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SOME FIXED POINT THEOREMS ON MODULAR METRIC SPACES

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Abstract. In recent years, there has been a great interest in the study of the fixed point property in modular metric spaces. In this article, we study and prove some fixed point theorems for contraction mappings in modular metric spaces.

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1. Introduction

The classical modular spaces introduced by Nakano in 1950 [13], on vector spaces and then Musielak and Orlicz introduced the modular function spaces ([11],[10] [14]).

In 2010, V.V.Chystyakov ([3],[4]) introduced the concept of modular metric spaces on an arbitrary set that is generalization of modular spaces. Fixed point theorems in modular spaces, generalizing the classical Banach fixed point theorem in metric spaces, have been studied extensively, see ([5],[9],[1],[2]). In recent years, there has been a great interest in the study of the fixed point property in modular metric spaces, see [15]. In this article, we study and prove some fixed point theorems for contraction mappings in modular metric spaces which are natural generalization of classical modulars over linear spaces like Lebesgue, Orlicz, Musielake-Orlicz, Lorentz, Calderon-Lozanovskii spaces and many others. For a current review of the theory of Musielak-Orlicz spaces and modular spaces , for further details reader is referred to the books of Musielak [12] and Koslowski [8].

Let X be a nonempty set , A function $\omega : (0,\infty) \times X \times X \longrightarrow [0,\infty]$ that will be written as $\omega_\lambda(x,y) = \omega(\lambda,x,y)$ is said to be a (metric) pseudomodular on X, if it satisfies the following conditions:
(i) given $x, y \in X$, $\omega_\lambda(x,y) = 0$ for all $\lambda > 0$ if and only if $x = y$;
(ii) $\omega_\lambda(x,y) = \omega_\lambda(y,x)$, for all $\lambda > 0$ and $x, y \in X$;
(iii) $\omega_{\lambda+\mu}(x,y) \leq \omega_\lambda(x,z) + \omega_\mu(z,y)$ for all $\lambda, \mu > 0$ and $x, y, z \in X$.

If instead of (i), we have only the condition
(i_1) $\omega_\lambda(x,x) = 0$, then ω is said to be a (metric) pseudomodular on X and if ω satisfies (i_1) and
(i_2) given $x, y \in X$, if there exists $\lambda > 0$, possibly depending on x and y such that $\omega_\lambda(x,y) = 0$ then $x = y$,
then ω is called a strict modular on X.

Remark 1. *Given a modular ω on a set X, by $0 < \lambda \longrightarrow \omega_\lambda(x,y) \in [0,\infty]$ for given $x, y \in X$, is non-increasing on $(0,\infty)$. Indeed,*
if $0 < \lambda < \mu$, then we have

$$\omega_\mu(x,y) \leq \omega_{\mu-\lambda}(x,x) + \omega_\lambda(x,y) = \omega_\lambda(x,y)$$

for all $x, y \in X$.

Definition 1. *A sequence $\{x_n\} \equiv \{x_n\}_{n=1}^{\infty}$ in X_ω is said to be ω-convergent to $x \in X$ if for all $\lambda > 0$ we have $\lim_{n\to\infty} \omega_\lambda(x_n, x) = 0$ or $x_n \overset{\omega}{\to} x$ (as $n \to \infty$).*

Definition 2. *A sequence in X_ω is said to be ω-Cauchy if for all $\varepsilon > 0$ and all $\lambda > 0$ there exists a number $n_\circ(\varepsilon) \in \mathbb{N}$ such that for all $n, m \geq n_\circ(\varepsilon)$ we have $\omega_\lambda(x_n, x_m) \leq \varepsilon$.*

Definition 3. *The modular space X_ω is said to be ω-complete if each modular ω-Cauchy sequence from X_ω is ω-convergent to an element $x \in X_\omega$.*

Definition 4. *Given a modular ω on X a subset $C \subseteq X_\omega$ is said to be ω-closed if for each sequence $\{x_n\} \in C$ with $x_n \overset{\omega}{\to} x$, we have $x \in C$.*

