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COEFFICIENT BOUNDS FOR A SUBCLASS OF BI-UNIVALENT FUNCTIONS USING SALAGEAN OPERATOR

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Abstract. In the present paper, we introduce new subclasses $ST_{\Sigma}(b,\phi)$ and $CV_{\Sigma}(b,\phi)$ of bi-univalent functions defined in the open disk. Furthermore, we find upper bounds for the second and third coefficients for functions in these new subclasses using Salagean operator.

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1. Introduction,Definitions And Preliminaries

Let $\mathcal{A}$ denote the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathcal{C} : |z| < 1\}$. Further, by $\mathcal{S}$ we shall denote the class of functions $f \in \mathcal{A}$ which are univalent in $\mathbb{U}$.

Since univalent functions are one-to-one, they are invertible and the inverse functions need not be defined on the entire unit disk $\mathbb{U}$. However, the famous Koebe one-quarter theorem ensures that the image of the unit disk $\mathbb{U}$ under every function $f \in \mathcal{A}$ contains a disk of radius 1/4. Thus every univalent function f has an inverse f^{-1} satisfying $f^{-1}(f(z)) = z$, $(z \in \mathbb{U})$ and $f\left(f^{-1}(w)\right) = w$, $\left(|w| < r_0(f), r_0(f) \geq \frac{1}{4}\right)$ where

$$f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots . \tag{2}$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathbb{U}$ if both $f(z)$ and $f^{-1}(z)$ are univalent in $\mathbb{U}$. We let Σ to denote the class of bi-univalent functions in $\mathbb{U}$ given by (1). If $f(z)$ is bi-univalent, it must be analytic in the boundary of the domain and such that it can be continued across the boundary of the domain so that $f^{-1}(z)$ is defined and analytic throughout $|w| < 1$. Examples of functions in the class Σ are

$$\frac{z}{1-z}, -log\,(1-z)$$

and so on.

The coefficient estimate problem for the class $\mathcal{S}$, known as the Bieberbach conjecture, is settled by de-Branges [4], who proved that for a function $f\,(z) = z + \sum_{n=2}^{\infty} a_n z^n$ in the class $\mathcal{S}$, $|a_n| \leq n$, for $n = 2, 3, \cdots$, with equality only for the rotations of the Koebe function

$$K_0(z) = \frac{z}{(1-z)^2}.$$

In 1967, Lewin [7] introduced the class Σ of bi-univalent functions and showed that $|a_2| < 1.51$ for the functions belonging to Σ. It was earlier believed that for $f \in \Sigma$, the bound was $|a_n| < 1$ for every n and the extremal function in the class was $\frac{z}{1-z}$. E.Netanyahu [9] in 1969, ruined this conjecture by proving that in the set Σ, $\max_{f\in\Sigma} |a_2| \leq 4/3$. In 1969, Suffridge [13] gave an example of $f \in \Sigma$ for which $a_2 = 4/3$ and conjectured that $|a_2| \leq 4/3$. In 1981, Styer and Wright [12] disproved the conjecture that $|a_2| > 4/3$. Brannan and Clunie [2] conjectured that $|a_2| \leq \sqrt{2}$. Kedzierawski [6] in 1985 proved this conjecture for a special case when the function f and f^{-1} are starlike functions. Brannan and Clunie [2] conjectured that $|a_2| \leq \sqrt{2}$. Tan [14] in proved that $|a_2| \leq 1.485$ which is the best known estimate for functions in the class of bi-univalent functions.

Brannan and Taha [3] introduced certain subclasses of the bi-univalent function class Σ similar to the familiar subclasses $S^*\,(\alpha)$ and $C\,(\alpha)$ of the univalent function class Σ. Recently, Ali et al.[1] extended the results of Brannan and Taha [3] by generalising their classes using subordination.

An analytic function f is subordinate to an analytic function g,written $f(z) \prec g(z)$, provided there is a Schwarz function w defined on $\mathbb{U}$ with $w(0) = 0$ and $|w(z)| < 1$ satisfying $f\,(z) = g\,(w(z))$. Ma and Minda [8], unified various subclasses of starlike and convex functions for which either of the quantity $\frac{zf'(z)}{f(z)}$ or $1 + \frac{zf''(z)}{f'(z)}$ is subordinate to a more general superordinate function. For this purpose, they considered an analytic function ϕ with positive real part in the unit disk U, $\phi(0) = 1$, $\phi'(0) > 0$ and ϕ maps U onto a region starlike with respect to 1 and symmetric with respect to the real axis. Such a function has a series expansion of the form

$$\phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots, (B_1 > 0). \tag{3}$$

