|}{1-\sum_{n=2}^{\infty} n^{2m}\left(\lambda(n-1)+1\right)^m|a_n|}.
\end{aligned}$$

On the other hand from the inequality (3) we have

$$\begin{aligned}
&\sum_{n=2}^{\infty} n^{2m}(n-\alpha)(\lambda(n-1)+1)^m |a_n| \\
= &\sum_{n=2}^{\infty} n^{2m}(n-1)(\lambda(n-1)+1)^m |a_n| + (1-\alpha)\left(\sum_{n=2}^{\infty} n^{2m}(\lambda(n-1)+1)^m\right)|a_n| \\
\leq &\ 1-\alpha
\end{aligned}$$

and therefore

$$\frac{\sum_{n=2}^{\infty} n^{2m}(n-1)(\lambda(n-1)+1)^m |a_n|}{1-\sum_{n=2}^{\infty} n^{2m}(\lambda(n-1)+1)^m |a_n|} \leq 1-\alpha. \tag{5}$$

Thus from (4) and (5) we obtain

$$\left|\frac{z\left(D_\lambda^m f(z)\right)'}{D_\lambda^m f(z)} - 1\right| \leq 1-\alpha,$$

and theorem is proved. Note that the denominator in last inequality of (4) is positive provided that (5) holds.

Corollary 2. *For $\lambda \geq 0$ and $0 \leq \alpha < 1$, let $f \in \mathcal{A}$ be defined by (1). If*

$$\sum_{n=2}^{\infty} n^{2m+1}(n-\alpha)(\lambda(n-1)+1)^m |a_n| \leq 1-\alpha, \tag{6}$$

then $f \in \mathcal{C}_m(\alpha,\lambda)$ where $m \in \mathbb{N}_0$.

Proof. It is well known that $D_\lambda^m f(z) \in \mathcal{C}_m(\alpha,\lambda)$ if and only if $z\left(D_\lambda^m f(z)\right)' \in \mathcal{S}_m^*(\alpha,\lambda)$. Since

$$z\left(D_\lambda^m f(z)\right)' = z + \sum_{n=2}^{\infty} n^{2m+1}(\lambda(n-1)+1)^m a_n z^n,$$

we may replace a_n with na_n in the Theorem 1.

For functions in $\mathcal{TS}_m^*(\alpha,\lambda)$, the converse of Theorem 1 is true.

Theorem 3. *For $\lambda \geq 0$ and $0 \leq \alpha < 1$, let $f \in \mathcal{T}$ be defined by (2). Then $f \in \mathcal{TS}_m^*(\alpha,\lambda)$ if and only if the inequality (3) is satisfied. The result is sharp with the extremal function f given by*

$$f(z) = z - \frac{1-\alpha}{4^m(2-\alpha)(\lambda+1)^m} z^2.$$

Proof. We only prove the right-hand side, since the other side can be justified using similar arguments in proof of Theorem 1. Since $f \in \mathcal{TS}_m^* (\alpha, \lambda)$, we have that

$$\Re \left\{ \frac{z \left(D_\lambda^m f(z) \right)'}{D_\lambda^m f(z)} \right\} = \Re \left\{ \frac{z - \sum_{n=2}^{\infty} n^{2m+1} \left(\lambda (n-1) + 1 \right)^m |a_n| z^n}{z - \sum_{n=2}^{\infty} n^{2m} \left(\lambda (n-1) + 1 \right)^m |a_n| z^n} \right\} > \alpha \qquad (7)$$

for $z \in \mathcal{U}$. Choose values of z on the real axis so that $z \left(D_\lambda^m f(z) \right)' \diagdown D_\lambda^m f\ (z)$ is real. Upon clearing the denominator in (7) and letting $z \to 1^-$ through real values, we obtain

$$\frac{1 - \sum_{n=2}^{\infty} n^{2m+1} \left(\lambda (n-1) + 1 \right)^m |a_n|}{1 - \sum_{n=2}^{\infty} n^{2m} \left(\lambda (n-1) + 1 \right)^m |a_n|} \geq \alpha.$$

Thus we obtain

$$\sum_{n=2}^{\infty} n^{2m} (n - \alpha) \left(\lambda (n-1) + 1 \right)^m |a_n| \leq 1 - \alpha,$$

and the proof is complete.

