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ON UNIVALENCE CRITERIA FOR ANALYTIC FUNCTIONS DEFINED BY SĂLĂGEAN OPERATOR AND RUSCHEWEYH DERIVATIVE

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Abstract. In this paper we obtain sufficient conditions for univalence of analytic functions defined by the linear operator $L_\alpha^n : \mathcal{A} \to \mathcal{A}$, $L_\alpha^n f(z) = (1-\alpha)R^n f(z) + \alpha S^n f(z)$, $z \in U$, where $R^n f(z)$ is the Ruscheweyh derivative, $S^n f(z)$ the Sălăgean operator and $\mathcal{A}_n = \{f \in \mathcal{H}(U) : f(z) = z + a_{n+1}z^{n+1} + \dots,\ z \in U\}$ is the class of normalized analytic functions with $\mathcal{A}_1 = \mathcal{A}$.

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1. Introduction

Denote by U the unit disc of the complex plane, $U = \{z \in \mathbb{C} : |z| < 1\}$ and $\mathcal{H}(U)$ the space of holomorphic functions in U.

Let $\mathcal{A}_n = \{f \in \mathcal{H}(U) : f(z) = z + a_{n+1}z^{n+1} + \dots,\ z \in U\}$ with $\mathcal{A}_1 = \mathcal{A}$ and $\mathcal{S}$ the subclass of functions that are univalent in U.

Definition 1. *Sălăgean [10]) For $f \in \mathcal{A}$, $n \in \mathbb{N}$, the operator S^n is defined by $S^n : \mathcal{A} \to \mathcal{A}$,*

$$\begin{aligned} S^0 f(z) &= f(z) \\ S^1 f(z) &= z f'(z) \\ &\dots \\ S^{n+1} f(z) &= z\left(S^n f(z)\right)', \quad z \in U. \end{aligned}$$

Remark 1. *If $f \in \mathcal{A}$, $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$, then $S^n f(z) = z + \sum_{j=2}^{\infty} j^n a_j z^j$, $z \in U$.*

Definition 2. *(Ruscheweyh [9]) For $f \in \mathcal{A}$, $n \in \mathbb{N}$, the operator R^n is defined by $R^n : \mathcal{A} \to \mathcal{A}$,*

$$\begin{aligned} R^0 f(z) &= f(z) \\ R^1 f(z) &= z f'(z) \\ &\ldots \\ (n+1) R^{n+1} f(z) &= z (R^n f(z))' + n R^n f(z), \quad z \in U. \end{aligned}$$

Remark 2. *If $f \in \mathcal{A}$, $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$, then $R^n f(z) = z + \sum_{j=2}^{\infty} \frac{(n+j-1)!}{n!(j-1)!} a_j z^j$, $z \in U$.*

Definition 3. *[1], [2] Let $\alpha \geq 0$, $n \in \mathbb{N}$. Denote by L_γ^n the operator given by $L_\gamma^n : \mathcal{A} \to \mathcal{A}$,*

$$L_\alpha^n f(z) = (1-\alpha) R^n f(z) + \alpha S^n f(z), \qquad z \in U.$$

Remark 3. *If $f \in \mathcal{A}$, $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$, then*
$L_\alpha^n f(z) = z + \sum_{j=2}^{\infty} \left\{ \alpha j^n + (1-\alpha) \frac{(n+j-1)!}{n!(j-1)!} \right\} a_j z^j$, $z \in U$.

Our considerations are based on the following results.

Lemma 1. *[4] Let $f \in \mathcal{A}$. If for all $z \in U$*

$$\left(1 - |z|^2\right) \left| \frac{z f''(z)}{f'(z)} \right| \leq 1,$$

then the function f is univalent in U.

Lemma 2. *[7] Let $f \in \mathcal{A}$. If for all $z \in U$*

$$\left| \frac{z^2 f'(z)}{f^2(z)} - 1 \right| \leq 1,$$

then the function f is univalent in U.

Lemma 3. *[11] Let μ be a real number, $\mu > \frac{1}{2}$ and $f \in \mathcal{A}$. If for all $z \in U$*

$$\left| \left(1 - |z|^{2\mu}\right) \frac{z f''(z)}{f'(z)} + 1 - \mu \right| \leq \mu,$$

then the function f is univalent in U.

Lemma 4. *[6]If $f(z) \in \mathcal{S}$ and*

$$\frac{z}{f(z)} = 1 + \sum_{n=1}^{\infty} b_n z^n,$$

then

$$\sum_{n=1}^{\infty} (n-1) |b_n|^2 \leq 1.$$

Lemma 5. *[8] Let $\nu \in \mathbb{C}$, $Re(\nu) \geq 0$ and $f \in \mathcal{A}$. If for all $z \in U$*

$$\frac{\left(1 - |z|^{2Re(\nu)}\right)}{Re(\nu)} \left| \frac{z f''(z)}{f'(z)} \right| \leq 1,$$

then the function

$$F_\nu(z) = \left(\nu \int_0^z u^{\nu-1} f'(u)\, du \right)^{\frac{1}{\nu}}$$

is univalent in U.