Let a differential operator be defined [11] on a class of analytic functions of the form (1) as follows:

$$D^0 f(z) = f(z), \quad D^1 f(z) = Df(z) = zf'(z)$$

and in general

$$D^n f(z) = D\left(D^{n-1} f(z)\right) \qquad \left(n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}\right).$$

We easily find that

$$D^k f(z) = z + \sum_{n=2}^{\infty} n^k a_n z^n \qquad (n \in \mathbb{N}_0). \tag{4}$$

Definition 1. *Let b be a non-zero complex number. A function $f(z)$ given by (1) is said to be in the class $ST_{\Sigma}(b, \phi)$ if the following conditions are satisfied:*

$$f \in \Sigma \quad \text{and} \quad 1 + \frac{1}{b}\left(\frac{z\left(D^m f(z)\right)'}{D^m f(z)} - 1\right) \prec \phi(z), \quad z \in \mathbb{U} \tag{5}$$

$$\text{and} \quad 1 + \frac{1}{b}\left(\frac{w\left(D^m g(w)\right)'}{D^m g(w)} - 1\right) \prec \phi(w), \quad w \in \mathbb{U}, \tag{6}$$

where the function g is given by (2).

Definition 2. *Let b be a non-zero complex number. A function $f(z)$ given by (1) is said to be in the class $CV_{\Sigma}(b, \phi)$ if the following conditions are satisfied:*

$$f \in \Sigma \quad \text{and} \quad 1 + \frac{1}{b}\left(\frac{z\left(D^m f(z)\right)''}{\left(D^m f(z)\right)'}\right) \prec \phi(z), z \in \mathbb{U} \tag{7}$$

$$\text{and} \quad 1 + \frac{1}{b}\left(\frac{w\left(D^m g(w)\right)''}{\left(D^m g(w)\right)'}\right) \prec \phi(w), w \in \mathbb{U}, \tag{8}$$

where the function g is given by (2).

2. Coefficient estimates

Lemma 1. *[10] If $p \in \wp$, then $|c_k| \leq 2$ for each k, where $\wp$ is the family of functions p analytic in $\mathbb{U}$ for which $\Re p(z) > 0$, $p(z) = 1 + c_1 z + c_2 z^2 + \cdots$ for $z \in \mathbb{U}$.*

Theorem 2. *Let the function $f(z) \in \mathcal{A}$ be given by (1). If $f \in ST_\Sigma(b,\phi)$, then*

$$|a_2| \leq \frac{B_1\sqrt{B_1}\,|b|}{\sqrt{\left|\left(2(3^m)-2^{2m}\right)B_1^2 b+(B_1-B_2)\,2^{2m}\right|}} \quad \textit{and} \quad |a_3| \leq \frac{(B_1+|B_2-B_1|)\,|b|}{2(3^m)-2^{2m}}. \tag{9}$$

Proof. Since $f \in ST_\Sigma(b,\phi)$, there exists two analytic functions $r, s : \mathbb{U} \to \mathbb{U}$, with $r(0) = 0 = s(0)$, such that

$$1+\frac{1}{b}\left(\frac{z\,(D^m f(z))'}{D^m f(z)}-1\right)=\phi\,(r(z)) \quad \text{and} \quad 1+\frac{1}{b}\left(\frac{w\,(D^m g(w))'}{D^m g(w)}-1\right)=\phi\,(s(z)). \tag{10}$$

Define the functions p and q by

$$p(z)=\frac{1+r(z)}{1-r(z)}=1+p_1z+p_2z^2+\cdots \quad \text{and} \quad q(z)=\frac{1+s(z)}{1-s(z)}=1+q_1z+q_2z^2+\cdots. \tag{11}$$

Or equivalently,

$$r(z)=\frac{p(z)-1}{p(z)+1}=\frac{1}{2}\left(p_1z+\left(p_2-\frac{p_1^2}{2}\right)z^2+\left(p_3+\frac{p_1}{2}\left(\frac{p_1^2}{2}-p_2\right)-\frac{p_1p_2}{2}\right)z^3+\cdots\right) \tag{12}$$

and

$$s(z)=\frac{q(z)-1}{q(z)+1}=\frac{1}{2}\left(q_1z+\left(q_2-\frac{q_1^2}{2}\right)z^2+\left(q_3+\frac{q_1}{2}\left(\frac{q_1^2}{2}-q_2\right)-\frac{q_1q_2}{2}\right)z^3+\cdots\right). \tag{13}$$

It is clear that p and q are analytic in $\mathbb{U}$ and $p(0) = 1 = q(0)$. Also p and q have positive real part in $\mathbb{U}$ and hence $|p_i| \leq 2$ and $|q_i| \leq 2$. In the view of (11), (12)and (13), clearly,

$$1+\frac{1}{b}\left(\frac{z\,(D^m f(z))'}{D^m f(z)}-1\right)=\phi\left(\frac{p(z)-1}{p(z)+1}\right) \quad \text{and} \quad 1+\frac{1}{b}\left(\frac{w\,(D^m g(w))'}{D^m g(w)}-1\right)=\phi\left(\frac{q(w)-1}{q(w)+1}\right). \tag{14}$$