Corollary 4. *If* $f \in \mathcal{TS}_m^* (\alpha, \lambda)$ *then,*

$$|a_n| \leq \frac{1 - \alpha}{n^{2m} (n - \alpha) \left(\lambda (n-1) + 1 \right)^m},$$

with equality only for functions of the form

$$f_n(z) = z - \frac{1 - \alpha}{n^{2m} (n - \alpha) \left(\lambda (n-1) + 1 \right)^m} z^n.$$

Corollary 5. *A function* $f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n$ *is in* $\mathcal{TC}_m^* (\alpha, \lambda)$ *if and only if the inequality (6) is satisfied. The result is sharp with the extremal function f given by*

$$f(z) = z - \frac{1 - \alpha}{2^{2m+1} (2 - \alpha) (\lambda + 1)^m} z^2.$$

Theorem 6. *Let* $0 \leq \lambda_1 \leq \lambda_2$ *and* $0 \leq \alpha < 1$, $m \in \mathbb{N}_0$. *Then* $\mathcal{TS}_m^* (\alpha, \lambda_2) \subseteq \mathcal{TS}_m^* (\alpha, \lambda_1)$ *and* $\mathcal{TC}_m^* (\alpha, \lambda_2) \subseteq \mathcal{TC}_m^* (\alpha, \lambda_1)$.

Proof. Since by assumption we have

$$\sum_{n=2}^{\infty} n^{2m} (n - \alpha) \left(\lambda_1 (n-1) + 1 \right)^m |a_n| \leq \sum_{n=2}^{\infty} n^{2m} (n - \alpha) \left(\lambda_2 (n-1) + 1 \right)^m |a_n| \leq 1 - \alpha.$$

Thus $f \in \mathcal{TS}_m^* (\alpha, \lambda_2)$ implies that $f \in \mathcal{TS}_m^* (\alpha, \lambda_1)$. Similarly $\mathcal{TC}_m^* (\alpha, \lambda_2) \subseteq \mathcal{TC}_m^* (\alpha, \lambda_1)$.

2.2. Covering theorems

In this section, growth and distortion theorems will be considered and covering property for function in the class will also be given.

Theorem 7. *If* $f \in \mathcal{TS}_m^*(\alpha, \lambda)$, *then,*

$$r - \frac{1-\alpha}{(2-\alpha)(4(\lambda+1))^m} r^2 \leq |f(z)| \leq r + \frac{1-\alpha}{(2-\alpha)(4(\lambda+1))^m} r^2 \quad (|z| = r),$$

where $m \in \mathbb{N}_0$, $\lambda \geq 0$ *and* $0 \leq \alpha < 1$. *The result is sharp with the extremal function* f *given by*

$$f(z) = z - \frac{1-\alpha}{(2-\alpha)(4(\lambda+1))^m} z^2.$$

Proof. Since $f \in \mathcal{TS}_m^*(\alpha, \lambda)$ by Theorem 3 we have that

$$(2-\alpha) 4^m (\lambda+1)^m \sum_{n=2}^{\infty} |a_n| \leq \sum_{n=2}^{\infty} n^{2m}(n-\alpha)(\lambda(n-1)+1)^m |a_n| \leq 1-\alpha$$

or

$$\sum_{n=2}^{\infty} |a_n| \leq \frac{1-\alpha}{(2-\alpha)(4(\lambda+1))^m}. \tag{8}$$

Thus from (8) we obtain

$$\begin{aligned} |f(z)| &= \left| z - \sum_{n=2}^{\infty} |a_n| z^n \right| \leq |z| + \sum_{n=2}^{\infty} |a_n| |z|^n \leq r + r^2 \sum_{n=2}^{\infty} |a_n| \\ &\leq r + r^2 \frac{1-\alpha}{(2-\alpha)(4(\lambda+1))^m} \end{aligned}$$

and similarly,

$$\begin{aligned} |f(z)| &= \left| z - \sum_{n=2}^{\infty} |a_n| z^n \right| \geq |z| - \sum_{n=2}^{\infty} |a_n| |z|^n \geq r - r^2 \sum_{n=2}^{\infty} |a_n| \\ &\geq r - r^2 \frac{1-\alpha}{(2-\alpha)(4(\lambda+1))^m}. \end{aligned}$$

Corollary 8. *If* $f \in \mathcal{TC}_m(\alpha, \lambda)$, *then*

$$r - \frac{1-\alpha}{2(2-\alpha)(4(\lambda+1))^m} r^2 \leq |f(z)| \leq r + \frac{1-\alpha}{2(2-\alpha)(4(\lambda+1))^m} r^2 \quad (|z| = r),$$

where $m \in \mathbb{N}_0,\ \lambda \geq 0$ *and* $0 \leq \alpha < 1$.*The result is sharp with the extremal function* f *given by*

$$f(z) = z - \frac{1-\alpha}{2\left(2-\alpha\right)\left(4(\lambda+1)\right)^m} z^2.$$

