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MATRIX FREE SUPER-IMPLICIT SECOND DERIVATIVE MULTISTEP METHODS FOR STIFF INITIAL VALUE PROBLEMS IN ODES

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Abstract. In this paper, a new class of second derivative multistep methods that possesses a implementation feather is presented. The formulas, which we call them matrix free super-implicit second derivative multistep methods (MF-SISDMM), are of more implicitness than the so-called implicit formulas in the sense that they require the knowledge of functions not only at the past and present step-points, but also at the future ones. Moreover, with a simple modification we take advantage of calling for the solution of algebraic equations with the same coefficient matrix in each step. Their accuracy and stability characteristics are investigated and the new class of general linear methods is shown to be A-stable, A(α)-stable and L-stable of higher order and so is appropriate for the solution of certain ordinary differential and stiff differential equations.

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1. Introduction

Let us consider the stiff initial value problem

$$y'(x) = f(x, y(x)), \quad y(x_0) = y_0, \tag{1}$$

on the finite interval $I = [x_0, x_N]$ where $y : I \to R^m$ and $f : I \times R^m \to R^m$ is continuous and differentiable. For several decades, there has been a strong interest in deriving more advanced and efficient methods to integrate this initial value problem. A potentially good numerical method for the solution of stiff systems of ODEs must have good accuracy and some reasonably wide region of absolute stability [3]. One of the first and most important stability requirement, particulary for linear multistep method, is that A-stability which was proposed in [4]. However, the requirement of A-stability puts a sever limitation on the choice of suitable linear multistep methods. This is articulated in the so-called Dahlquist second barrier which says, among other things, that the order of an A-stable linear multistep method must be ≤ 2 and that an A-stable linear multistep method must be implicit. This pessimistic result has encouraged researchers to seek other classes of numerical methods for solving stiff equations. The search to improve the accuracy and extend the stability region, finding a high accurate and high efficient A-stable multistep method, is carried out in the two main directions:

• Using higher order derivatives of the solutions.
• Throw in additional stages, off-step points, super-future points and like. This leads into the large field general
linear methods[5].

One successful proposal in this direction was introduced by Enright[?] that used second derivative of solution in his algorithm. Cash [1] , Ismail [7] , Hojjati [6], Mehdizadeh [9] introduced second derivative multistep methods(SDMMs) that have good stability properties. By following a appropriate modifications, we were able to improve the stability regions and computational efficiency of SDMMs approach. The new class of second derivative formulas, has great advantage in accuracy and it is A-stable of order 8. Also our technique allows stiffly stable regions of higher order.

The paper is constructed as follows. In the next section, second derivative multistep method (SDMM) is described. In the third section, in the same lines of MF-SDMM$_{\texttt{new}}$[6] the MF-SISDMM is also introduced. The stability behavior of our approach is analyzed in the forth section, and a comparison is made with existing methods for A-stability and A(α)-stability orders. The numerical solutions and a comparison have been shown with some methods for results in the final section.

2.Second derivative multistep methods

Assume that the solution of the initial value problem(1) has the desired continuous derivatives. A SDMM can be written in the form:

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h \sum_{j=0}^{k} \beta_j f_{n+j} + h^2 \sum_{j=0}^{k} \gamma_j g_{n+j}, \tag{2}$$

where α_j, β_j and γ_j are parameters to be determined $g_{n+j} = f^{(1)}_{n+j}$. If either β_k or γ_k is nonzero, the formula will be implicit. Taylor expansion shows that method (2) is of order p if and only if

$$\sum_{j=0}^{k}\alpha_j j^q = q\sum_{j=0}^{k}\beta_j j^{q-1} + q(q+1)\sum_{j=0}^{k}\gamma_j j^{q-2} \quad , \quad 0 \le p \le q.$$

Some known important SDMM schemes that will be used for comparison are as follows:

• The Enright[4] k-step formulas of order $k+2$ which takes the following form:

$$y_{n+1} - y_n = h\sum_{j=0}^{k}\beta_j f_{n+j-k+1} + h^2\gamma_k g_{n+1}.$$

• Second derivative extended backward differentiation formulas (E2BD), that was introduced by Cash[1] with the following form:

Class 1 :

$$Predictor: \quad y_{n+k} - y_{n+k-1} = h\textstyle\sum_{j=0}^{k}\beta_j f_{n+j} + h^2\gamma_k g_{n+k},$$

$$Corrector: \quad y_{n+k} - y_{n+k-1} = h\textstyle\sum_{j=0}^{k+1}\bar{\beta}_j f_{n+j} + h^2(\bar{\gamma}_k g_{n+k} + \bar{\gamma}_{k+1} g_{n+k+1}),$$

Class 2 :

$$Predictor: \quad y_{n+k} - y_{n+k-1} = h\textstyle\sum_{j=0}^{k}\beta_j f_{n+j} + h^2\gamma_k g_{n+k},$$

$$Corrector: \quad y_{n+k} - y_{n+k-1} = h\textstyle\sum_{j=0}^{k+1}\bar{\beta}_j f_{n+j} + h^2\bar{\gamma}_k g_{n+k}.$$

These formulas are of order $k+2$.

• Ismail and Ibrahim [7] introduced special class of SDMM as follows:

$$\sum_{j=0}^{k}\alpha_j y_{n+j} = h\beta_k(f_{n+k} - \beta^* f_{n+k-1}) + h^2\gamma_k(g_{n+k} - \gamma^* g_{n+k-1}).$$

For $\beta^* = 0$,$\gamma^* = 0$ this is the same SDBDF[7]method. This class of methods turns out to be A-stable for k=1,2,3 (hence for p=2,3 and 4) and are stiffly stable

	Enright method		E2BDF Class 1		E2BDF Class 2		Ismail method		Hojjati method	
k	p	$\alpha(^\circ)$	p	$\alpha(^\circ)$	p	$\alpha(^\circ)$	p	$\alpha(^\circ)$	p	$\alpha_{max}(^\circ)$
1	3	90	4	90	4	90	2	90	3	90
2	4	90	5	90	5	90	3	90	4	90
3	5	87.88	6	90	6	90	4	90	5	90
4	6	82.03	7	90	7	89	5	89.9	6	90
5	7	73.10	8	90	8	87	6	87.3	7	89.8
6	8	59.95	9	89	9	83	7	84.2	8	88.3

Table 1: The A(α)-stability of some mentioned methods

for k=4,5,6,7,8 and 9 whenever $\beta^* = -0.5$,$\gamma^* = 0.9$.

• Hojjati[6] introduced second derivative multistep method as follows:

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h\beta_k f_{n+k} + h^2(\gamma_k g_{n+k} - \gamma_{k+1} g_{n+k+1}).$$

This method is of order $k+2$. The A(α)-stability of these methods are tabulate in Table 1.

3.Matrix free super-implicit second derivative multistep methods

We are going to introduce a new class of super-implicit second derivative multistep methods(SISDMM) with the following general form:

$$\sum_{j=0}^{k} \hat{\alpha}_j y_{n+j} = h(\hat{\beta}_k f_{n+k} + \hat{\beta}_{k+1} f_{n+k+1} + \hat{\beta}_{k+2} f_{n+k+2}) + h^2 \hat{\gamma}_k g_{n+k}. \qquad (3)$$

where $g(x,y) = y'' = f_x + f_y f$, $\hat{\beta}_j, \hat{\gamma_k}$ are parameters to be determined. Coefficients are chosen so that (3) has order $k+3$. The coefficients of method (3) are given in Table 2, for $k = 1, 2, \ldots, 8$. It has used two super-future points technique and designed so that to have good stability properties with high order of accuracy. Starting from given data $y_n, y_{n+1}, \ldots, y_{n+k-1}$, a predictor is first used to predict y_{n+k+1},y_{n+k+2}, the derivative approximations y'_{n+k+1},y'_{n+k+2} are then computed and finally y_{n+k} is computed from $y_n, y_{n+1}, \ldots, y_{n+k-1}, y'_{n+k+1}, y'_{n+k+2}$. The (k+1)th-order predictor we have used here, is the SDBDF method. The way in which (3) is used in practice is by carry out the following computations:

Stage 1 : Use the SDBDF to compute the first predictor $\bar{y}_{n+k}$, assuming that approximate solutions y_{n+j} have been computed at x_{n+j}, for $0 \le j \le k-1$

$$\bar{y}_{n+k} + \sum_{j=0}^{k-1} \alpha_j y_{n+j} = h\beta_k f(x_{n+k}, \bar{y}_{n+k}) + h^2\gamma_k g(x_{n+k}, \bar{y}_{n+k}), \tag{4}$$

where α_j, β_k and γ_k are the SDBDF coefficients. See Table 3.

Stage 2 : Compute the second predictor $\bar{y}_{n+k+1}$ by solving the following algebraic equation

$$\bar{y}_{n+k+1} + \alpha_{k-1}\bar{y}_{n+k} + \sum_{j=0}^{k-2} \alpha_j y_{n+j+1} = h\beta_k f(x_{n+k+1}, \bar{y}_{n+k+1}) + h^2\gamma_k g(x_{n+k+1}, \bar{y}_{n+k+1}), \tag{5}$$

Stage 3 : Compute $\bar{y}_{n+k+2}$ as the solution of

$$\bar{y}_{n+k+2} + \alpha_{k-1}\bar{y}_{n+k+1} + \alpha_{k-2}\bar{y}_{n+k} + \sum_{j=0}^{k-3} \alpha_j y_{n+j+2} = h\beta_k f(x_{n+k+2}, \bar{y}_{n+k+2}) + h^2\gamma_k g(x_{n+k+2}, \bar{y}_{n+k+2}), \tag{6}$$

Stage 4 : Compute a corrected solution of order (k+3) at x_{n+k} using

$$y_{n+k} + \sum_{j=0}^{k-1} \hat{\alpha}_j y_{n+j} = h(\hat{\beta}_k f_{n+k} + \hat{\beta}_{k+1}\bar{f}_{n+k+1} + \hat{\beta}_{k+2}\bar{f}_{n+k+2}) + h^2\hat{\gamma}_k g_{n+k}. \tag{7}$$

Note that at each stages 1,2,3 and 4 a system of nonlinear equation must be solved in order that the desired approximation can be computed. Usually, to solve these nonlinear systems, a modified Newton method is used. Then a direct method is used to solve any resulting system of linear equations. Hence, in each stage, it is necessary to obtain the Jacobian matrix, the related LU factorization matrices and a forward elimination and back substitution to solve a linear system. Thus causing more computational cost and running time. To avoid this, the approach described above can be modified (to give the so-called MF-SISDMM approach) such that the Jacobian matrix is not used explicitly. In this modification the inexact Newton method is used and then the IOM algorithm [11] is applied to solve the resulting system of linear equations. We observed that the solution of system of ODEs (1) reduced to the solution of the following system of (generally) nonlinear equations:

$$y_{n+k} - h\beta_k f(x_{n+k}, y_{n+k}) - h^2\gamma_k g(x_{n+k}, y_{n+k}) - a_{n+k} = 0,$$

where $a_{n+k} = -\sum_{j=0}^{k-1} a_j y_{n+j}$. If we let

$$X_{n+k} = h\beta_k f(x_{n+k}, y_{n+k}) + h^2\gamma_k g(x_{n+k}, y_{n+k}) = y_{n+k} - a_{n+k},$$

then we have the following system of nonlinear equations to be solved:

$$F(X_{n+k}) = X_{n+k} - h\beta_k f(x_{n+k}, a_{n+k} + X_{n+k}) - h^2\gamma_k g(x_{n+k}, a_{n+k} + X_{n+k}) = 0.$$

