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ON CLASS OF NONHOMOGENEOUS DISCRETE DIRICHLET PROBLEMS

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Abstract. This work deals with the nonhomogeneous discrete Dirichlet eigenvalue problem

$$-\Delta\Big(\Big(|\Delta u(k-1)|^{p_1(k-1)-2}+|\Delta u(k-1)|^{p_2(k-1)-2}\Big)\Delta u(k-1)\Big) = \lambda|u|^{q(k)-2}u, \quad k \in [1,T].$$

Through an approach based on variational methods and critical points, intervals of a numerical parameter λ are derived for which the existence and nonexistence results are obtained.

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1. Introduction

In the recent mathematical literature a great deal of work has been devoted to the study of discrete boundary value problems. The studies of such kind of problems can be placed at the interface of certain mathematical fields, such as nonlinear differential equations and numerical analysis. More, they are strongly motivated by their applicability to various fields of research, such as computer science, mechanical engineering, control systems, artificial or biological neural networks, economics and many others. For this reasons, in these last years, many authors have widely developed various methods and techniques, such as fixed points theorems, lower and upper solutions, and Brower degree. Very recently, also the critical point theory has aroused the attention of many authors in the study of these problems. Far from being exhaustive, further details can be found in [1, 3, 4, 8, 11, 12, 14] and the reference therein.

Let $T \geq 2$ be a positive integer, $[a, b]$ be the discrete interval $\{a, a+1, ..., b\}$ where a and b are integers and $a < b$. and λ be a positive parameter. The main aim of this paper is to study the existence of solutions for following discrete boundary value problem

$$\begin{gathered} -\Delta\Big(\Big(|\Delta u(k-1)|^{p_1(k-1)-2}+|\Delta u(k-1)|^{p_2(k-1)-2}\Big)\Delta u(k-1)\Big)=\lambda|u|^{q(k)-2}u, \\ k\in[1,T], u(0)=u(T+1)=0, \end{gathered} \tag{1}$$

where $p_1, p_2 : [0,T] \to [2,\infty)$ and $q : [1,T] \to [2,\infty)$ are bounded functions, $\Delta u(k) = u(k+1) - u(k)$ is the forward difference operator while λ is a positive real number.

Continuous versions of problems such as (1) are known to be mathematical models of various phenomena arising in the study of elastic mechanics [18], electrorheological fluids [15] or image restoration [5]. Variational continuous anisotropic problems have been started by Fan and Zhang in [7] and later considered by many methods and authors, see [10] for an extensive survey of such boundary value problems.

However, to the best of our knowledge, discrete problems like (1) involving anisotropic exponents have been discussed for the first time by Mihailescu, Radulescu and Tersian [14] and for the second time by Koné and Ouaro [12], where known tools from the critical point theory are applied in order to get the existence of solutions.

From now onwards, we will use the following notations:

$$\begin{aligned} &p_{\min}(k) := \min_{i=1,2} p_i(k), \quad p_{\max}(k) := \max_{i=1,2} p_i(k), \quad \text{for all } k \in [0,T]; \\ &p^-_{\min} := \min_{k\in[0,T]} p_{\min}(k), \quad p^+_{\max} := \min_{k\in[0,T]} p_{\max}(k); \\ &p^-_i := \min_{k\in[0,T]} p_i(k), \quad p^+_i := \max_{k\in[0,T]} p_i(k), \quad \text{for } i = 1,2; \\ &q^- := \min_{k\in[1,T]} q(k) \quad \text{and } q^+ := \max_{k\in[1,T]} q(k) \end{aligned}$$

Inspired by the paper [14] and the ideas introduced in [8, 13], our analysis mainly concern existence and nonexistence of weak solutions to problem (1), under appropriate assumptions. More precisely, we aim to provide intervals for a parameter λ for which problem (1) has, or not, non trivial solutions. For these results we use some known tools such as a direct variational method, mountain pass geometry and Ekeland's variational principle.