Theorem 9. *The disk* $|z| < 1$ *is mapped onto a domain that cantains the disk* $|w| < 1 - \frac{1-\alpha}{(2-\alpha)(4(\lambda+1))^m}$ *by any* $f \in \mathcal{TS}_m^*\left(\alpha,\lambda\right)$ *and onto a domain that contains the disk* $|w| < 1 - \frac{1-\alpha}{2(2-\alpha)(4(\lambda+1))^m}$ *by any* $f \in \mathcal{TC}_m\left(\alpha,\lambda\right)$.

Proof. The results follow upon letting $r \to 1$ in Theorem 7 and its corollary.

Theorem 10. *If* $f \in \mathcal{TS}_m^*\left(\alpha,\lambda\right)$, *then*

$$1 - \frac{2(1-\alpha)}{\left(2-\alpha\right)\left(4(\lambda+1)\right)^m} r \leq \left|f'(z)\right| \leq 1 + \frac{2(1-\alpha)}{\left(2-\alpha\right)\left(4(\lambda+1)\right)^m} r \qquad \left(|z| = r\right),$$

where $m \in \mathbb{N}_0,\ \lambda \geq 0$ *and* $0 \leq \alpha < 1$.

Proof. Since $f \in \mathcal{TS}_m^*\left(\alpha,\lambda\right)$ by Theorem 3 we have that

$$\left(2-\alpha\right) 2^{2m-1} \left(\lambda+1\right)^m \sum_{n=2}^{\infty} n\left|a_n\right| \leq \sum_{n=2}^{\infty} n^{2m}(n-\alpha)\left(\lambda\left(n-1\right)+1\right)^m \left|a_n\right| \leq 1-\alpha$$

or

$$\sum_{n=2}^{\infty} n\left|a_n\right| \leq \frac{2(1-\alpha)}{\left(2-\alpha\right)\left(4(\lambda+1)\right)^m}. \tag{9}$$

In view of the inequalities (8) we obtain

$$\begin{aligned} \left|f'(z)\right| &\leq 1 + \sum_{n=2}^{\infty} n\left|a_n\right|\left|z\right|^{n-1} \qquad (10) \\ &\leq 1 + r\sum_{n=2}^{\infty} n\left|a_n\right| \\ &\leq 1 + \frac{2(1-\alpha)}{\left(2-\alpha\right)\left(4(\lambda+1)\right)^m} r. \end{aligned}$$

which is right-hand inequality in Theorem 10. On the other hand, similarly

$$\left|f'(z)\right| \geq 1 - \frac{2(1-\alpha)}{\left(2-\alpha\right)\left(4(\lambda+1)\right)^m} r$$

and thus proof is completed.

Corollary 11. *If $f \in \mathcal{TC}_m(\alpha,\lambda)$, then*

$$1-\frac{1-\alpha}{(2-\alpha)(4(\lambda+1))^m}r \leq \left|f'(z)\right| \leq 1+\frac{1-\alpha}{(2-\alpha)(4(\lambda+1))^m}r \quad (|z|=r),$$

where $m \in \mathbb{N}_0$, $\lambda \geq 0$ and $0 \leq \alpha < 1$.

2.3. Radii of Starlikeness and Convexity

Next, we obtain the radii of starlikeness and convexity of order δ $(0 \leq \delta < 1)$ of the class $\mathcal{TS}_m^*(\alpha,\lambda)$.

Theorem 12. *If $f \in \mathcal{TS}_m^*(\alpha,\lambda)$, then f is starlike of order δ $(0 \leq \delta < 1)$ in the disk*

$$|z| < r_1 = r_1(\alpha,\lambda,\delta,m) = \inf_n \left(\frac{(n-\alpha)(1-\delta)\,n^{2m}(\lambda(n-1)+1)^m}{(n-\delta)(1-\alpha)}\right)^{\frac{1}{n-1}}, \quad (n=2,3,...).$$