After applying a modified Newton method, we have

$$\big(I - h\beta_k \frac{\partial f}{\partial y}(x_{n+k}, y_{n+k}^{(m)})\big)\Delta y_{n+k}^{(m)} = h\beta_k f(x_{n+k}, y_{n+k}^{(m)}) + h^2\gamma_k g(x_{n+k}, y_{n+k}^{(m)}) - X_{n+k},$$

$$X_{n+k}^{(m+1)} = \Delta y_{n+k}^{(m)} + X_{n+k}^{(m)}.$$

In each step, we predict a value $y_{n+k}^{(0)}$ using a suitable one-step method say, one of the Runge-Kutta methods, and then using $X_{n+k}^{(0)} = y_{n+k}^{(0)} - a_{n+k}$, we predict $X_{n+k}^{(0)}$. Hence, the first system of linear equations to be solved in the n^{th} step is $AX = b$ where

$$A = F'(X_{n+k}^{(0)}) = I - h\beta_k \frac{\partial f}{\partial y}(x_{n+k}, y_{n+k}^{(0)}) - h^2\gamma_k g(x_{n+k}, y_{n+k}^{(0)}),$$

$$b = -F(X_{n+k}^{(0)}) = h\beta_k f(x_{n+k}, y_{n+k}^{(0)}) + h^2\gamma_k g(x_{n+k}, y_{n+k}^{(0)}) - X_{n+k}^{(0)}.$$

In stages 1, 2 and 3 the Jacobian matrix is $I - h\beta_k \frac{\partial f}{\partial y} - h^2\gamma_k \frac{\partial g}{\partial y}$ and for step 4 the Jacobian matrix is $I - h\hat{\beta}_k \frac{\partial f}{\partial y} - h^2\hat{\gamma}_k \frac{\partial g}{\partial y}$. By changing stage 4 to

Stage 4* : $y_{n+k} - h\beta_k f_{n+k} - h^2\gamma_k g_{n+k} = -\sum_{j=0}^{k-1} \hat{\alpha}_j y_{n+j} + h(\hat{\beta}_k - \beta_k)\bar{f}_{n+k} + h(\bar{f}_{n+k+1} + \bar{f}_{n+k+2})$

$$+h^2(\hat{\gamma}_k - \gamma_k)\bar{g}_{n+k},$$

the Jacobian matrix in each of 4 steps 1,2,3 and 4* is the same as $I - h\beta_k \frac{\partial f}{\partial y} - h^2\gamma_k \frac{\partial g}{\partial y}$.

It is important to notice that to apply the IOM to solve the linear system $AX = b$ the matrix of A is not needed explicitly, only the action of A times a vector v is necessary, for more details see [10]. If we let $F = \beta_k f + h\gamma_k g$, therefore by using $A = I - h\frac{\partial F}{\partial y}$, we require value of product $\frac{\partial F}{\partial y}v$. For this value, approximation is made using the difference quotient

$$\frac{\partial F}{\partial y}v \approx \frac{F(x_n, y_n + \sigma v) - F(x_n, y_n)}{\sigma}$$

for suitably chosen scalar σ. Note that if $f(x_n, y_n)$ and $g(x_n, y_n)$ have been saved, then this only requires one additional f and g evaluation. With these arrangements