Firstly, in the situation when $p^+_{\min} < q^- \le q^+ < p^-_{\max}$ and following along the same lines as in [13], we will prove the existence of two positive constants λ_* and $\widehat{\lambda}$ with $\lambda_* < \widehat{\lambda}$ such that any $\lambda \in [\widehat{\lambda}, +\infty)$ is an eigenvalue, while and $\lambda \in (0, \lambda_*)$ is not an eigenvalue of our problem.

Secondly, in the case $q^- < p^-_{\min}$ or $q^- > p^+_{\max}$, and by applying mountain pass Lemma and Ekeland's variational principle, we will show that there exists a positive constant λ_* such that any $\lambda \in (0, \lambda_*)$ is an eigenvalue of problem (1).

The remaining part of this article is organized as follows. The second section is devoted to mathematical preliminaries and statement of main results which are proved in the third section.

2. Preliminaries and statement of main results

Solutions to (1) will be investigated in a space

$$W = \Big\{u : [0, T+1] \to \mathbb{R} \text{ s.t } u(0) = u(T+1) = 0\Big\},$$

which is a T-dimensional Hilbert space, see [2], with the inner product

$$(u, v) = \sum_{k=1}^{T+1} \Delta u(k-1)\Delta v(k-1), \qquad \text{for all } u, v \in W.$$

The associated norm is defined by

$$\|u\| = \Big(\sum_{k=1}^{T+1} |\Delta u(k-1)|^2\Big)^{\frac{1}{2}}.$$

Moreover, it is useful to introduce other norms on W, namely

$$|u|_m = \Big(\sum_{k=1}^{T} |u(k)|^m\Big)^{\frac{1}{m}}, \quad \forall u \in W \text{ and } m \geq 2.$$

It can be verified (see [4]) that

$$T^{\frac{2-m}{2m}} |u|_2 \leq |u|_m \leq T^{\frac{1}{m}} |u|_2, \quad \forall u \in W \text{ and } m \geq 2. \tag{2}$$

Next, we list some inequalities that will be are used later. For $(a)-(c)$ see [8, 14], for (d) and (e) see [8] and for (g) see [17].

Lemma 1. *For every $u \in W$, we have*

(a) $$\sum_{k=1}^{T} |u(k)|^m \leq T(T+1)^{m-1} \sum_{k=1}^{T+1} |\Delta u(k-1)|^m, \quad \forall m \geq 2.$$

(b) $$\sum_{k=1}^{T+1} |\Delta u(k-1)|^{p(k-1)} \geq T^{\frac{2-p^-}{2}} \|u\|^{p^-} - (T+1), \quad \text{with } \|u\| > 1.$$

(c) $\sum_{k=1}^{T+1} |\Delta u(k-1)|^{p(k-1)} \geq T^{\frac{p^+-2}{2}} \|u\|^{p^+}$, *with* $\|u\| < 1$.

(d) $(T+2)^{\frac{2-m}{2}} \|u\|^m \leq \sum_{k=1}^{T+1} |\Delta u(k-1)|^m \leq (T+1)\|u\|^m, \quad \forall m \geq 2.$

(e) $\sum_{k=1}^{T+1} |\Delta u(k-1)|^{p(k-1)} \leq (T+1)\|u\|^{p^+} + (T+1).$

(f) $\sum_{k=1}^{T+1} |\Delta u(k-1)|^{p(k-1)} \leq 2^m \sum_{k=1}^{T} |u(k)|^m, \quad \forall m \geq 2.$

(g) $\max_{k\in[1,T]} |u(k)| \leq (T+1)^{\frac{1}{q}} \Big(\sum_{k=1}^{T+1} |\Delta u(k-1)|^p \Big)^{\frac{1}{p}}$, *for all* $p, q > 1$ *such that*

$$\frac{1}{p} + \frac{1}{q} = 1.$$

We say that $\lambda \in \mathbb{R}$ is an eigenvalue of problem (1) if there exists $u \in W \setminus \{0\}$ such that

$$\sum_{k=1}^{T+1} \Big(|\Delta u(k-1)|^{p_1(k-1)-2} + |\Delta u(k-1)|^{p_2(k-1)-2} \Big) \Delta u(k-1) \Delta\varphi(k-1)$$
$$= \lambda \sum_{k=1}^{T} |u(k)|^{q(k)-2} u(k)\varphi(k),$$

for any $\varphi \in W$. We point out that if λ is an eigenvalue of problem (1), then the corresponding eigenfunction $u \in W \setminus \{0\}$ is a weak solution of problem (1).