Remark 2. *[5] Given a modular ω on X, the sets*

$$X_\omega \equiv X_\omega(x_\circ) = \{x \in X : \omega_\lambda(x, x_\circ) \to 0 \ as\ \lambda \to \infty\}$$

and

$$X_\omega^* \equiv X_\omega^*(x_\circ) = \{x \in X : \omega_\lambda(x, x_\circ) < \infty \ \ for\ some\ \ \lambda > 0\}$$

is said to be modular space (around $x_\circ$)Also the modular space X_ω and X_ω^* can be equipped with metrics d_ω and d_ω^*, generated by ω and given by

$$d_\omega(x,y) = \inf\{\lambda > 0 : \omega_\lambda(x,y) \leq \lambda\}, \quad x, y \in X_\omega$$

and

$$d_\omega^*(x,y) = \inf\{\lambda > 0 : \omega_\lambda(x,y) \leq 1\}, \quad x, y \in X_\omega^*$$

Definition 5. *[15] Given a modular metric spaces X_ω, we say that $T : X_\omega \to X_\omega$ is modular continuous (ω-continuous) if for each $\{x_n\} \in X$ when $x_n \overset{\omega}{\to} x$ as $n \to \infty$, then $Tx_n \overset{\omega}{\to} Tx$ as $n \to \infty$.*

Definition 6. *Given a modular ω on X, the ω-closure of a subset E of X_ω is denoted by $\overline{E}$ and defined by the set of all $x \in X_\omega$ such that there exists a sequence $\{x_n\}$ of elements of E such that $x_n \overset{\omega}{\to} x$.*
The subset E is ω-dense in X_ω if $\overline{E} = X_\omega$.

2. Main result

In this section we study and prove some fixed point theorems for contraction mappings in modular metric space.

Theorem 1. *Let X_ω be a modular metric space and let $T : X_\omega \longrightarrow X_\omega$ be a mapping such that T satisfies that*
(a) $\omega_\lambda(Tx, Ty) \leq \alpha\omega_\lambda(x, Tx) + \beta\omega_\lambda(y, Ty)$ for all $x, y \in X_\omega, \lambda > 0$ where $0 < \alpha + \beta < 1$,
(b) T is $\omega-$continuous at a point $u \in X_\omega$,
(c) there exists $x \in X_\omega$ such that $\{T^n(x)\}_{n\in\mathbb{N}}$ has a subsequence $\{T^{n_i}(x)\}_{n\in\mathbb{N}}$ that is $\omega-$convergent to u.
Then u is unique fixed point.

Proof. Since T is $\omega-$continuous at u so $\{T^{n_i+1}(x)\}_{n\in\mathbb{N}}$ is $\omega-$convergent to $T(u) = u$. Suppose $T(u) \neq u$, by hypothesis $T^{n_i}(x) \overset{\omega}{\to} u$ so $T^{n_i+1}(x) \overset{\omega}{\to} Tu$, there exist N_1 such that for $i \geq N_1$ we have $\omega_\lambda(T^{n_i}(x), u) \leq \varepsilon$ and $\omega_\lambda(T^{n_i+1}(x), Tu) \leq \varepsilon$ for all $\lambda > 0$. We supposed $T(u) \neq u$, that implies

$$\omega_\lambda(T^{n_i+1}(x), T^{n_i}(x)) > \varepsilon \quad for \;\; i \geq N_1 \tag{1}$$

Since,

$$\begin{aligned} \omega_\lambda(u, Tu) &\leq \omega_{\frac{\lambda}{3}}(T^{n_i}(x), u) + \omega_{\frac{\lambda}{3}}(T^{n_i+1}(x), T^{n_i}(x)) + \omega_{\frac{\lambda}{3}}(T^{n_i+1}(x), Tu) \\ &\leq \varepsilon + \omega_{\frac{\lambda}{3}}(T^{n_i+1}(x), T^{n_i}(x)) + \varepsilon \\ &\leq 2\,\varepsilon + \omega_{\frac{\lambda}{3}}(T^{n_i+1}(x), T^{n_i}(x)) \end{aligned}$$