k	1	2	3	4	5	6	7	8
d	48	1327	195989	1853431	141352313	2456058017	2593522395599	108883865938171
$\hat{\beta}_k$	$11/d$	$876/d$	$144384/d$	$1388172/d$	$105077940/d$	$1798199460/d$	$1865659618620/d$	$76926295023480/d$
$\hat{\beta}_{k+1}$	$44/d$	$400/d$	$29592/d$	$169344/d$	$8712000/d$	$109584000/d$	$88028892000/d$	$2916498816000/d$
$\hat{\beta}_{k+2}$	$-7/d$	$-46/d$	$-2646/d$	$-12336/d$	$-534000/d$	$-5787000/d$	$-4077927000/d$	$-120210249600/d$
$\hat{\gamma}_k$	$-9/d$	$-826/d$	$-88110/d$	$-668376/d$	$-43230600/d$	$-659273400/d$	$-625305277800/d$	$-23973496999200/d$
$\hat{\alpha}_0$	-1	$97/d$	$-2804/d$	$8009/d$	$-236688/d$	$1875350/d$	$-488308725/d$	$23704210845/d$
$\hat{\alpha}_1$		$-1424/d$	$30267/d$	$-83392/d$	$2548375/d$	$-21367392/d$	$12309944150/d$	$-307689004800/d$
$\hat{\alpha}_2$			$-223452/d$	$451008/d$	$-13280000/d$	$115089375/d$	$-70150486224/d$	$1869737178400/d$
$\hat{\alpha}_3$				$-2229056/d$	$47958000/d$	$-400144000/d$	$252006344625/d$	$-7105244407296/d$
$\hat{\alpha}_4$					$-178342000/d$	$1085174250/d$	$-657558097000/d$	$19150543041000/d$
$\hat{\alpha}_5$						$-3236685600/d$	$1428139684650/d$	$-39997397054720/d$
$\hat{\alpha}_6$							$-3557256704400/d$	$72456943624800/d$
$\hat{\alpha}_7$								$-154974463526400/d$

Table 2: Coefficients in (3)

k	1	2	3	4	5	6	7	8
d	2	7	85	415	12019	13489	726301	3144919
β_k	1	$6/d$	$66/d$	$300/d$	$8220/d$	$8820/d$	$457380/d$	$1917720/d$
γ_k	$-1/d$	$-2/d$	$-18/d$	$-72/d$	$-1800/d$	$-1800/d$	$-88200/d$	$-352800/d$
α_0	-1	$1/d$	$-4/d$	$9/d$	$-144/d$	$100/d$	$-3600/d$	$11025/d$
α_1		$-8/d$	$27/d$	$-64/d$	$1125/d$	$-864/d$	$34300/d$	$-115200/d$
α_2			$-108/d$	$216/d$	$-4000/d$	$3375/d$	$-148276/d$	$548800/d$
α_3				$-576/d$	$9000/d$	$-8000/d$	$385875/d$	$-1580544/d$
α_4					$-18000/d$	$13500/d$	$-686000/d$	$3087000/d$
α_5						$-21600/d$	$926100/d$	$-4390400/d$
α_6							$-1234800/d$	$4939200/d$
α_7								$-5644800/d$

Table 3: Coefficients in (4)

mentioned above, we expect MF-SISDMM to perform much better than E2BD-class 1, E2BD-Class 2 and SISDMM[9]. Our numerical results will confirm this expectation.

Lemma 1. *Let*

(i) formula (3) is of order $k+1$,

(ii) formula (2) is of order $k+3$,

(iii) the implicit algebra equations defining $\bar{y}_{n+k}, \bar{y}_{n+k+1}$ and $\bar{y}_{n+k+2}$ are solved exactly,

then scheme (3) has order $k+2$.

Proof. Suppose the values $y_n, y_{n+1}, \ldots, y_{n+k-1}$ be exact. From (4) we have

$$y(x_{n+k}) - \bar{y}_{n+k} = C_1 h^{k+2} y^{(k+2)}(x_{n+k}) + \mathcal{O}(h^{k+3}),$$

and for one super-future point we have

$$\begin{aligned} y(x_{n+k+1}) - \bar{y}_{n+k+1} &= C_1 h^{k+2} y^{(k+2)}(x_{n+k+1}) + \mathcal{O}(h^{k+3}) \\ &= C_1 h^{k+2} (y^{(k+2)}(x_{n+k}) + h y^{(k+3)}(x_{n+k}) + \tfrac{h^2}{2} y^{(k+4)}(x_{n+k}) + \cdots) + \mathcal{O}(h^{k+3}) \\ &= C_1 h^{k+2} y^{(k+2)}(x_{n+k}) + \mathcal{O}(h^{k+3}). \end{aligned}$$

