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A SUBCLASS OF ANALYTIC FUNCTIONS AND A GENERALIZED LINEAR DIFFERENTIAL OPERATOR

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Abstract. In this article a sub class of analytic function in introduced, which is defined using a generalized differential operator and various properties of this sub class are discussed.

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1. Introduction

Let $\mathcal{A}$ denote the class of analytic functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

defined on the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$. Silverman introduced and studied about a sub class $\mathcal{T}$ of $\mathcal{A}$ consisting of functions of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n \qquad (a_n \geq 0). \tag{1.2}$$

Let $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ be a function in $\mathcal{A}$ then the convolution or Hadamard product of f given by (1.1) and $g(z)$ is defined as

$$(f * g)(z) := z + \sum_{n=2}^{\infty} a_n b_n z^n.$$

Modified Hadamard product for functions with negative coefficients $f(z)$ given by (1.2) and $g(z) = z - \sum_{n=2}^{\infty} b_n z^n \quad (b_n \geq 0)$ is defined as

$$(f * g)(z) := z - \sum_{n=2}^{\infty} a_n b_n z^n.$$

For complex numbers $\alpha_1, \alpha_2, \ldots, \alpha_q$ and $\beta_1, \beta_2, \ldots, \beta_s; (\beta_j \in \mathbb{C} \setminus \mathcal{Z}_0^-; \mathcal{Z}_0^- = \{0, -1, -2, \ldots\}$; for $j = 1, 2, .., s)$, the generalized hypergeometric function, denoted as ${}_qF_s(\alpha_1, \alpha_2, \ldots, \alpha_q; \beta_1, \beta_2, \ldots, \beta_s; z)$, defined as

$${}_qF_s(\alpha_1, \alpha_2, \ldots, \alpha_q; \beta_1, \beta_2, \ldots, \beta_s; z) := \sum_{n=0}^{\infty} \frac{(\alpha_1)_n(\alpha_2)_n \ldots (\alpha_q)_n z^n}{(\beta_1)_n(\beta_2)_n \ldots (\beta_s)_n n!}$$

where$q \leq s+1; q, s \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}; z \in U$ and $\mathbb{N}$ denotes the set of all positive integers and $(x)_n$ is the Pochhammer symbol defined in terms of gamma function, as

$$(x)_n := \frac{\Gamma(x+n)}{\Gamma(x)} = \begin{cases} 1 & \text{if} \quad n = 0 \\ x(x+1)\ldots(x+n-1) & \text{if} \quad n \in \mathbb{N}. \end{cases}$$

Corresponding to the function $g_{q,s}(\alpha_1, \beta_1; z)$, defined by

$$g_{q,s}(\alpha_1, \beta_1; z) := z\,{}_qF_s(\alpha_1, \alpha_2, \ldots, \alpha_q; \beta_1, \beta_2, \ldots, \beta_s; z),$$

let us introduce a generalized differential operator $\mathcal{D}^m_{\lambda,\mu}(\alpha_1, \beta_1)f(z) : \mathcal{A} \to \mathcal{A}$ as follows

$$\begin{aligned} \mathcal{D}^0_{\lambda,\mu}(\alpha_1, \beta_1)f(z) &:= f(z) * g_{q,s}(\alpha_1, \beta_1; z) \\ \mathcal{D}^1_{\lambda,\mu}(\alpha_1, \beta_1)f(z) &:= (1-\lambda+\mu)(f(z) * g_{q,s}(\alpha_1, \beta_1; z)) + \\ &\quad (\lambda-\mu)z(f(z) * g_{q,s}(\alpha_1, \beta_1; z))' + \lambda\mu z^2(f(z) * g_{q,s}(\alpha_1, \beta_1; z))'' \\ \mathcal{D}^m_{\lambda,\mu}(\alpha_1, \beta_1)f(z) &:= \mathcal{D}^1_{\lambda,\mu}(\mathcal{D}^{m-1}_{\lambda,\mu}(\alpha_1, \beta_1)f(z)) \end{aligned}$$

where $0 \leq \mu \leq \lambda \leq 1$ and $m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$. It is easy to observe that

$$\mathcal{D}^m_{\lambda,\mu}(\alpha_1, \beta_1)f(z) = z + \sum_{n=2}^{\infty}[1+(n-1)(\lambda-\mu+n\mu\lambda)]^m \frac{(\alpha_1)_{n-1}(\alpha_2)_{n-1}\ldots(\alpha_q)_{n-1}}{(\beta_1)_{n-1}(\beta_2)_{n-1}\ldots(\beta_s)_{n-1}(n-1)!} a_n z^n.$$