Define

$$\widehat{\lambda} = \inf_{u\in W\setminus\{0\}} \frac{J(u)}{I(u)},$$

where

$$J(u) = \sum_{k=1}^{T+1} \Big(\frac{1}{p_1(k-1)} |\Delta u(k-1)|^{p_1(k-1)} + \frac{1}{p_2(k-1)} |\Delta u(k-1)|^{p_2(k-1)} \Big)$$

and

$$I(u) = \sum_{k=1}^{T} \frac{1}{q(k)} |u(k)|^{q(k)}.$$

3. Main results and Proofs

Theorem 2. *If*

$$p^+_{\min} < q^- \leq q^+ < p^-_{\max}, \tag{3}$$

then $\widehat{\lambda} > 0$. *Moreover, any* $\lambda \in [\widehat{\lambda}, +\infty)$ *is an eigenvalue of problem* (1). *Furthermore, there exists a positive constant* λ_* *such that* $\lambda_* \leq \widehat{\lambda}$ *and any* $\lambda \in (0, \lambda_*)$ *is not an eigenvalue of problem* (1).

Proof. Our proof, follows as [13], is divide in a series of steps in order to clarify the exposition.

Step 1 we show $\widehat{\lambda} > 0$.

By (3), for all $k \in [1, T]$ and $u \in W$, we point out that

$$|\Delta u(k)|^{q^+} + |\Delta u(k)|^{q^-} \leq 2\big(|\Delta u(k)|^{p_1(k)} + |\Delta u(k)|^{p_2(k)}\big),$$

and

$$|u(k)|^{q(k)} \leq |u(k)|^{q^+} + |u(k)|^{q^-}.$$

Thus, summing for k from 1 to T, for $u \in W$ we obtain

$$\sum_{k=1}^{T} \Big(|\Delta u(k)|^{q^+} + |\Delta u(k)|^{q^-}\Big) \leq 2 \sum_{k=1}^{T} \Big(|\Delta u(k)|^{p_1(k)} + |\Delta u(k)|^{p_2(k)}\Big), \tag{4}$$

and

$$\sum_{k=1}^{T} |u(k)|^{q(k)} \leq \sum_{k=1}^{T} \Big(|u(k)|^{q^+} + |u(k)|^{q^-}\Big). \tag{5}$$

Combining (5) and Lemma 1 (a), we have

$$\begin{aligned}\sum_{k=1}^{T} |u(k)|^{q(k)} &\leq \sum_{k=1}^{T} \Big(|u(k)|^{q^+} + |u(k)|^{q^-}\Big) \\ &\leq c_1 \sum_{k=1}^{T+1} |\Delta u(k-1)|^{q^+} + c_2 \sum_{k=1}^{T+1} |\Delta u(k-1)|^{q^-}. \end{aligned} \tag{6}$$

Thus, according to (4), it follows

$$\sum_{k=1}^{T} |u(k)|^{q(k)} \leq 2c_3 \sum_{k=1}^{T} \Big(|\Delta u(k)|^{p_1(k)} + |\Delta u(k)|^{p_2(k)}\Big). \tag{7}$$

Consequently, from (7), we deduce

$$\begin{aligned}
\widehat{\lambda} &\geq \inf_{u\in W\setminus\{0\}} \frac{\frac{1}{p_2^+}\Big(\sum_{k=1}^{T+1}|\Delta u(k-1)|^{p_1(k-1)}+\sum_{k=1}^{T+1}|\Delta u(k-1)|^{p_2(k-1)}\Big)}{\frac{1}{q^-}\sum_{k=1}^{T}|u(k)|^{q(k)}}\\
&= \frac{q^-}{p_{\max}^+}\inf_{u\in W\setminus\{0\}} \frac{\sum_{k=1}^{T}|\Delta u(k)|^{p_1(k)}+\sum_{k=1}^{T}|\Delta u(k)|^{p_2(k)}}{\sum_{k=1}^{T}|u(k)|^{q(k)}}\\
&\geq \frac{q^-}{2cp_{\max}^+}>0.
\end{aligned}$$

So, Step 1 is verified.