We have from (a),

$$\omega_\lambda(T^{n_i+1}(x), T^{n_i+2}(x)) \leq \alpha\omega_\lambda(T^{n_i}(x), T^{n_i+1}(x)) + \beta\omega_\lambda(T^{n_i+1}(x), T^{n_i+2}(x))$$

so,

$$(1-\beta)\omega_\lambda(T^{n_i+1}(x), T^{n_i+2}(x)) \le \alpha\omega_\lambda(T^{n_i}(x), T^{n_i+1}(x)) \tag{2}$$

for all $\lambda > 0$. Whence , from (2.2) , we get

$$\begin{aligned}
\omega_\lambda(T^{n_i+1}(x), T^{n_i+2}(x)) &\le \frac{\alpha}{1-\beta}\omega_\lambda(T^{n_i}(x), T^{n_i+1}(x)) \\
&\le (\frac{\alpha}{1-\beta})^2\omega_\lambda(T^{n_i-1}(x), T^{n_i}(x)) \\
&\le ... \\
&\le (\frac{\alpha}{1-\beta})^{n_i-n_j}\omega_\lambda(T^{n_j+1}(x), T^{n_j+2}(x))
\end{aligned}$$

for all $\lambda > 0$ where $\frac{\alpha}{1-\beta} < 1$. Thus $\omega_\lambda(T^{n_i}(x), T^{n_{i+1}}(x)) \xrightarrow{\omega} 0$ as $i \to \infty$ for all $\lambda > 0$, which contradict (2.1), therefore $Tu = u$. Suppose there is $z \in X_\omega$ such that $Tz = z$, from (a) , we have

$$\omega_\lambda(u, z) = \omega_\lambda(Tu, Tz) \le \alpha\omega_\lambda(u, Tu) + \beta\omega_\lambda(z, Tz) = 0$$

for all $\lambda > 0$. This implies that u is unique.

Theorem 2. *Let X_ω be a $\omega-$complete modular metric space and let $T : X_\omega \to X_\omega$ be a mapping such that T satisfies following conditions for all $x, y \in X_\omega$*

$$\omega_\lambda(Tx, Ty) \le \alpha\omega_\lambda(x, Tx) + \beta\omega_\lambda(y, Ty) + \gamma\omega_\lambda(x, y) \tag{3}$$

for all $\lambda > 0$ where $0 \le \alpha + \beta + \gamma < 1$. Then T has unique fixed point u, and T is $\omega-$continuous at u.

Proof. Let $x_\circ \in X_\omega$ be an arbitrary point and we define the sequence $\{x_n\}_{n\in\mathbb{N}}$ by $x_n = T^n(x_\circ)$. By (2.3) we have

$$\omega_\lambda(x_n, x_{n+1}) = \omega_\lambda(Tx_{n-1}, Tx_n) \le \alpha\omega_\lambda(x_{n-1}, Tx_{n-1}) + \beta\omega_\lambda(x_n, Tx_n) + \gamma\omega_\lambda(x_{n-1}, x_n)$$

for all $\lambda > 0$. So

$$(1-\beta)\omega_\lambda(x_n, x_{n+1}) \le \alpha\omega_\lambda(x_{n-1}, x_n) + \gamma\omega_\lambda(x_{n-1}, x_n)$$

for all $\lambda > 0$. Let $r = \frac{\alpha+\gamma}{1-\beta}$ then $0 \le r < 1$. This implies

$$\omega_\lambda(x_n, x_{n+1}) \le r\omega_\lambda(x_{n-1}, x_n)$$

for all $\lambda > 0$. By induction we have

$$\omega_\lambda(x_n, x_{n+1}) \le r^n\omega_\lambda(x_\circ, x_1)$$

for all $\lambda > 0$. Moreover for all $n, m \in \mathbb{N}$; $n < m$ we have

$$\begin{aligned} \omega_\lambda(x_n, x_m) &\leq \omega_{\frac{\lambda}{m-n}}(x_n, x_{n+1}) + \omega_{\frac{\lambda}{m-n}}(x_{n+1}, x_{n+2}) + ... + \omega_{\frac{\lambda}{m-n}}(x_{m-1}, x_m) \\ &\leq r^n \omega_{\frac{\lambda}{m-n}}(x_\circ, x_1) + r^{n+1}\omega_{\frac{\lambda}{m-n}}(x_\circ, x_1) + ... + r^{m-1}\omega_{\frac{\lambda}{m-n}}(x_\circ, x_1) \\ &= (r^n + r^{n+1} + ... + r^{m-1})\omega_{\frac{\lambda}{m-n}}(x_\circ, x_1) \\ &= \frac{r^n - r^m}{1-r}\omega_{\frac{\lambda}{m-n}}(x_\circ, x_1) \end{aligned}$$