For brevity let us take

$$B_n = \frac{(\alpha_1)_{n-1}(\alpha_2)_{n-1}\ldots(\alpha_q)_{n-1}}{(\beta_1)_{n-1}(\beta_2)_{n-1}\ldots(\beta_s)_{n-1}(n-1)!}. \tag{1.3}$$

Hence

$$\mathcal{D}^m_{\lambda,\mu}(\alpha_1, \beta_1)f(z) = z + \sum_{n=2}^{\infty}[1+(n-1)(\lambda-\mu+n\mu\lambda)]^m B_n a_n z^n.$$

This operator $\mathcal{D}^m_{\lambda,\mu}(\alpha_1, \beta_1)f(z)$ generalizes several earlier operators for proper choices of the parameters. For $\mu = 0$, we find $\mathcal{D}^m_{\lambda,0}(\alpha_1, \beta_1)f(z)$ reduces to the operator introduced and studied by Selvaraj et al., [27]. For $q = 2, s = 1, \alpha_1 = \beta_1$

we see that this operator reduces to the operator introduced and studied by Dorina Răducanu et al., [4]. For $\lambda = 1, \mu = 0, q = 2, s = 1, \alpha_1 = \beta_1$ we obtain the differential operator defined by Al Oboudi [16]. For $\lambda = \mu = 0$ we obtain Dziok-Srivatsava operator [5].

Further by specializing the parameters we can find Ruscheweyh derivative operator [25], Carlson-Shaffer operator [3], fractional calculus operators [18, 19], Hohlov linear operator [8] and the generalized Bernardi-Libera-Livingston linear integral operator [2, 11, 13] and Sălăgean derivative operator [26].

Definition 1.1. Let $0 \leq \gamma \leq 1, \alpha \geq 1, k \geq 0$ and $0 \leq \beta < 1$. A function $f \in \mathcal{A}$ is said to be in the class $S(\lambda, \mu, m, \gamma, \alpha, k, \beta)$, if it satisfies

$$\Re\left\{\alpha \frac{zG'(z)}{G(z)} - (\alpha - 1)\right\} > k\left|\alpha \frac{zG'(z)}{G(z)} - \alpha\right| + \beta \tag{1.4}$$

where

$$G(z) = (1-\gamma)\mathcal{D}^m_{\lambda,\mu}(\alpha_1, \beta_1)f(z) + \gamma z[\mathcal{D}^m_{\lambda,\mu}(\alpha_1, \beta_1)f(z)]' \tag{1.5}$$

Also we define $TS(\lambda, \mu, m, \gamma, \alpha, k, \beta) = \mathcal{T} \cap S(\lambda, \mu, m, \gamma, \alpha, k, \beta)$.

By specializing the parameters involved in $S(\lambda, \mu, m, \gamma, \alpha, k, \beta)$ and $TS(\lambda, \mu, m, \gamma, \alpha, k, \beta)$ one could result in known classes of analytic functions which were studied earlier such as Starlike functions, parabolic starlike functions and k-starlike functions. Further several new subclasses of analytic functions could be defined by specializing the parameters involved.

In this investigation various properties of the functions belonging to the classes $S(\lambda, \mu, m, \gamma, \alpha, k, \beta)$ and $TS(\lambda, \mu, m, \gamma, \alpha, k, \beta)$.

2. Coefficient estimates

Lemma 2.1. [4] Let β be a real number and let w be a complex number. Then $\Re w \geq \beta$ if and only if

$$|w + (1-\beta)| - |w - (1+\beta)| \geq 0.$$

Theorem 2.2. Let $f(z) \in \mathcal{A}$ as given by (1.1). If

$$\sum_{n=2}^{\infty} [1 - \beta + \alpha(n-1)(1+k)]B_n|a_n| \leq 1 - \beta \tag{2.1}$$

then $f \in S(\lambda, \mu, m, \gamma, \alpha, k, \beta)$.