Step 2 $\widehat{\lambda}$ is an eigenvalue of problem (1).

We need to establish the following two auxiliary results which will be used. Claim:

$$\lim_{\|u\|\to 0}\frac{J(u)}{I(u)}=+\infty. \tag{8}$$

$$\lim_{\|u\|\to \infty}\frac{J(u)}{I(u)}=+\infty. \tag{9}$$

Indeed, using (6) and 2, we infer

$$\begin{aligned}
|u|_{q^-}^{q^-}+|u|_{q^+}^{q^+} &\leq c_{q^-}\sum_{k=1}^{T+1}|\Delta u(k-1)|^{q^-}+c_{q^+}\sum_{k=1}^{T+1}|\Delta u(k-1)|^{q^+}\\
&\leq c_{q^-}T\|u\|^{q^-}+c_{q^+}T\|u\|^{q^+}.
\end{aligned}\tag{10}$$

Then, combining relations (5) and (10), it follows

$$\sum_{k=1}^{T}|u(k)|^{q(k)}\leq c_{q^-}T\|u\|^{q^-}+c_{q^+}T\|u\|^{q^+} \tag{11}$$

On the other hand, from Lemma 1 (c), there exists a positive constant C such that

$$\sum_{k=1}^{T+1}|\Delta u(k-1)|^{p_{\min}(k-1)}\geq C\|u\|^{p_{\min}^+},\quad \forall u\in W \text{ with } \|u\|<1. \tag{12}$$

We focus on the case, when $u \in W$ with $\|u\| < 1$. Thus, $|\Delta u(k)| < 1$ for any $k \in [0, T]$. So, for any $u \in W$ with $\|u\| < 1$ small enough, by relations (11) and (12) we get

$$\begin{aligned}\frac{J(u)}{I(u)} &\geq \frac{\frac{1}{p_{\min}^{+}} \sum_{k=1}^{T+1} |\Delta u(k-1)|^{p_{\min}(k-1)}}{\frac{T\left(c_{q^-}\|u\|^{q^-} + c_{q^+}\|u\|^{q^+}\right)}{q^-}} \\ &\geq \frac{q^-}{p_{\min}^{+}} \frac{C\|u\|^{p_{\min}^{+}}}{T\left(c_{q^-}\|u\|^{q^-} + c_{q^+}\|u\|^{q^+}\right)}.\end{aligned}$$

In view of 3, passing to the limit as $\|u\| \to 0$ in the above inequality, (8) follows.

Similarly, from Lemma 1 (b), there exist two positive constants C_1 and C_2 such that

$$\sum_{k=1}^{T+1} \Big(|\Delta u(k-1)|^{p_{\max}(k-1)} \geq C_1 \|u\|^{p_{\max}^{-}} - C_2, \quad \forall u \in W \text{ with } \|u\| > 1. \tag{13}$$

Thus, for any $u \in W$ with $\|u\| > 1$, from (11) and (13), we obtain

$$\begin{aligned}\frac{J(u)}{I(u)} &\geq \frac{\frac{1}{p_{\max}^{-}} \sum_{k=1}^{T+1} |\Delta u(k-1)|^{p_{\max}(k-1)}}{\frac{T\left(c_{q^-}\|u\|^{q^-} + c_{q^+}\|u\|^{q^+}\right)}{q^-}} \\ &\geq \frac{q^-}{p_{\max}^{-}} \frac{C_1\|u\|^{p_{\max}^{-}} - C_2}{T\left(c_{q^-}\|u\|^{q^-} + c_{q^+}\|u\|^{q^+}\right)}.\end{aligned}$$

Using again 3, passing to the limit as $\|u\| \to +\infty$, (9) holds true.