for all $\lambda > 0$. Therefore $\{x_n\}_{n\in\mathbb{N}}$ is $\omega-$Cauchy sequence, and since X_ω is $\omega-$complete there exists $u \in X_\omega$ such that $\{x_n\}_{n\in\mathbb{N}}$ is $\omega-$convergent to u. Suppose that $Tu \neq u$, then we have

$$\omega_\lambda(x_n, Tu) = \omega_\lambda(Tx_{n-1}, Tu) \leq \alpha\omega_\lambda(x_{n-1}, Tx_{n-1}) + \omega_\lambda(u, Tu) + \gamma\omega_\lambda(x_{n-1}, u)$$

for all $\lambda > 0$. Taking the limit as $n \longrightarrow \infty$ then $\omega_\lambda(u, Tu) \leq \beta\omega_\lambda(u, Tu)$ for all $\lambda > 0$. This contradiction implies that $Tu = u$.
To show that u is unique , suppose that $Tu = u$, $Tz = z$ and $u \neq z$, then

$$\omega_\lambda(u, z) = \omega_\lambda(Tu, Tz) \leq \alpha\omega_\lambda(u, Tu) + \beta\omega_\lambda(z, Tz) + \gamma\omega_\lambda(u, z)$$

for all $\lambda > 0$. This contradiction implies that $u = z$.
Now we show that T is $\omega-$continuous at u. Let $\{y_n\}_{n\in\mathbb{N}}$ be $\omega-$convergent sequence such that $y_n \overset{\omega}{\to} u$ as $(n \to \infty)$. So we have

$$\begin{aligned} \omega_\lambda(u, Ty_n) &= \omega_\lambda(Tu, Ty_n) \\ &\leq \alpha\omega_\lambda(u, Tu) + \beta\omega_\lambda(y_n, Ty_n) + \gamma\omega_\lambda(u, y_n) \\ &= \beta\omega_\lambda(y_n, Ty_n) + \gamma\omega_\lambda(u, y_n) \end{aligned}$$

for all $\lambda > 0$.The modular ω is non-increasing on $(0, \infty)$,so

$$\begin{aligned} \omega_\lambda(u, Ty_n) &\leq \beta(\omega_{\frac{\lambda}{2}}(y_n, u) + \omega_{\frac{\lambda}{2}}(u, Ty_n)) + \gamma\omega_\lambda(u, y_n) \\ &\leq \beta\omega_{\frac{\lambda}{2}}(y_n, u) + \beta\omega_\lambda(u, Ty_n)) + \gamma\omega_\lambda(u, y_n) \end{aligned}$$

for all $\lambda > 0$. So we have $(1-\beta)\omega_\lambda(u, Ty_n) \leq \beta\omega_{\frac{\lambda}{2}}(y_n, u) + \gamma\omega_\lambda(u, y_n) \longrightarrow 0$, (as $n \to \infty$). So

$$Ty_n \overset{\omega}{\to} u = Tu$$

Therefore T is $\omega-$continuous.

Theorem 3. *Let X_ω be a $\omega-$complete modular metric space and let $T : X_\omega \to X_\omega$ be a $\omega-$continuous mapping such that T satisfies following conditions for all $x, y \in X_\omega$*
(a) : $\omega_\lambda(Tx, Ty) \leq k\{\omega_\lambda(x, Tx) + \omega_\lambda(y, Ty)\}$ for all $x, y \in M$ and all $\lambda > 0$, where M is $\omega-$dense subset of X_ω and $0 < k < \frac{1}{2}$.
(b) : there is $x \in X_\omega$; $\{T^n(x)\}_{n\in\mathbb{N}} \overset{\omega}{\to} u$.
Then u is unique fixed point .

