Acta Universitatis Apulensis No. 35/2013
ISSN: 1582-5329 pp. 111-123

THE EXTENDED F-IMPLICIT COMPLEMENTARITY AND VARIATIONAL INEQUALITY PROBLEMS IN SEMI-INNER PRODUCT SPACES

N. K. Sahu, C. Nahak, S. Nanda

Abstract. In this paper, the extended F-implicit complementarity problem (extended (F-ICP)) and extended F-implicit variational inequality problem (extended (F-IVIP)) are introduced in semi-inner product spaces. Under certain assumptions, we prove the equivalence between the extended (F-ICP) and extended (F-IVIP). Moreover, we establish some new existence theorems of solutions for the extended (F-ICP) and extended (F-IVIP). We present an alternative proof for the existence theorem of solutions of the extended (F-IVIP).

2000 *Mathematics Subject Classification*: 47J20, 90C33, 46C50.

Keywords: variational inequality, complementarity, semi-inner product space.

1. Introduction and Preliminaries

Variational inequalities introduced in the early sixties have witnessed an explosive growth in theoretical advances, algorithmic development and applications across all the discipline of pure and applied mathematics. It combines novel theoretical and algorithmic advances with new domain of applications. The variational inequality, as an important subject of current mathematics, has not only stimulated new results dealing with partial differential equations, but also has been used in a large variety of problems arising in mechanics, physics, optimization and control, economic and transportation equilibrium, and engineering sciences. Variational inequalities theory, the origin of which can be traced back to Stampacchia [12], provides us with a simple, natural, general and unified framework to study a wide class of problems arising in pure and applied sciences. Because of its wide applications, the classical variational inequality has been well studied and generalized in various directions.

Complementarity theory provides us with a natural and elegant framework for the study of many unrelated free boundary value and equilibrium problems. The linear complementarity problem is applied for the development of theory and algorithms in linear and quadratic programming problems. Nonlinear complementarity problems have been applied to the solution of Nash equilibrium, traffic assignment or network equilibrium, spatial price equilibrium and the general equilibrium problems. The complementarity and variational inequality theory are powerful tools for the development of the modern science and technological age. The classical complementarity problem can be considered as equivalent form of the variational inequality problem. The classical complementarity and variational inequality problems have been extended and generalized to study a wide class of problems arising in mechanics, physics, optimization and control, nonlinear programming, game theory, economics, finance, regional, structural, transportation and elasticity etc.

Lumer [11] introduced the concept of semi-inner product in the following sense.

Semi-inner product. Let X be a vector space over the field F of real or complex numbers. A functional $[\ ,\ ] : X \times X \to F$ is called a semi-inner product if it satisfies the following conditions:

(i) $[x + y, z] = [x, z] + [y, z]$ for all $x, y, z \in X$

(ii) $[\lambda x, y] = \lambda[x, y]$ for all $\lambda \in F$ and $x, y \in X$

(iii) $[x, x] > 0$ for $x \neq 0$

(iv) $|[x, y]|^2 \leq [x, x][y, y]$ for all $x, y \in X$.

The pair $(X, [\ ,\ ])$ is called a semi-inner product space.

A semi-inner product space is a normed linear space with the norm $\|x\| = [x, x]^{\frac{1}{2}}$. Every normed linear space can be made into a semi-inner product space in infinitely many different ways. Giles [4] had shown that if the underlying space X is a uniformly convex smooth Banach space then it is possible to define a semi-inner product uniquely. Also the unique semi-inner product has the following nice properties:

(i) $[x, \lambda y] = \overline{\lambda}[x, y]$ for all scalars λ.

(ii) $[x, y] = 0$ if and only if y is orthogonal to x, that is if and only if $\|y\| \leq \|y+\lambda x\|$ for all scalars λ.

(iii) Generalized Riesz representation theorem:- If f is a continuous linear functional on X then there is a unique vector $y \in X$ such that $f(x) = [x, y]$ for all $x \in X$.

(iv) The semi-inner product is continuous, that is for each $x, y \in X$, we have $Re[y, x + \lambda y] \to Re[y, x]$ as $\lambda \to 0$.