Now, we verify step 2. For this, we will prove that there exists $u_0 \in W \setminus \{0\}$ such that

$$\frac{J(u_0)}{I(u_0)} = \widehat{\lambda}. \tag{14}$$

In fact, Let $(u_n) \subset W \setminus \{0\}$ be a minimizing sequence for $\widehat{\lambda}$, that is,

$$\lim_{n\to\infty} \frac{J(u_n)}{I(u_n)} = \widehat{\lambda}. \tag{15}$$

In view of the 9 above, we infer that (u_n) is bounded in W. That information combined with the fact that W is a finite dimensional Hilbert space implies that there exists a subsequence, still denoted by (u_n), and $u_0 \in W$ such that u_n converges to u_0 in W. As consequence,

$$\frac{J(u_0)}{I(u_0)} = \widehat{\lambda}, \quad \text{if } u_0 \neq 0.$$

It remains to show that u_0 is nontrivial. In fact, if not, (8) implies

$$\lim_{n\to\infty} \frac{J(u_n)}{I(u_n)} = \infty.$$

But, this contradicts (15). Thus, $u_0 \neq 0$ and (14) holds true.

Therefore, for any $v \in W$, we infer

$$\frac{d}{dt}\frac{J(u_0+tv)}{I(u_0+tv)}\Big|_{t=0} = 0.$$

A simple computation yields,

$$\sum_{k=1}^{T+1}\Big(|\Delta u_0(k-1)|^{p_1(k-1)-2}+|\Delta u_0(k-1)|^{p_2(k-1)-2}\Big)\Delta u_0(k-1)\Delta v(k-1)I(u_0)$$
$$= \sum_{k=1}^{T} |u(k)|^{q(k)-2}u(k)v(k)J(u_0).$$

for any $v \in W$. The above relation combined with (14) and the fact that $I(u_0) \neq 0$ implies the fact that $\widehat{\lambda}$ is an eigenvalue of problem (1). Thus, step 2 is verified.

Remark 1. *Define*

$$\lambda_* = \inf_{u\in W\setminus\{0\}} \frac{\displaystyle\sum_{k=1}^{T+1}\Big(|\Delta u(k-1)|^{p_1(k-1)}+|\Delta u(k-1)|^{p_2(k-1)}\Big)}{\displaystyle\sum_{k=1}^{T}|u(k)|^{q(k)}}. \tag{16}$$

One has

$$\widehat{\lambda} > \lambda_*.$$

In fact, as u_0 is an eigenfunction corresponding to the eigenvalue $\widehat{\lambda}$ of problem (1), *then*

$$\sum_{k=1}^{T+1}\Big(|\Delta u_0(k-1)|^{p_1(k-1)-2}+|\Delta u_0(k-1)|^{p_2(k-1)-2}\Big)\Delta u_0(k-1)\Delta v(k-1)$$
$$= \widehat{\lambda}\sum_{k=1}^{T} |u(k)|^{q(k)-2}u(k)v(k),$$

for every $v \in W$. Taking $v = u_0$, we obtain

$$\sum_{k=1}^{T+1}\Big(|\Delta u_0(k-1)|^{p_1(k-1)} + |\Delta u_0(k-1)|^{p_2(k-1)}\Big) = \widehat{\lambda}\sum_{k=1}^{T}|u_0(k)|^{q(k)}.$$

Step 3 Any $\lambda \in (\widehat{\lambda}, \infty)$ is an eigenvalue of problem (1).

Fix $\lambda \in (\widehat{\lambda}, +\infty)$, let us define a functional $T_\lambda : W \to \mathbb{R}$ by

$$T_\lambda(u) = J(u) - \lambda I(u).$$

Standard arguments assure that $T_\lambda \in C^1(W, \mathbb{R})$ and its Gâteaux derivative $T_\lambda^{'}$ at u reads

$$\begin{aligned}\langle T_\lambda^{'}(u), v\rangle = \sum_{k=1}^{T+1}\Big(|\Delta u(k-1)|^{p_1(k-1)-2} + |\Delta u(k-1)|^{p_2(k-1)-2}\Big)\Delta u(k-1)\Delta v(k-1)\\ - \lambda\sum_{k=1}^{T}|u(k)|^{q(k)-2}u(k)v(k),\end{aligned}$$

for all $v \in W$.