Proof. It is enough to show that condition (a) in theorem 2.1 holds for any $x, y \in X_\omega$ and $\lambda > 0$.
Case 1 : If $x, y \in X_\omega \setminus M$, let $\{x_n\}, \{y_n\}$ be a sequence in M such that $x_n \overset{\omega}{\to} x$ and $y_n \overset{\omega}{\to} y$. So we have

$$\begin{aligned}\omega_\lambda(Tx, Ty) &\leq \omega_{\frac{\lambda}{2}}(Tx, Tx_n) + \omega_{\frac{\lambda}{2}}(Tx_n, Ty)\\ &\leq \omega_{\frac{\lambda}{2}}(Tx, Tx_n) + \omega_{\frac{\lambda}{4}}(Tx_n, Ty_n) + \omega_{\frac{\lambda}{4}}(Ty_n, Ty)\\ &\leq \omega_{\frac{\lambda}{2}}(Tx, Tx_n) + k\{\omega_{\frac{\lambda}{4}}(x_n, Tx_n) + \omega_{\frac{\lambda}{4}}(y_n, Ty_n)\} + \omega_{\frac{\lambda}{4}}(Ty_n, Ty)\end{aligned}$$

for all $\lambda > 0$. Since T is $\omega-$continous as $n \longrightarrow \infty$ in the above inequality we obtain

$$\omega_\lambda(Tx, Ty) \leq k\{\omega_{\frac{\lambda}{4}}(x, Tx) + \omega_{\frac{\lambda}{4}}(y, Ty)\}$$

for all $\lambda > 0$.
Case 2 : If $x \in M$ and $y \in X_\omega \setminus M$, let $\{y_n\}$ be a sequence in M such that $y_n \overset{\omega}{\to} y$, then we have

$$\begin{aligned}\omega_\lambda(Tx, Ty) &\leq \omega_{\frac{\lambda}{2}}(Tx, Ty_n) + \omega_{\frac{\lambda}{2}}(Ty_n, Ty)\\ &\leq k\{\omega_{\frac{\lambda}{2}}(x, Tx) + \omega_{\frac{\lambda}{2}}(y_n, Ty_n)\} + \omega_{\frac{\lambda}{2}}(Ty_n, Ty)\end{aligned}$$

for all $\lambda > 0$. T is $\omega-$continous as $n \longrightarrow \infty$ in the above inequality we obtain

$$\omega_\lambda(Tx, Ty) \leq k\{\omega_{\frac{\lambda}{2}}(x, Tx) + \omega_{\frac{\lambda}{2}}(y, Ty)\}$$

for all $\lambda > 0$.
Case 3 : If $x, y \in M$ then we have

$$\omega_\lambda(Tx, Ty) \leq k\{\omega_\lambda(x, Tx) + \omega_\lambda(y, Ty)\}$$

for all $\lambda > 0$.
So in any three case for all $x, y \in X_\omega$ and all $\lambda > 0$, since $0 < k < \frac{1}{2}$ by theorem (2.1) , T has a unique fixed point.

Definition 7. *Let X_ω be a $\omega-$complete modular metric space. A mapping $T : X_\omega \to X_\omega$ is said to be $\varepsilon-$contractive if there exists $0 < \alpha < 1$ such that*

$$0 < \omega_\lambda(x, y) < \varepsilon \Rightarrow \omega_\lambda(Tx, Ty) \leq \alpha\omega_\lambda(x, y)$$

for all $\lambda > 0$.

Theorem 4. *Let X_ω be a $\omega-$complete modular metric space and $T : X_\omega \to X_\omega$ be an $\varepsilon-$contractive mapping , and let $x_\circ$ be a point of X_ω such that the sequence $\{T^n(x_\circ)\}$ has a $\omega-$convergent subsequence that convergent to a point u of X_ω. Then u is a periodic point of T, i.e. there is a positive integer k such that $T^k u = u$.*