Proof. It is sufficient to show that

$$\left|\alpha\frac{zG'(z)}{G(z)}-(\alpha-1)-k\left|\alpha\frac{zG'(z)}{G(z)}-\alpha\right|-(1+\beta)\right|$$
$$\leq\left|\alpha\frac{zG'(z)}{G(z)}-(\alpha-1)-k\left|\alpha\frac{zG'(z)}{G(z)}-\alpha\right|+(1-\beta)\right|. \tag{2.2}$$

Consider

$$\left|\alpha\frac{zG'(z)}{G(z)}-(\alpha-1)-k\left|\alpha\frac{zG'(z)}{G(z)}-\alpha\right|+(1-\beta)\right|$$

$$= \frac{1}{G(z)}\left|\alpha zG'(z)-(\alpha-1)G(z)-ke^{i\theta}|\alpha zG'(z)-\alpha G(z)|+(1-\beta)G(z)\right|$$
$$> \frac{|z|}{|G(z)|}[2-\beta-\sum_{n=2}^{\infty}[2-\beta+\alpha(n-1)(1+k)]B_n|a_n|].$$

In similar manner

$$\left|\alpha\frac{zG'(z)}{G(z)}-(\alpha-1)-k\left|\alpha\frac{zG'(z)}{G(z)}-\alpha\right|-(1+\beta)\right|$$
$$< \frac{|z|}{|G(z)|}[\beta+\sum_{n=2}^{\infty}[\alpha(n-1)(1+k)-\beta]B_n|a_n|].$$

Therefore

$$\left|\alpha\frac{zG'(z)}{G(z)}-(\alpha-1)-k\left|\alpha\frac{zG'(z)}{G(z)}-\alpha\right|+(1-\beta)\right|$$
$$-\left|\alpha\frac{zG'(z)}{G(z)}-(\alpha-1)-k\left|\alpha\frac{zG'(z)}{G(z)}-\alpha\right|-(1+\beta)\right|$$
$$> \frac{|z|}{|G(z)|}[2(1-\beta)-2\sum_{n=2}^{\infty}[1-\beta+\alpha(n-1)(1+k)]B_n|a_n|\geq 0.$$

Hence the proof.

Theorem 2.3. If $f\in\mathcal{T}$ as given in (1.2), then $f\in TS(\lambda,\mu,m,\gamma,k,\alpha_1,\beta_1)$ if and only if

$$\sum_{n=2}^{\infty}[1-\beta+\alpha(n-1)(k+1)]B_n|a_n|\leq 1-\beta. \tag{2.3}$$

The result is sharp.

Proof. Assume that the condition (2.3) holds. In view of Theorem 2.2 and by the definition of $TS(\lambda, \mu, m, \gamma, k, \alpha_1, \beta_1)$, we see that $f \in TS(\lambda, \mu, m, \gamma, k, \alpha_1, \beta_1)$. Conversely suppose that $f \in TS(\lambda, \mu, m, \gamma, k, \alpha_1, \beta_1)$, then (1.4) reduces to

$$\frac{1 - \sum_{n=2}^{\infty}[1 + \alpha(n-1)]B_n a_n z^{n-1}}{1 - \sum_{n-2}^{\infty} B_n a_n z^{n-1}} - \beta > k \left| \frac{\sum_{n=2}^{\infty} \alpha(n-1) B_n a_n z^{n-1}}{1 - \sum_{n=2}^{\infty} B_n a_n z^{n-1}} \right|. \qquad (2.4)$$

By letting $z \to 1^-$ through real axis we get the desired result.

Also the result is sharp for the functions given by

$$f(z) = z - \frac{1-\beta}{[1 - \beta + \alpha(n-1)(1+k)]B_n} z^n \qquad \text{for} \quad n \geq 2. \qquad (2.5)$$

Corollary 2.4. If $f \in TS(\lambda, \mu, m, \gamma, k, \alpha_1, \beta_1)$, then

$$a_n \leq \frac{1-\beta}{[1 - \beta + \alpha(n-1)(1+k)]B_n} \qquad \text{for} \quad n \geq 2.$$