Suppose that u is a critical point to T_λ, i.e. $\langle T_\lambda^{'}(u), v\rangle = 0$ for all $v \in W$. Summing by parts and taking boundary values into account, see [9], we observe that

$$\begin{aligned}\sum_{k=1}^{T+1}\Delta\Big(|\Delta u(k-1)|^{p_1(k-1)-2} + |\Delta u(k-1)|^{p_2(k-1)-2}\Big)\Delta u(k-1)v(k-1)\\ - \lambda\sum_{k=1}^{T}|u(k)|^{q(k)-2}u(k)v(k) = 0.\end{aligned}$$

Since $v \in W$ is arbitrary, we see that u satisfies (1).

As W is finite dimensional and T_λ is Gâteaux differentiable and continuous it suffices to show that it is coercive. By Lemma 1 (b) and (11), for sufficiently large $\|u\|$, we obtain

$$T_\lambda(u) \geq C_1\|u\|^{p_{\max}^-} - C_2 - \lambda T\big(c_{q^-}\|u\|^{q^-} + c_{q^+}\|u\|^{q^+}\big),$$

where $c_i, i = 1, 2$ are positive constants. Thanks to $q^- \leq q^+ < p_2^-$, we conclude $T_\lambda(u) \to +\infty$ as $\|u\| \to +\infty$. Applying [16, Theorem 1.2] in order to prove that there exists $u_\lambda \in W$ a global minimum point of T_λ and thus, a critical point of T_λ.

In order to finish the proof of step 3 it is enough to show that u_λ is not trivial. Indeed, since $\lambda > \widehat{\lambda} = \inf_{u\in W\setminus\{0\}} \dfrac{J(u)}{I(u)}$ it follows that there exists $v_\lambda \in W$ such that

$$J(v_\lambda) < \lambda I(v_\lambda), \quad \text{that is} \quad T_\lambda(v_\lambda) < 0.$$

Thus,

$$\inf_{u\in W} T_\lambda(v) < 0$$

and we conclude that u_λ is a nontrivial critical point of T_λ. Thus, step 3 is verified.

Step 4 Any $\lambda \in (0, \lambda_*)$ is not an eigenvalue of problem (1) with λ_* is given by (16).

Precise that, from Remark 1, we have $\lambda_* < \widehat{\lambda}$. By contradiction, if we assume that there exists $\lambda \in (0, \lambda_*)$ an eigenvalue of problem (1), then there exists $u_\lambda \in W \setminus \{0\}$ such that

$$\langle J^{'}(u_\lambda), v\rangle = \lambda\langle I^{'}(u_\lambda), v\rangle, \quad \forall v \in W.$$

In particular, for $v = u_\lambda$ we get

$$\sum_{k=1}^{T+1}\Big(|\Delta u_\lambda(k-1)|^{p_1(k-1)} + |\Delta u_\lambda(k-1)|^{p_2(k-1)}\Big) = \lambda \sum_{k=1}^{T} |u_\lambda(k)|^{q(k)}.$$

The fact that $u_\lambda \in W \setminus \{0\}$ assures that $\sum_{k=1}^{T} |u_\lambda(k)|^{q(k)} > 0$. Since $\lambda < \lambda_*$, the above information yields

$$\begin{aligned}\sum_{k=1}^{T+1}\Big(|\Delta u_\lambda(k-1)|^{p_1(k-1)}+|\Delta u_\lambda(k-1)|^{p_2(k-1)}\Big) &\geq \lambda_* \sum_{k=1}^{T} |u_\lambda(k)|^{q(k)} \\ &> \lambda \sum_{k=1}^{T} |u_\lambda(k)|^{q(k)} \\ &= \sum_{k=1}^{T+1}\Big(|\Delta u_\lambda(k-1)|^{p_1(k-1)} + |\Delta u_\lambda(k-1)|^{p_2(k-1)}\Big).\end{aligned}$$

Clearly, the above inequalities lead to a contradiction. Thus, step 4 is verified and the proof of Theorem 2 is ended.

Theorem 3. *If*

$$q^- < p^-_{\min} \quad \textit{or} \quad q^- > p^+_{\max} \tag{17}$$

then, there exists a positive constant λ^* such that any $\lambda \in (0, \lambda^*)$ is an eigenvalue of problem (1).