But since in (5) we apply $\bar{y}_{n+k}$, we must add the error of $\alpha_{k-1}(y(x_{n+k}) - \bar{y}_{n+k})$ to the above expression. Hence

$$\begin{aligned} y(x_{n+k+1}) - \bar{y}_{n+k+1} &= C_1 h^{k+2} y^{(k+2)}(x_{n+k}) + \mathcal{O}(h^{k+3}) - \alpha_{k-1} C_1 h^{k+2} y^{(k+2)}(x_{n+k}) \\ &= C_1 (1 - \alpha_{k-1}) h^{k+2} y^{(k+2)}(x_{n+k}) + \mathcal{O}(h^{k+3}), \end{aligned} \tag{8}$$

and for two super-future point we have

$$\begin{aligned} y(x_{n+k+2}) - \bar{y}_{n+k+2} &= C_1 h^{k+2} y^{(k+2)}(x_{n+k+2}) + \mathcal{O}(h^{k+3}) \\ &= C_1 h^{k+2} (y^{(k+2)}(x_{n+k}) + 2h y^{(k+3)}(x_{n+k}) + \cdots) + \mathcal{O}(h^{k+3}) \\ &= C_1 h^{k+2} y^{(k+2)}(x_{n+k}) + \mathcal{O}(h^{k+3}). \end{aligned}$$

But since in (6) we apply $\bar{y}_{n+k}$ and $\bar{y}_{n+k+1}$, we must add the error of $\alpha_{k-2}(y(x_{n+k}) -$

$\bar{y}_{n+k})$ and $\alpha_{k-1}(y(x_{n+k+1})-\bar{y}_{n+k+1})$ to the above expression. Hence

$$\begin{aligned} y(x_{n+k+2})-\bar{y}_{n+k+2} &= C_1h^{k+2}y^{(k+2)}(x_{n+k})+\mathcal{O}(h^{k+3}) \\ &\quad -\alpha_{k-2}C_1h^{k+2}y^{(k+2)}(x_{n+k})-\alpha_{k-1}C_1h^{k+2}y^{(k+2)}(x_{n+k}) \\ &= C_1(1-\alpha_{k-2}-\alpha_{k-1})h^{k+2}y^{(k+2)}(x_{n+k})+\mathcal{O}(h^{k+3}). \end{aligned} \tag{9}$$

If now $C_2h^{k+4}y^{(k+4)}(x_{n+k})+\mathcal{O}(h^{k+5})$ is the defect of formula (3), replacing $f(x_{n+k+1},y_{n+k+1})$ by $f(x_{n+k+1},\bar{y}_{n+k+1})$ and $f(x_{n+k+2},y_{n+k+2})$ by $f(x_{n+k+2},\bar{y}_{n+k+2})$, we obtain

$$\begin{aligned} y(x_{n+k})-y_{n+k} &= C_2h^{k+4}y^{(k+4)}(x_{n+k})-\hat{\beta}_{k+1}h(f(x_{n+k+1},y(x_{n+k+1})) \\ &\quad -f(x_{n+k+1},\bar{y}_{n+k+1}))-\hat{\beta}_{k+2}h(f(x_{n+k+2},y(x_{n+k+2})) \\ &\quad -f(x_{n+k+2},\bar{y}_{n+k+2})). \end{aligned}$$