3. Growth and Distortion Theorem

Theorem 3.1. Let $f \in TS(\lambda, \mu, m, \gamma, k, \alpha_1, \beta_1)$. Then for $|z| < 1$,

$$r - \frac{1-\beta}{[1 - \beta + \alpha(1+k)]B_2} r^2 \leq |f(z)| \leq r + \frac{1-\beta}{[1 - \beta + \alpha(1+k)]B_2} r^2$$

and

$$1 - \frac{2(1-\beta)}{[1 - \beta + \alpha(1+k)]B_2} r \leq |f'(z)| \leq 1 + \frac{1-\beta}{[1 - \beta + \alpha(1+k)]B_2} r.$$

The result is sharp for the function $f(z)$ given by

$$f(z) = z - \frac{1-\beta}{[1 - \beta + \alpha(1+k)]B_2} z^2.$$

Proof. By Theorem(2.3) we have for $f(z) \in TS(\lambda, \mu, m, \gamma, k, \alpha_1, \beta_1)$,

$$\sum_{n=2}^{\infty} [1 - \beta + \alpha(n-1)(k+1)]B_n a_n \leq 1 - \beta.$$

Note that

$$[1 - \beta + \alpha(k+1)]B_2 \sum_{n=2}^{\infty} a_n = \sum_{n=2}^{\infty} [1 - \beta + \alpha(k+1)]B_2 a_n$$

$$\leq \sum_{n=2}^{\infty}[1-\beta+\alpha(n-1)(k+1)]B_n a_n \leq 1-\beta.$$

Hence

$$\sum_{n=2}^{\infty} a_n \leq \frac{1-\beta}{[1-\beta+\alpha(1+k)]B_2}. \tag{3.1}$$

Therefore

$$\begin{aligned} |f(z)| &\leq r+\sum_{n=2}^{\infty} a_n r^n \\ &\leq r+r^2\sum_{n=2}^{\infty} a_n \\ &\leq r+\frac{1-\beta}{[1-\beta+\alpha(1+k)]B_2}r^2 \end{aligned}$$

and

$$\begin{aligned} |f(z)| &\geq r-\sum_{n=2}^{\infty} a_n r^n \\ &\geq r-r^2\sum_{n=2}^{\infty} a_n \\ &\geq \frac{1-\beta}{[1-\beta+\alpha(1+k)]B_2}r^2. \end{aligned}$$

In view of Theorem 2.3 we have,

$$\begin{aligned} \frac{[1-\beta+\alpha(1+k)]B_2}{2}\sum_{n=2}^{\infty} n a_n &= \sum_{n=2}^{\infty}\frac{[1-\beta+\alpha(1+k)]B_2}{2} n a_n \\ &\leq \sum_{n=2}^{\infty}\frac{[1-\beta+\alpha(n-1)(1+k)]B_n n a_n}{n} \\ &\leq 1-\beta. \end{aligned}$$

Therefore

$$\begin{aligned} |f'(z)| &\leq 1+\sum_{n=2}^{\infty} n a_n r^{n-1} \\ &\leq 1+r\sum_{n=2}^{\infty} n a_n \\ &\leq 1+\frac{2(1-\beta)}{[1-\beta+\alpha(1+k)]B_2}r. \end{aligned}$$

Similarly we can prove

$$|f'(z)| \geq 1 - \frac{2(1-\beta)}{[1-\beta+\alpha(1+k)]B_2} r.$$

4. Extreme points

Theorem 4.1. Let

$$f_1(z) := z \qquad \text{and} \qquad f_n(z) := z - \frac{1-\beta}{[1-\beta+\alpha(n-1)(1+k)]B_n} z^n. \tag{4.1}$$

Then $f \in TS(\lambda, \mu, m, \gamma, k, \alpha_1, \beta_1)$ if and only if

$$f(z) = \sum_{n=1}^{\infty} \lambda_n f_n(z) \qquad (z \in U). \tag{4.2}$$

where $\lambda_n \geq 0$, $(n \geq 1)$ and $\sum_{n=1}^{\infty} \lambda_n = 1$. Also the extreme points of $TS(\lambda, \mu, m, \gamma, k, \alpha_1, \beta_1)$ are given by (4.1).