Proof. **Case** $q^- > p^+_{\max}$.

In order to use a mountain pass lemma, we start by proving that T_λ satisfies the Palais-Smale condition. Let $(u_n) \subset W$ be a sequence such that $\{T_\lambda(u_n)\}$ is bounded and $J(u_n) \to 0$. As W is finitely dimensional, it is enough to show that (u_n) is bounded. If not, we may assume that $\|u_n\| \to \infty$ as $n \to \infty$. Thus, we may consider that $\|u_n\| > 1$ for any integer n. Then, by inequality $(f), (d)$ and (e) in Lemma 1, we get

$$T_\lambda(u_n) \leq \frac{1}{p^-_{\max}}\big((T+1)\|u_n\|^{p^+_{\max}} + (T+1)\big) - \frac{2^{q^-}\lambda}{q^+}(T+1)^{\frac{2-q^-}{2}}\|u_n\|^{q^-}$$

which implies that $T_\lambda(u_n) \to -\infty$ as $\|u_n\| \to +\infty$ because $q^- > p^+_{\max}$. Thus, we obtain a contradiction with the fact that $T_\lambda(u_n)$ is bounded. Hence, the sequence (u_n) is bounded.

Now, we will verify the other assumptions. Put

$$\Omega = \Big\{u \in W : \quad \|u\| \leq (T+1)^{\frac{-1}{2}}\Big\}.$$

Then, from Lemma 1 (g), it follows

$$|u(k)| \leq \max_{s\in[1,T]} |u(s)| \leq (T+1)^{\frac{1}{2}}\|u\| \leq 1, \quad \forall u \in \Omega, \quad \forall k \in [1,T],$$

so,

$$I(u) \leq \sum_{k=1}^{T} \frac{1}{q(k)}, \quad \forall u \in \Omega.$$

Therefore, in view Lemma 1 (c), we deduce

$$T_\lambda(u) \geq \frac{1}{p^+_{\max}} T^{\frac{p^+_{\max}-2}{2}}(T+1)^{\frac{-p^+_{\max}}{2}} - \lambda \sum_{k=1}^{T}\frac{1}{q(k)}, \quad \forall u \in \partial\Omega.$$

Consequently, if we set

$$\lambda^* = \frac{T^{\frac{p^+_{\max}-2}{2}}(T+1)^{\frac{-p^+_{\max}}{2}}}{p^+_{\max}\sum_{k=1}^{T}\frac{1}{q(k)}},$$

then for any $\lambda \in (0, \lambda^*)$, we have

$$T_\lambda(u) > 0, \quad \forall u \in \partial\Omega. \tag{18}$$

It remains to verify that there exists h_0 such that

$$u_{h_0} \notin \Omega \text{ and } T_\lambda(u_{h_0}) < \min_{u\in\partial\Omega} T_\lambda(u). \tag{19}$$

In fact, let $u_h \in W$ be defined by

$$\begin{cases} u_h(k) = h & \text{for} \quad k \in [1,T], \\ u_h(0) = u_h(T) = 0, \end{cases}$$

Therefore, for $h > 1$, we get

$$\begin{aligned} T_\lambda(u_h) \leq & 2\Big(\frac{h^{p_{\max}(0)}}{p_{\max}(0)} + \frac{h^{p_{\max}(T)}}{p_{\max}(T)}\Big) - \lambda \sum_{k=1}^{T} \frac{h^{q(k)}}{q(k)} \\ \leq & \frac{4}{p_{\max}^-} h^{p_{\max}^+} - \lambda \frac{T}{q^+} h^{q^-} \end{aligned}$$

Since $q^- > p^+_{\max}$, we deduce $\lim\limits_{h\to\infty} T_\lambda(u_h) = -\infty$. So, the assertion (19) holds true. Applying a mountain pass lemma, problem (1) has at least one solution, that is, there exists $u^* \in W$ such that $T_\lambda^{'}(u^*) = 0$ and $T_\lambda(u^*) = c$, where $c > \max\big(T_\lambda(0), T_\lambda(u_{h_0})\big)$. As, $T_\lambda(0) = 0$, $u^* \neq 0$.