Using (13) and (14) together with (3), one can easily verify that

$$\phi\left(\frac{p(z)-1}{p(z)+1}\right)=1+\frac{B_1p_1}{2}z+\left(\frac{B_1}{2}\left(p_2-\frac{p_1^2}{2}\right)+\frac{1}{4}B_2p_1^2\right)z^2+\cdots \tag{15}$$

and

$$\phi\left(\frac{q(w)-1}{q(w)+1}\right)=1+\frac{B_1q_1}{2}w+\left(\frac{B_1}{2}\left(q_2-\frac{q_1^2}{2}\right)+\frac{B_2q_1^2}{4}\right)w^2+\cdots. \tag{16}$$

Since $f\in\Sigma$ has the Maclaurin series given by (1), computation shows that its inverse $g=f^{-1}$ has the expansion given by (2). It follows from (14), (15) and (16) that

$$2^m a_2=\frac{1}{2}B_1p_1b, \tag{17}$$

$$2\left(3^m\right)a_3-\left(2^{2m}\right)a_2^2=\frac{1}{2}bB_1\left(p_2-\frac{1}{2}p_1^2\right)+\frac{1}{4}bB_2p_1^2 \tag{18}$$

and

$$-2^m a_2=\frac{1}{2}B_1bq_1, \tag{19}$$

$$\left(4\left(3^m\right)-\left(2^{2m}\right)\right)a_2^2-2\left(3^m\right)a_3=\frac{1}{2}bB_1\left(q_2-\frac{1}{2}q_1^2\right)+\frac{1}{4}bB_2q_1^2. \tag{20}$$

From (17) and (19), it follows that

$$p_1=-q_1. \tag{21}$$

Now (18), (20) and (21) gives

$$a_2^2=\frac{B_1^3\left(p_2+q_2\right)b}{4\left(\left(2.3^m-2^{2m}\right)B_1^2b+2^{2m}\left(B_1-B_2\right)\right)}. \tag{22}$$

Using the fact that $|p_2|\le 2$ and $|q_2|\le 2$ gives the desired estimate on $|a_2|$,

$$|a_2|\le\frac{B_1\sqrt{B_1}\,|b|}{\sqrt{\left|\left(2.3^m-2^{2m}\right)B_1^2b+\left(B_1-B_2\right)2^{2m}\right|}}.$$

From (18)-(20), gives

$$a_3=\frac{\frac{bB_1}{2}\left(\left(4(3^m)-2^{2m}\right)p_2+2^{2m}q_2\right)+3^mp_1^2\left(B_2-B_1\right)b}{4\left(2(3^{2m})-3^m2^{2m}\right)}.$$

Using the inequalities $|p_1|\le 2$, $|p_2|\le 2$ and $|q_2|\le 2$ for functions with positive real part yields the desired estimation of $|a_3|$.

For a choice of $\phi(z) = \dfrac{1+Az}{1+Bz}$, $-1 \leq B < A \leq 1$, we have the following corollary.

Corollary 3. *Let $-1 \leq B < A \leq 1$. If $f \in ST_\Sigma\left(b, \frac{1+Az}{1+Bz}\right)$, then*

$$|a_2| \leq \frac{|b|(A-B)}{\sqrt{|(2(3^m) - 2^{2m})(A-B)b + (1+B)2^{2m}|}}$$

and

$$|a_3| \leq \frac{|A-B|(1+|1+B|)|b|}{(2(3^m) - 2^{2m})}.$$

If we let $\phi(z) = \left(\dfrac{1+z}{1-z}\right)^\alpha = 1 + 2\alpha z + 2\alpha^2 z^2 + \cdots$, $0 < \alpha \leq 1$, in the above theorem, we get the following:

Corollary 4. *Let $0 < \alpha \leq 1$. If $f \in ST_\Sigma(b, \alpha)$, then*

$$|a_2| \leq \frac{|b|\, 2\alpha}{\sqrt{|2\alpha(2(3^m) - 2^{2m})b + (1-\alpha)2^{2m}|}}$$

and

$$|a_3| \leq \frac{(1+|\alpha - 1|)2\alpha|b|}{2(3^m) - 2^{2m}}.$$

Theorem 5. *Let the function $f(z) \in \mathcal{A}$ be given by (1). If $f \in CV_\Sigma(b, \phi)$, then*

$$|a_2| \leq \frac{B_1\sqrt{B_1}\,|b|}{\sqrt{2\left|(3^{m+1} - 2^{2m+1})B_1^2 b + 2(B_1 - B_2)2^{2m}\right|}} \quad \text{and} \quad |a_3| \leq \frac{(B_1 + |B_2 - B_1|)|b|}{2(3^{m+1} - 2^{2m+1})}. \tag{23}$$