Proof. Suppose (4.2) holds for $f \in TS(\lambda, \mu, m, \gamma, k, \alpha_1, \beta_1)$, then

$$f(z) = z + \sum_{n=2}^{\infty} \lambda_n \frac{1-\beta}{[1-\beta+\alpha(n-1)(1+k)]B_n} z^n.$$

Since

$$\begin{aligned} \sum_{n=2}^{\infty} \frac{[1-\beta+\alpha(n-1)(1+k)]B_n \lambda_n (1-\beta)}{[1-\beta+\alpha(n-1)(1+k)]B_n} &= (1-\beta)\sum_{n=2}^{\infty} \lambda_n \\ &= (1-\beta)(1-\lambda_1) \\ &\leq 1-\beta, \end{aligned}$$

we have $f \in TS(\lambda, \mu, m, \gamma, k, \alpha_1, \beta_1)$.

Conversely, suppose that $f \in TS(\lambda, \mu, m, \gamma, k, \alpha_1, \beta_1)$ and take

$$\begin{aligned} \lambda_n &= \frac{[1-\beta+\alpha(n-1)(1+k)]B_n}{1-\beta} a_n \qquad \text{for} \qquad n \geq 2 \\ \text{and} \qquad \lambda_1 &= 1 - \sum_{n=2}^{\infty} \lambda_n. \\ \text{Then} \qquad f(z) &= \sum_{n=1}^{\infty} \lambda_n f_n(z). \end{aligned}$$

Hence the proof.

5. Closure Theorem

Theorem 5.1. Let the functions $f_j \in TS(\lambda, \mu, m, \gamma, k, \alpha_1, \beta_1)$ be defined by

$$f_j(z) = z - \sum_{n=2}^{\infty} a_{n,j} z^n \qquad (z \in U) \tag{5.1}$$

and let $c_j \geq 0 \quad (j = 1, 2, \ldots, p)$ such that $\sum_{j=1}^{p} c_j = 1$. Then $h(z) = \sum_{j=1}^{p} c_j f_j(z) \in TS(\lambda, \mu, m, \gamma, k, \alpha_1, \beta_1)$.

Proof. Now $h(z)$ can be written as

$$\begin{aligned} h(z) &= \sum_{j=1}^{p} c_j \left[z - \sum_{n=2}^{\infty} a_{n,j} z^n \right] \\ &= z - \sum_{n=2}^{\infty} \left[\sum_{j=1}^{p} c_j a_{n,j} \right] z^n. \end{aligned}$$

Since $f_j \in TS(\lambda, \mu, m, \gamma, k, \alpha_1, \beta_1)$ for every $j = 1, 2, \ldots, p$ we have

$$\sum_{n=2}^{\infty} [1 - \beta + \alpha(n-1)(k+1)] B_n a_{n,j} \leq 1 - \beta.$$

Therefore

$$\sum_{n=2}^{\infty} [1 - \beta + \alpha(n-1)(k+1)] B_n \left[\sum_{j=1}^{p} c_j a_{n,j} \right] \leq \sum_{j=1}^{p} c_j (1 - \beta) = 1 - \beta.$$

Hence $h \in TS(\lambda, \mu, m, \gamma, k, \alpha_1, \beta_1)$.

Corollary 5.2. The class $TS(\lambda, \mu, m, \gamma, k, \alpha_1, \beta_1)$ is closed under convex linear combination.

6. Convolution and Integral properties

Theorem 6.1. Let $g(z) \in \mathcal{T}$ of the form

$$g(z) = z - \sum_{n=2}^{\infty} b_n z^n \qquad (0 \leq b_n \leq 1 \quad \text{for} \quad n \geq 2)$$

be analytic in U. If the function $f \in TS(\lambda, \mu, m, \gamma, k, \alpha_1, \beta_1)$, then $(f * g)(z) \in TS(\lambda, \mu, m, \gamma, k, \alpha_1, \beta_1)$, where $*$ denotes the modified Hadamard product.

Proof. Consider

$$\sum_{n=2}^{\infty}[1-\beta+\alpha(n-1)(k+1)]B_n a_n b_n$$

$$\leq \sum_{n=2}^{\infty}[1-\beta+\alpha(n-1)(k+1)]B_n a_n \leq 1-\beta$$

and hence $f * g \in TS(\lambda, \mu, m, \gamma, k, \alpha_1, \beta_1)$.