Proof. Let $\{n_i\}$ be a strictly increasing sequence of positive integers such that $T^{n_i}x_\circ \overset{\omega}{\to} u$ as $i \longrightarrow \infty$, and let $x_i = T^{n_i}x_\circ$. For each $\varepsilon_\circ > 0$ there exists $N = N(\varepsilon_\circ)$ such that

$$\omega_\lambda(T^{n_i}(x_\circ), u) = \omega_\lambda(x_i, u) \leq \frac{\varepsilon_\circ}{4} \tag{4}$$

for all $\lambda > 0$ and $i \geq N$. Choose any $i \geq N$ and let $k = n_{i+1} - n_i$, then

$$\omega_\lambda(x_{i+1}, T^k u) = \omega_\lambda(T^k x_i, T^k u) \leq \alpha\frac{\varepsilon_\circ}{4} < \frac{\varepsilon_\circ}{4} \tag{5}$$

for all $\lambda > 0$, and

$$\omega_{2\lambda}(T^k u, u) \leq \omega_\lambda(T^k u, x_{i+1}) + \omega_\lambda(x_{i+1}, u) < \frac{\varepsilon_\circ}{4} + \frac{\varepsilon_\circ}{4} = \frac{\varepsilon_\circ}{2} \tag{6}$$

for all $\lambda > 0$, and $i \geq N$. So $T^k u = u$. Suppose that $v = T^k u \neq u$, where $0 < \omega_\lambda(u, v) < \frac{\varepsilon}{2} < \varepsilon$ for given $\varepsilon > 0$. Then since T is $\varepsilon-$contractive

$$\omega_\lambda(Tu, Tv) \leq \alpha\omega_\lambda(u, v)$$

for all $\lambda > 0$. Since $\omega_\lambda(x_r, u) = \omega_\lambda(T^{n_r}x_\circ, u) \longrightarrow 0$, as $r \longrightarrow \infty$ and T is $\omega-$continous we have

$$\omega_\lambda(T^k x_r, T^k u) = \omega_\lambda(T^k x_r, v) \longrightarrow 0 \quad as \quad r \to \infty$$

The inequalities (2.4),(2.5) and (2.6)are hold for each $\varepsilon_\circ > 0$ such as for given $\varepsilon > 0$. So for given $\varepsilon > 0$ there exists $N' \geq N$ such that

$$\omega_\lambda(x_r, u) \leq \frac{\varepsilon}{4} < \varepsilon \quad and \quad \omega_\lambda(T^k x_r, v) \leq \frac{\varepsilon}{4} < \varepsilon$$

for all $\lambda > 0$ and $r \geq N'$. Since T is $\varepsilon-$contractive

$$\omega_\lambda(Tx_r, Tu) \leq \alpha\omega_\lambda(x_r, u) \quad and \quad \omega_\lambda(TT^k x_r, Tv) \leq \alpha\omega_\lambda(T^k x_r, v)$$

so,

$$\begin{aligned}\omega_{3\lambda}(Tx_r, TT^k x_r) &\leq \omega_\lambda(Tx_r, Tu) + \omega_\lambda(Tu, Tv) + \omega_\lambda(TT^k x_r, Tv)\\ &\leq \alpha\omega_\lambda(x_r, u) + \alpha\omega_\lambda(u, v) + \alpha\omega_\lambda(T^k x_r, v)\\ &< \frac{\varepsilon}{4} + \frac{\varepsilon}{2} + \frac{\varepsilon}{4} = \varepsilon\end{aligned}$$

for all $\lambda > 0$ and $r \geq N'$. By $\varepsilon-$contractivity of T ,

$$\omega_\lambda(T^2 x_r, T^2 T^k x_r) \leq \alpha\omega_\lambda(Tx_r, TT^k x_r) < \alpha^2\omega_\lambda(x_r, u)$$

and so,

$$\omega_\lambda(T^p x_r, T^p T^k x_r) \leq \alpha^p(x_r, u).$$

Setting $p = n_{r+1} - n_r$ then ,

$$\omega_\lambda(x_{r+1}, T^k x_{r+1}) \leq \alpha^p\omega_\lambda(x_r, u)$$

hence

$$\omega_\lambda(x_s, T^k x_s) \leq \alpha^{p(s-r)}(x_r, u)$$

for all $\lambda > 0$, and so

$$\omega_{3\lambda}(u, v) \leq \omega_\lambda(u, x_s) + \omega_\lambda(x_s, T^k x_s) + \omega_\lambda(T^k x_s, v) \longrightarrow 0 \quad as \quad (s \to \infty)$$

This contradicts the assumption that $\omega_\lambda(u, v) > 0$. Thus $u = v = T^k u$.

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