Proof. Since $f \in CV_\Sigma(b, \phi)$, there exists two analytic functions $r, s : \mathbb{U} \to \mathbb{U}$, with $r(0) = 0 = s(0)$, such that

$$1 + \frac{1}{b}\left(\frac{z(D^m f(z))''}{(D^m f(z))'}\right) = \phi(r(z)) \quad \text{and} \quad 1 + \frac{1}{b}\left(\frac{w(D^m g(w))''}{(D^m g(w))'}\right) = \phi(s(z)). \tag{24}$$

Using (11), (12), (15) and (16), one can easily verified that

$$2^{m+1}a_2 = \frac{1}{2}B_1 p_1 b, \tag{25}$$

$$6(3^m)a_3 - 4\left(2^{2m}\right)a_2^2 = \frac{1}{2}bB_1\left(p_2 - \frac{1}{2}p_1^2\right) + \frac{1}{4}bB_2 p_1^2 \tag{26}$$

and

$$-2^{m+1}a_2 = \frac{1}{2}B_1 b q_1, \tag{27}$$

$$\left(12\left(3^m\right)-4\left(2^{2m}\right)\right)a_2^2 - 6\left(3^m\right)a_3 = \frac{1}{2}bB_1\left(q_2-\frac{1}{2}q_1^2\right)+\frac{1}{4}bB_2q_1^2. \tag{28}$$

From (25) and (27), it follows that

$$p_1 = -q_1. \tag{29}$$

Now (26), (28) and (29) gives

$$a_2^2 = \frac{B_1^3\left(p_2+q_2\right)b}{8\left(\left(3.3^m-2.2^{2m}\right)B_1^2 b+2\left(B_1-B_2\right)\left(2^{2m}\right)\right)}. \tag{30}$$

Using the fact that $|p_2| \leq 2$ and $|q_2| \leq 2$ gives the desired estimate on $|a_2|$,

$$|a_2| \leq \frac{B_1\sqrt{B_1}\,|b|}{\sqrt{2\left|\left(3^{m+1}-2^{2m+1}\right)B_1^2 b+2\left(B_1-B_2\right)2^{2m}\right|}}.$$

From (26)-(28), gives

$$a_3 = \frac{\frac{bB_1}{2}\left(\left(12(3^{2m})-4(2^{2m})\right)p_2+4(2^{2m})q_2\right)+\left(B_2-B_1\right)bp_1^2 3^{m+1}}{24(3^m)\left(3^{m+1}-2^{2m+1}\right)}.$$

Using the inequalities $|p_1| \leq 2$, $|p_2| \leq 2$ and $|q_2| \leq 2$ for functions with positive real part yields

$$|a_3| \leq \frac{\left(B_1+|B_2-B_1|\right)|b|}{2(3^{m+1}-2^{2m+1})}.$$

For a choice of $\phi(z) = \dfrac{1+Az}{1+Bz}$, $-1 \leq B < A \leq 1$, we have the following corollary.

Corollary 6. *Let $-1 \leq B < A \leq 1$. If $f \in ST_\Sigma\left(b, \frac{1+Az}{1+Bz}\right)$, then*

$$|a_2| \leq \frac{|b|\left(A-B\right)}{\sqrt{2\left|\left(3^{m+1}-2^{2m+1}\right)\left(A-B\right)b+2\left(1+B\right)2^{2m}\right|}}$$

and

$$|a_3| \leq \frac{|A-B|\left(1+|1+B|\right)|b|}{2\left(3^{m+1}-2^{2m+1}\right)}.$$

If we let $\phi(z) = \left(\frac{1+z}{1-z}\right)^{\alpha} = 1 + 2\alpha z + 2\alpha^2 z^2 + \cdots$, $0 < \alpha \leq 1$, in the above theorem, we get the following:

Corollary 7. *Let* $0 < \alpha \leq 1$. *If* $f \in ST_{\Sigma}(b, \alpha)$, *then*

$$|a_2| \leq \frac{|b|\,\alpha}{\sqrt{|(3^{m+1} - 2^{2m+1})\,\alpha b + (1-\alpha)\,2^{2m}|}}$$

and

$$|a_3| \leq \frac{(1 + |\alpha - 1|)\,\alpha\,|b|}{(3^{m+1} - 2^{2m+1})}.$$

Remark 1. *If we let* $b = 1, m = 0$, *Theorem 2.2 and Theorem 2.5 reduce to the result of R.M.Ali et.al [1], corollary* 2.1 *and corollary* 2.2.

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