Definition 6.1. Let $I_c : \mathcal{T} \to \mathcal{T}$ be an integral operator defined as

$$I_c(f(z)) = \frac{c+1}{z^c}\int_0^z t^{c-1} f(t)dt \qquad (c > -1, z \in U). \tag{6.1}$$

Note that for $f(z)$ given by (1.2),

$$I_c(f(z)) = z - \sum_{n=2}^{\infty}\frac{c+1}{c+n}a_n z^n.$$

By taking $g(z) = z - \sum_{n=2}^{\infty}\frac{c+1}{c+n}z^n$ where $0 \leq \frac{c+1}{c+n} \leq 1$ in Theorem 6.1, we have the following result.

Corollary 6.2. If the function $f \in TS(\lambda, \mu, m, \gamma, k, \alpha_1, \beta_1)$ then $I_c(f(z)) \in TS(\lambda, \mu, m, \gamma, k, \alpha_1, \beta_1)$.

7. Radii of Starlikeness, Convexity and Close to convexity

Theorem 7.1. Let the function $f \in TS(\lambda, \mu, m, \gamma, k, \alpha_1, \beta_1)$. Then f is starlike of order $\rho(0 \leq \rho < 1)$ in $|z| < r_1$ where

$$r_1 = \inf_{n\geq 2}\left[\frac{(1-\rho)[1-\beta+\alpha(n-1)(k+1)]B_n}{(n-\rho)(1-\beta)}\right]^{\frac{1}{n-1}}.$$

Proof. To prove the result we have to show that

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq 1-\rho \qquad (0 \leq \rho \leq 1)$$

for $z \in U$ with $|z| < r_1$. We have $\left|\frac{zf'(z)}{f(z)} - 1\right| \leq 1-\rho$ if $\sum_{n=2}^{\infty}\frac{n-\rho}{1-\rho}a_n z^{n-1} \leq 1$. By (2.3) we have

$$\sum_{n=2}^{\infty}\frac{[1-\beta+\alpha(n-1)(k+1)]B_n|a_n|}{1-\beta} \leq 1.$$

Hence the result will follow if

$$\begin{aligned} \frac{n-\rho}{1-\rho}|z|^{n-1} &\leq \frac{[1-\beta+\alpha(n-1)(k+1)]B_n|a_n|}{1-\beta} \\ \text{or if} \quad |z| &\leq \left[\frac{(1-\rho)[1-\beta+\alpha(n-1)(k+1)]B_n}{(n-\rho)(1-\beta)}\right]^{\frac{1}{n-1}}. \end{aligned}$$

Corollary 7.2. Let the function $f \in TS(\lambda, \mu, m, \gamma, k, \alpha_1, \beta_1)$. Then f is convex of order $\rho(0 \leq \rho < 1)$ in $|z| < r_2$ where

$$r_2 = \inf_{n\geq 2}\left[\frac{(1-\rho)[1-\beta+\alpha(n-1)(k+1)]B_n}{n(n-\rho)(1-\beta)}\right]^{\frac{1}{n-1}}.$$

Corollary 7.3. Let the function $f \in TS(\lambda, \mu, m, \gamma, k, \alpha_1, \beta_1)$. Then f is close to covex of order $\rho(0 \leq \rho < 1)$ in $|z| < r_3$ where

$$r_3 = \inf_{n\geq 2}\left[\frac{(1-\rho)[1-\beta+\alpha(n-1)(k+1)]B_n}{n(1-\beta)}\right]^{\frac{1}{n-1}}.$$

References

[1] O. P. Ahuja, G. Murugusundaramoorthy, N. Magesh, *Integral Means for uniformly convex and starlike functions associated with generalized hypergeometric functions*, J. Inequal. Pure Appl. Math. 8(4)(2007), art. 118,9pp.

[2] S. D. Bernardi, *Convex and starlike univalent functions*, Trans. Amer. Math. Soc. 135(1969), 429-446.

[3] B. C. Carlson, D. B. Shaffer, *Starlike and prestarlike hypergeometric fucntions*, SIAM. J. Math. Anal. 15(4)(1984), 737-745.

[4] Dorina Răducanu, Halit Orhan, *Subclasses of analytic functions defined by a generalized differential operator*, Intern. J. Math. Anal. 4(1)(2010), 1-15.

[5] J. Dziok, H. M. Srivatsava, *Classes of analytic functions associated with the generalized hypergeometric function*, Appl. Math. Comput. 103(1)(1999), 1- 13.