From (8) and (9)

$$\begin{aligned} f(x_{n+k+1},y(x_{n+k+1}))-f(x_{n+k+1},\bar{y}_{n+k+1}) &= \tfrac{\partial f}{\partial y}(\eta_1)(y(x_{n+k+1})-\bar{y}_{n+k+1}) \\ &= \tfrac{\partial f}{\partial y}(\eta_1)C_1(1-\alpha_{k-1})h^{k+2}y^{(k+2)}(x_{n+k})+\mathcal{O}(h^{k+3}), \\ f(x_{n+k+2},y(x_{n+k+2}))-f(x_{n+k+2},\bar{y}_{n+k+2}) &= \tfrac{\partial f}{\partial y}(\eta_2)(y(x_{n+k+2})-\bar{y}_{n+k+2}) \\ &= \tfrac{\partial f}{\partial y}(\eta_2)C_1(1-\alpha_{k-2}-\alpha_{k-1})h^{k+2}y^{(k+2)}(x_{n+k})+\mathcal{O}(h^{k+3}). \end{aligned}$$

This yields

$$\begin{aligned} y(x_{n+k})-\bar{y}_{n+k} &= C_2h^{k+4}y^{(k+4)}(x_{n+k}) \\ &\quad -\hat{\beta}_{k+1}h(\tfrac{\partial f}{\partial y}(\eta_1)C_1(1-\alpha_{k-1})h^{k+2}y^{(k+2)}(x_{n+k})+\mathcal{O}(h^{k+3})) \\ &\quad -\hat{\beta}_{k+2}h(\tfrac{\partial f}{\partial y}(\eta_2)C_1(1-\alpha_{k-2}-\alpha_{k-1})h^{k+2}y^{(k+2)}(x_{n+k})+\mathcal{O}(h^{k+3})) \\ &= h^{k+3}(C_2hy^{(k+4)}(x_{n+k})-\tfrac{\partial f}{\partial y}(\eta_1)C_1(1-\alpha_{k-1})\hat{\beta}_{k+1}y^{(k+2)}(x_{n+k}) \\ &\quad -\tfrac{\partial f}{\partial y}(\eta_2)C_1(1-\alpha_{k-2}-\alpha_{k-1})\hat{\beta}_{k+2}y^{(k+2)}(x_{n+k})+\mathcal{O}(h^{k+3}), \end{aligned}$$

which shows that the order of scheme (3) is $k+2$.

x	y_i	Theoretical Solution	Error in MF-SISDMM
1	y_1	2.53836814408295E-1	0.31E-7
	y_2	1.10363832351433E-1	0.43E-7
10	y_1	2.67859585598661E-4	0.51E-10
	y_2	1.81599719049931E-4	0.63E-10
20	y_1	1.21608063723875E-8	0.17E-14
	y_2	8.24461448975422E-9	0.33E-14

Table 4: Numerical results for Example 1

4.Numerical Results

In this section, we present some numerical results to compare our new class of methods with that of other second derivative multistep methods. What we shall be attempting to do, is to show the superior performance of new method for a given fixed stepsize over some special methods for a small selection of examples.

Example 1. In our first experiment, we have used the MF-SISDMM to solve the following initial value problem:

$$y_1' = -41y_1 + 56y_2 - 2x^3(x^2 - 50x - 2)e^{-x^2},$$

$$y_2' = 40y_1 - 60y_2 + 2x^3(x^2 - 50x - 2)e^{-x^2},$$

with initial value $y(0) = (9.91, 0)^T$. The theoretical solution is

$$y_1(x) = 4e^{-100x} + 5.9e^{-x} + x^4e^{-x^2},$$

$$y_2(x) = -4e^{-100x} + 4e^{-x} - x^4e^{-x^2}.$$

We have solved this problem for $x = 1$, $x = 10$ and $x = 20$. A fixed stepsize $h = 0.001$ has used here and the order of method is five. We list the absolute errors in Table 5.