Proof. For $0 \leq \delta < 1$ we need to show that

$$\left|\frac{zf'(z)}{f(z)}-1\right| < 1-\delta,$$

that is,

$$\begin{aligned}\left|\frac{zf'(z)}{f(z)}-1\right| &\leq \frac{\sum_{n=2}^{\infty}(n-1)|a_n||z|^{n-1}}{1-\sum_{n=2}^{\infty}|a_n||z|^{n-1}} \\ &< 1-\delta\end{aligned}$$

or

$$\sum_{n=2}^{\infty}\left(\frac{n-\delta}{1-\delta}\right)|a_n||z|^{n-1} < 1.$$

By using Theorem 3, the above inequality holds if

$$|z|^{n-1} < \frac{(1-\delta)(n-\alpha)\,n^{2m}(\lambda(n-1)+1)^m}{(1-\alpha)(n-\delta)}.$$

This completes the proof of theorem.

We now determine the radius of convexity for functions in $\mathcal{TS}_m^*(\alpha,\lambda)$.

Theorem 13. *If* $f \in \mathcal{TS}_m^*(\alpha,\lambda)$, *then* f *is convex of order* δ $(0 \leq \delta < 1)$ *in the disk*

$$|z| < r_2 = r_2(\alpha,\lambda,\delta,m) = \inf_n \left(\frac{(n-\alpha)(1-\delta)\, n^{2m-1}(\lambda(n-1)+1)^m}{(n-\delta)(1-\alpha)} \right)^{\frac{1}{n-1}}, \quad (n = 2,3,...).$$

Proof. It suffices to show that $\left|\frac{zf''(z)}{f'(z)}\right| \leq 1-\delta$ for $|z| \leq r_2$. Then proof is similar to the proof of Theorem 12 and therefore we omit the details.

2.4. Extreme Points

The extreme points of the classes $\mathcal{TS}_m^*(\alpha,\lambda)$ and $\mathcal{TC}_m(\alpha,\lambda)$ are given by the following theorem.

Theorem 14. *Let* $f_1(z) = z$ *and*

$$f_n(z) = z - \frac{(1-\alpha)\, z^n}{n^{2m}(n-\alpha)(\lambda(n-1)+1)^m}, \quad (n = 2,3,...).$$

Then $f \in \mathcal{TS}_m^*(\alpha,\lambda)$ *if and only if it can be expressed in the form* $f(z) = \sum_{n=1}^{\infty} \gamma_n f_n(z)$ *where* $\gamma_n > 0$ *and* $\sum_{n=1}^{\infty} \gamma_n = 1$.

Proof. Suppose

$$f(z) = \sum_{n=1}^{\infty} \gamma_n f_n(z) = z - \sum_{n=2}^{\infty} \gamma_n \frac{1-\alpha}{n^{2m}(n-\alpha)(\lambda(n-1)+1)^m} z^n.$$

Then we have

$$\begin{aligned} &\sum_{n=2}^{\infty} \frac{n^{2m}(n-\alpha)(\lambda(n-1)+1)^m}{1-\alpha} \left(\gamma_n \frac{1-\alpha}{n^{2m}(n-\alpha)(\lambda(n-1)+1)^m} \right) \\ = &\sum_{n=2}^{\infty} \gamma_n = 1 - \gamma_1 \leq 1. \end{aligned}$$

Thus, $f \in \mathcal{TS}_m^*(\alpha,\lambda)$.

Conversselly, suppose $f \in \mathcal{TS}_m^*(\alpha,\lambda)$.Since

$$|a_n| \leq \frac{1-\alpha}{n^{2m}(n-\alpha)(\lambda(n-1)+1)^m}, \quad (n = 2,3,...),$$

we may set

$$\gamma_n = \frac{n^{2m}(n-\alpha)(\lambda(n-1)+1)^m}{1-\alpha} |a_n|, \quad (n = 2,3,...),$$

and $\gamma_1 = 1 - \sum_{n=2}^{\infty} \gamma_n$.

Then

$$f(z) = \sum_{n=1}^{\infty} \gamma_n f_n(z).$$

This complete the proof.

Corollary 15. *The extreme points of* $\mathcal{TS}_m^* (\alpha, \lambda)$ *are the functions* $f_n(z) \quad (n = 2, 3, ...)$ *in Theorem 14*

Corollary 16. *The extreme points of* $\mathcal{TC}_m (\alpha, \lambda)$ *are given by* $f_1(z) = z$ *and*

$$f_n(z) = z - \frac{1-\alpha}{n^{2m+1}(n-\alpha) (\lambda (n-1) + 1)^m} z^n, \quad (n = 2, 3, ...) .$$

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