2. The main result

Following the paper of M. Darus and R. Ibrahim [5], we establish the sufficient conditions to obtain a univalence for analytic function involving the differential operator $RD^n_{\lambda,\alpha}$.

Theorem 6. *Let $f \in \mathcal{A}$. If for all $z \in U$,*

$$\sum_{j=2}^{\infty} \left\{ \alpha j^n + (1-\alpha) \frac{(n+j-1)!}{n!\,(j-1)!} \right\} [j(2j-1)] |a_j| \leq 1. \quad (1)$$

Then $L^n_\alpha f(z)$ is univalent in U.

Proof. Let $f \in \mathcal{A}$. Assume that (1) is hold. Then for all $z \in U$ we have

$$\left(1 - |z|^2\right) \frac{z \left(L^n_\alpha f(z)\right)''}{\left(L^n_\alpha f(z)\right)'} \leq \left(1 + |z|^2\right) \left| \frac{z \left(L^n_\alpha f(z)\right)''}{\left(L^n_\alpha f(z)\right)'} \right| =$$

$$\left(1 + |z|^2\right) \left| \frac{z \sum_{j=2}^{\infty} \left\{ \alpha j^n + (1-\alpha) \frac{(n+j-1)!}{n!(j-1)!} \right\} j(j-1) a_j z^{j-2}}{\left(1 + \sum_{j=2}^{\infty} \left\{ \alpha j^n + (1-\alpha) \frac{(n+j-1)!}{n!(j-1)!} \right\} j a_j z^{j-1}\right)} \right| \leq$$

$$\frac{2 \sum_{j=2}^{\infty} \left\{ \alpha j^n + (1-\alpha) \frac{(n+j-1)!}{n!(j-1)!} \right\} j(j-1) |a_j|}{1 - \sum_{j=2}^{\infty} \left\{ \alpha j^n + (1-\alpha) \frac{(n+j-1)!}{n!(j-1)!} \right\} j |a_j|} \leq 1.$$

Thus, in view of Lemma 1, $L^n_\alpha f(z)$ is univalent in U.

Theorem 7. *Let $f \in \mathcal{A}$. If for all $z \in U$,*

$$\sum_{j=2}^{\infty}\left\{\alpha j^{n}+(1-\alpha)\frac{(n+j-1)!}{n!\,(j-1)!}\right\}|a_j| \leq \frac{1}{\sqrt{7}}. \tag{2}$$

Then $L_{\alpha}^{n}f(z)$ is univalent in U.

Proof. Let $f \in \mathcal{A}$. Assume that (2) is hold. It is sufficient to show that

$$\left|\frac{z^{2}\left(L_{\alpha}^{n}f(z)\right)'}{\left(L_{\alpha}^{n}f(z)\right)^{2}}-1\right| \leq 1,$$

which is equivalent to show that

$$\left|\frac{z^{2}\left(L_{\alpha}^{n}f(z)\right)'}{2\left(L_{\alpha}^{n}f(z)\right)^{2}}\right| \leq 1.$$

We have

$$\left|\frac{z^{2}\left(L_{\alpha}^{n}f(z)\right)'}{2\left(L_{\alpha}^{n}f(z)\right)^{2}}\right| = \left|\frac{z^{2}\left(1+\sum_{j=2}^{\infty}\left\{\alpha j^{n}+(1-\alpha)\frac{(n+j-1)!}{n!(j-1)!}\right\}ja_j z^{j-1}\right)}{2\left(z+\sum_{j=2}^{\infty}\left\{\alpha j^{n}+(1-\alpha)\frac{(n+j-1)!}{n!(j-1)!}\right\}a_j z^{j}\right)^{2}}\right| =$$

$$\left|\frac{1+\sum_{j=2}^{\infty}\left\{\alpha j^{n}+(1-\alpha)\frac{(n+j-1)!}{n!(j-1)!}\right\}ja_j z^{j-1}}{2\left(1+\sum_{j=2}^{\infty}\left\{\alpha j^{n}+(1-\alpha)\frac{(n+j-1)!}{n!(j-1)!}\right\}a_j z^{j-1}+\left(\sum_{j=2}^{\infty}\left\{\alpha j^{n}+(1-\alpha)\frac{(n+j-1)!}{n!(j-1)!}\right\}a_j z^{j-1}\right)^{2}\right)}\right|$$

$$\leq \frac{1+\sum_{j=2}^{\infty}\left\{\alpha j^{n}+(1-\alpha)\frac{(n+j-1)!}{n!(j-1)!}\right\}j\,|a_j|}{2\left(1-2\sum_{j=2}^{\infty}\left\{\alpha j^{n}+(1-\alpha)\frac{(n+j-1)!}{n!(j-1)!}\right\}|a_j|-\left(\sum_{j=2}^{\infty}\left\{\alpha j^{n}+(1-\alpha)\frac{(n+j-1)!}{n!(j-1)!}\right\}|a_j|^{2}\right)\right)}$$

which is less than 1 if the assertion (2) is hold. Thus in view of Lemma 2, $L_{\alpha}^{n}f(z)$ is univalent in U.