Case $q^- < p^-_{\min}$.

Let $\lambda \in (0, \lambda^*)$ be fixed. From (18) and using the Weierstrass theorem, we obtain

$$\inf_{u\in\partial\Omega} T_\lambda(u) > 0.$$

Take $t \in [0,1]$ and define $u_0 \in W$ a function such that

$$\begin{cases} u_0(k) = 0 & \text{for} \quad k \in [1,T] \setminus \{k_0\}, \\ u_0(k_0) = t, \end{cases}$$

with $k_0 \in [1,T]$ is given such that $q(k_0) = q^-$. Then,

$$\begin{aligned} T_\lambda(u_0) & \leq 2\Big(\frac{t^{p_{\min}(k_0-1)}}{p_{\min}(k_0-1)} + \frac{t^{p_{\max}(k_0)}}{p_{\min}(k_0)}\Big) - \lambda \frac{t^{q(k_0)}}{q(k_0)} \\ & \leq \frac{4}{p^-_{\min}} t^{p^-_{\min}} - \frac{\lambda}{q^+} t^{q^-}. \end{aligned}$$

Hence, for $0 < t < \left(\frac{\lambda p^-_{\min}}{4q^+}\right)^{\frac{1}{p^-_{\min}-q^-}}$, we have

$$T_\lambda(u_0) < 0.$$

As $u_0 \in \mathrm{Int}\Omega$, we write

$$\inf_{u\in \mathrm{Int}\Omega} T_\lambda(u) < 0 < \inf_{u\in\partial\Omega} T_\lambda(u).$$

Let us choose $\epsilon > 0$ such that

$$0 < \epsilon < \inf_{\partial\Omega} T_\lambda - \inf_{\mathrm{Int}\Omega} T_\lambda. \tag{20}$$

Therefore, by applying the Ekeland's variational principle [6] to the functional $T_\lambda : \Omega \to \mathbb{R}$, there exists $u_\epsilon \in \Omega$ such that

$$T_\lambda(u_\epsilon) < \inf_{\Omega} T_\lambda + \epsilon \quad \text{ and } \quad T_\lambda(u_\epsilon) < T_\lambda(u) + \epsilon\|u - u_\epsilon\|, \quad \text{ for } u \neq u_\epsilon.$$

Hence, by (20), it follows that $T_\lambda(u_\epsilon) < \inf\limits_{\partial\Omega} T_\lambda$ and so, $u_\epsilon \in \mathrm{Int}\Omega$.

Now, let us define $\Phi_\lambda : \Omega \to \mathbb{R}$ by $\Phi(u) = T_\lambda(u) + \epsilon\|u - u_\epsilon\|$. It is easy to see that u_ϵ is a minimum point of Φ, and thus

$$\frac{\Phi(u_\epsilon + t.v) - \Phi(u_\epsilon)}{t} \geq 0,$$

for $t > 0$ small enough and any $v \in \Omega$. The above expression shows that

$$\frac{T_\lambda(u_\epsilon + t.v) - T_\lambda(u_\epsilon)}{t} + \epsilon\|v\| \geq 0.$$

Letting $t \to 0^+$, we deduce that

$$\langle T_\lambda^{'}(u_\epsilon), u\rangle + \epsilon\|u\| \geq 0,$$

that is,

$$\|T_\lambda^{'}(u_\epsilon)\| \leq \epsilon.$$

Therefore, there exists a sequence $(u_n) \subset \mathrm{Int}\Omega$ such that

$$T_\lambda(u_n) \to \underline{c} := \inf_{u\in\Omega} T_\lambda(u), \quad \text{ and } T_\lambda^{'}(u_n) \to 0 \quad \text{ as } n \to \infty.$$

Since The sequence (u_n) is bounded in W, there exists $u_0 \in W$ such that, up to a subsequence, (u_n) converges to u_0 in W. Thus,

$$T_\lambda(u_0) = \underline{c}, \quad \text{ and } T_\lambda^{'}(u_0) = 0.$$

So, u_0 is a non trivial weak solution of problem (1).

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