[6] A. W. Goodman, *On uniformly convex functions*, Ann. Polon. Math. 56(1)(1991), 87 - 92.

[7] A. W. Goodman, *On uniformly starlike functions*, J. Math. Anal. Appl. 155(2)(1991), 364-370.

[8] Ju. E. Hohlov, *Operators and operations on the class of univalent functions*, Izv. Vyssh. Uchebn. Zaved. Math., 10(197)(1978), 83-89.

[9] S. Kanas, H. M. Srivatsava, *Linear operators associated with k-uniformly convex functions*, Int. Transf. Spec. Funct. 9(2)(2000), 121-132.

[10] S. Kanas, A. Wisniowska, *Conic domains and starlike functions*, Rev. Roumaine Math. Pures. Appl. 45(2000), 647 - 657.

[11] R. J. Libera, *Some classes of regular univalent functons*, Proc. Amer. Math. Soc. 16(1965), 755-758.

[12] J. E. Littlewood, *On inequalities in the theory of functions*, Proc. London Math. Soc., 23(1925) 481 - 519.

[13] A. E. Livingston, *On the radius of univalence of certain analytic function*, Proc. Amer. Math. Soc. 17(1966), 352-357.

[14] W. Ma, D. Minda, *Uniformly convex functions*, Ann. Polon. Math. 57(1992), 165-175.

[15] W. Ma, D. Minda, *A Unified treatment of some special classes of univalent functions*, Z. Li, F. Ren, L. Yang, S.Zhang(Eds.) Proceedings of the conference on Complex Analysis(Cambridge, Massachusetts: International press) (1994), 157-169.

[16] F. M. Al-Oboudi, *On univalent functions defined by a generalized Sălăgean operator*, Int. J. Math. Math. Sci. 27(2004), 1429-1436.

[17] S, Owa, Y. Polatoglu, E. Yavuz, *Coefficient inequalities for classes of uniformly starlike and convex functions*, J. Inequal. Pure. Appl. Math. 7(5)(2006), Art 160.

[18] S. Owa, *On the distortion theorems I*, Kyungpook. Math. J. 18(1978), 53-59.

[19] S. Owa, H. M. Srivatsava,*Univalent and starlike generalized hypergeometric functions*, Canad. J. Math. 39(5)(1987), 1057-1077.

[20] R. K. Raina, D. Bansal,*Some properties of a new class of analytic functions defined in terms of a Hadamard product*, J. Inequql. Pure Appl. Math. 9(1)(2008)Art. 118.

[21] C. Ramachandran, T. N. Shanmugam, H. M. Srivatsava, A. Swaminathan, *A Unified class of k- uniformly convex functions defined by Dziok-Srivatsava linear operator*, Appl. Math. Comput. 190(2007), 1627 - 1636.

[22] F. Rønning, *Uniformly convex functions and a corresponding class of starlike functions*, Proc. Amer. Math. Soc. 118(1)(1993), 189-196.

[23] F. Rønning,*On starlike functions associated with parabolic regions*, Ann. Univ. Mariae Curie-Sklodowska Sect. 45(1991), 117-122.

[24] T. Rosy, K. G. Subramanian, G. Murugusundaramoorthy, *Neighbourhoods and partial sums of starlike functions based on Ruscheweyh derivatives*, J. Inequal. Pure Appl. Math. 4(4)(2003), Art. 64.

[25] S. Ruscheweyh, *New Criteria for univalent functions*, Proc. Amer. Math. Soc. 49(1975), 109-115.

[26] G. Ş. Sălăgean, *Subclasses of univalent functions*, Complex analysis—fifth Romanian-Finnish seminar, Part 1 (Bucharest, 1981), 362–372, Lecture Notes in Math. 1013 Springer, Berlin.

[27] C. Selvaraj, K. R. Karthikeyan,*Subclasses of analytic functions involving a certain family of linear operators*, Int. J. Contemp. Math. Sci. 3(13)(2008), 615-627.

[28] S. Shams, S. R. Kulkarni, J. M. Jahangiri,*Classes of uniformly starlike and convex functions*, Internat. J. Math. Math. Sci. 55(2004), 2959-2961.

[29] H. Silverman, *Univalent functions with negative coefficients*, Proc. Amer. Math. Soc. 51(1975), 109-116.

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