x	y_i	Error in MF-SISDMM	Error in E2BD-Class 1	Error in E2BD-Class 2
4.5	y_1	0.6E-14	<0.1E-10	<0.1E-10
	y_2	0.6E-14	<0.1E-10	<0.1E-10
9	y_1	0.6E-17	<0.1E-12	<0.1E-12
	y_2	0.6E-17	<0.1E-12	<0.1E-12
13.5	y_1	0.5E-18	<0.1E-15	0.1E-11
	y_2	0.4E-18	<0.1E-15	0.1E-11
18	y_1	0.4E-21	<0.1E-17	0.1E-11
	y_2	0.5E-21	<0.1E-17	0.1E-11

Table 5: Numerical results of Example 2, for the case $\alpha = 1$, $\beta = 30$

Example 2. In our second example, we consider the following stiff ODEs

$$\begin{aligned} y_1' &= -\alpha y_1 - \beta y_2 + (\alpha + \beta - 1)e^{-x}, \\ y_2' &= \beta y_1 - \alpha y_2 - (\alpha - \beta - 1)e^{-x}, \end{aligned}$$

with initial value $y(0) = (1, 1)^T$. In order to make this system homogeneous, we introduce an additional variable y_3 such that

$$y_3' = 1, y_3(0) = 0.$$

The eigenvalues of the Jacobian associated with the resulting system are $-\alpha \pm i\beta, 0$ and the required solution is

$$y_1(x) = y_2(x) = e^{-x}.$$

In Table 6 we give the results obtained for the integration of this problem using a stepsize $h = 0.01$ for the case $\alpha = 1$, $\beta = 15$. We solve this problem at $x = 4.5, 9, 13.5, 18$ using the new method of order five and a comparison is made with the results of E2BD-Class 1 and E2BD-Class 2 reported by Cash [1].

Example 3. We solve the van der Pol's equation

$$\begin{aligned} y_1' &= y_2, \\ y_2' &= \mu^2((1 - y_1^2)y_2 - y_1), \end{aligned}$$

x	y_i	MF-SISDMM
1	y_1	-1.865095095034
	y_2	0.7524845332931
5	y_1	1.898512781456
	y_2	-0.7289766066725
10	y_1	1.786196476523
	y_2	-0.8154281623431
20	y_1	1.504881812954
	y_2	-1.189933304390

Table 6: The results for Example 3

with initial value $y(0) = (2, 0)^T$. We have summarized the results at $x = 1, 5, 10, 20$ using a stepsize $h = 0.001$ in Table 7. It should also be noted that $\mu = 500$ has been used here and the order of method is five.

Example 4. In our last example, we ran our MF-SISDMM using $h = 0.001$ and compared the results with those of Ismail methods [7], SISDMM [9] and SDBDF [5] for solving the following stiff problem arose from a chemistry problem

$$\begin{aligned} y_1' &= -0.013y_2 - 1000y_1y_2 - 2500y_1y_3, \\ y_2' &= -0.013y_2 - 1000y_1y_2, \\ y_3' &= -2500y_1y_3, \end{aligned}$$

with initial value $y(0) = (0, 1, 1)^T$. We have solved this problem at $x = 2.0$ and the order of method is four. One can also use the smaller stepsize to get significantly more accurate than this results. For the numerical results, see Table 8.

x	y_i	Exact solution	Error in MF-SISDMM	Error in Ismail method	Error in SISDMM	Error in SDBDF
2.0	y_1	-0.3616933169289E-5	0.52E-15	0.82E-10	0.43E-14	0.31E-08
	y_2	0.9815029948230	0.78E-11	0.61E-05	0.17E-10	0.18E-05
	y_3	01.018493388244	0.63E-10	0.57E-05	0.37E-9	0.57E-05

Table 7: Numerical results for Example 4

5. Conclusions

In this paper we developed A-stable second derivative multistep methods. The advantage of MF-SISDMM is that they have extensive region of stability and particularity are A-stable up to order 8. This property, let us to apply the new method for numerical solution of stiff systems of ODEs with high accuracy.

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