Theorem 8. *Let $f \in \mathcal{A}$. If for all $z \in U$*

$$\sum_{j=2}^{\infty} j\left[2(j-1)+(2\mu-1)\right]\left\{\alpha j^{n}+(1-\alpha)\frac{(n+j-1)!}{n!\,(j-1)!}\right\}|a_j| \leq 2\mu-1, \quad \mu>\frac{1}{2}. \tag{3}$$

Then $L_{\alpha}^{n}f(z)$ is univalent in U.

Proof. Let $f \in \mathcal{A}$. Then for all $z \in U$ we have

$$\left|\left(1-|z|^{2\mu}\right)\frac{z\left(L_{\alpha}^{n}f\left(z\right)\right)''}{\left(L_{\alpha}^{n}f\left(z\right)\right)'}+1-\mu\right|\leq\left(1+|z|^{2\mu}\right)\left|\frac{z\left(L_{\alpha}^{n}f\left(z\right)\right)''}{\left(L_{\alpha}^{n}f\left(z\right)\right)'}\right|+|1-\mu|\leq$$

$$\frac{2\sum_{j=2}^{\infty}\left\{\alpha j^{n}+(1-\alpha)\frac{(n+j-1)!}{n!(j-1)!}\right\}j\left(j-1\right)|a_j|}{1-\sum_{j=2}^{\infty}\left\{\alpha j^{n}+(1-\alpha)\frac{(n+j-1)!}{n!(j-1)!}\right\}j\,|a_j|}+|1-\mu|$$

the last inequality is less that μ if the assertion (3) is hold. thus, in view of Lemma 3, $L_{\alpha}^{n}f\left(z\right)$ is univalent in U.

As applications of Theorems 6, 7 and 8 we have the following result

Theorem 9. *Let $f \in \mathcal{A}$. If for all $z \in U$ one of the inequalities (1-3) holds, then*

$$\sum_{j=1}^{\infty}\left(j-1\right)|b_j|^2\leq 1,$$

where

$$\frac{z}{L_{\alpha}^{n}f\left(z\right)}=1+\sum_{j=1}^{\infty}b_j z^j.$$

Proof. Let $f \in \mathcal{A}$. Then, in view of Theorems 6, 7 or 8, $L_{\alpha}^{n}f\left(z\right)$ is univalent in U. Hence, by Lemma 4 we obtain the result.

Theorem 10. *Let $f \in \mathcal{A}$. If for all $z \in U$,*

$$\sum_{j=2}^{\infty}j\left[2\left(j-1\right)+Re\left(\nu\right)\right]\left\{\alpha j^{n}+(1-\alpha)\frac{(n+j-1)!}{n!\left(j-1\right)!}\right\}|a_j|\leq Re\left(\nu\right),\quad Re\left(\nu\right)>0. \tag{4}$$

Then

$$G_{\nu}\left(z\right)=\left(\nu\int_0^z u^{\nu-1}\left(L_{\alpha}^{n}f\left(u\right)\right)'du\right)^{\frac{1}{\nu}}$$

is univalent in U.

Proof. Let $f \in \mathcal{A}$. Then for all $z \in U$, we have

$$\frac{\left(1-|z|^{2Re(\nu)}\right)}{Re\left(\nu\right)}\left|\frac{z\left(L_{\alpha}^{n}f\left(z\right)\right)''}{\left(L_{\alpha}^{n}f\left(z\right)\right)'}\right|\leq\frac{\left(1+|z|^{2Re(\nu)}\right)}{Re\left(\nu\right)}\left|\frac{z\left(L_{\alpha}^{n}f\left(z\right)\right)''}{\left(L_{\alpha}^{n}f\left(z\right)\right)'}\right|\leq$$

$$\frac{2}{Re\left(\nu\right)}\frac{\sum_{j=2}^{\infty}\left\{\alpha j^{n}+(1-\alpha)\frac{(n+j-1)!}{n!(j-1)!}\right\}j\left(j-1\right)|a_j|}{1-\sum_{j=2}^{\infty}\left\{\alpha j^{n}+(1-\alpha)\frac{(n+j-1)!}{n!(j-1)!}\right\}j\,|a_j|}.$$

The last inequality is less than 1 if the assertion (4) is hold. Thus, in view of Lemma 5, $G_\nu(z)$ is univalent.

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