The sequence space $l^p,\ p > 1$ and the function space $L^p,\ p > 1$ are uniformly convex smooth Banach spaces. So one can define semi-inner product on these spaces uniquely. Giles [4] had shown that the function space $L^p,\ p > 1$ is a semi-inner product space with the semi-inner product defined by:

$$[f,g] := \frac{1}{\|g\|_p^{p-2}} \int_X f(x)|g(x)|^{p-1} sgn(g(x))d\mu, \quad \forall f,g \in L^P(X,\mu).$$

Classical variational inequality and complementarity problem in Hilbert space. Let H be a Hilbert space, K be a nonempty closed convex subset of H. Let T be an operator defined on H. The problem is to find a vector $u \in K$ such that $\langle Tu, v-u\rangle \geq 0, \quad \forall v \in K$. This is known as the classical variational inequality problem introduced and studied by Stampacchia [12] in 1964. The classical complementarity problem is to find a vector $x \in K$ such that $\langle Tx, x\rangle = 0$ and $\langle Tx, y\rangle \geq 0, \quad \forall y \in K$ (see Borwein and Dempster [1]).

In 1995, Khan [9] proved the existence and uniqueness theorem of the solution for a class of mildly nonlinear complementarity problems in a uniformly convex and strongly smooth Banach space equipped with a semi-inner product. Ceng and Yao [2] in 2007, established an extragradient-like approximation method for solving variational inequality problems and fixed point problems in a Hilbert space. Recently in 2011, Junlouchai and Plubtieng [8] proved an existence theorem for the solutions of generalized variational inequality problem for upper semi-continuous multivalued mappings with compact contractible values over compact convex subsets in a reflexive Banach space. For some recent literature one may refer to He [5], Huang and Gu [7], Wong et al. [13], Yu and Yang [15], Zeng and Yao [16].

F-CP and F-VIP in Banach space. Let X be a Banach space and K be a closed convex cone in X. Yin et al. [14] introduced a class of F-complementarity problem (F-CP) which consists of finding an $x \in K$ such that

$$\langle Tx, x\rangle + F(x) = 0, \quad \langle Tx, y\rangle + F(y) \geq 0, \quad \forall y \in K,$$

where $T : K \to X^*$, X^* denotes the space of continuous linear functionals on X, $F : K \to \mathbb{R}$ and $\langle Tx, y\rangle$ denotes the value of the functional Tx at a point y. They proved that (F-CP) is equivalent to the following F-variational inequality problem (F-VIP): find $x \in K$ such that

$$\langle Tx, y-x\rangle + F(y) - F(x) \geq 0, \quad \forall y \in K,$$

where F is a positively homogeneous convex function. They also proved the existence of solutions for (F-CP) under the assumption of F-pseudo monotonicity.

Huang and Li [6] studied a new class of F-implicit complementarity problems (F-ICP) and F-implicit variational inequality (F-IVIP) problems in Banach spaces.

They proved the equivalence between the (F-ICP) and (F-IVIP) and established some new existence results of the said problems.

(F-ICP) and (F-IVIP) in Banach space: Let X be a real Banach space with dual space X^* and K be a nonempty closed convex cone of X. Let $T : K \to X^*$, $g : K \to K$ and $F : K \to \mathbb{R}$ be mappings. The (F-ICP) is to find $x \in K$ such that

$$\langle Tx, g(x)\rangle + F(g(x)) = 0, \quad \textit{and} \quad \langle Tx, y\rangle + F(y) \geq 0, \quad \forall y \in K.$$

The (F-IVIP) is to find $x \in K$ such that

$$\langle Tx, y - g(x)\rangle \geq F(g(x)) - F(y), \quad \forall y \in K.$$

In this paper, the extended F-implicit complementarity problem (extended (F-ICP)) and extended F-implicit variational inequality problem (extended (F-IVIP)) are introduced in a real continuous semi-inner product space. The equivalence between the two problems extended (F-ICP) and extended (F-IVIP) is proved under certain assumptions. Moreover, we establish some new existence theorems of solutions for the extended (F-ICP) and extended (F-IVIP) by using the KKM theorem [3] and Lin's result [10] without F-pseudo monotonicity condition. The extended (F-ICP) and extended (F-IVIP) are defined here as follows:

Extended F-implicit complementarity problem (Extended (F-ICP)): Let X be a continuous real semi-inner product space and K be a non empty closed convex subset of X. Let $F, f : K \to \mathbb{R}$ be two real valued functions on K, $g : K \to K$ be another function and c be a real positive constant. The extended (F-ICP) is to find $\overline{x} \in K$ such that

$$[g(\overline{x}), \overline{x}] + F(g(\overline{x})) + cf(\overline{x}) = 0, \quad [y, \overline{x}] + F(y) \geq 0, \quad \forall y \in K.$$

Extended F-implicit variational inequality problem (Extended (F-IVIP)): Let X be a continuous real semi-inner product space and K be a non empty closed convex subset of X. Let $F, f : K \to \mathbb{R}$ be two real valued functions on K, $g : K \to K$ be another function and c be a real positive constant. The extended (F-IVIP) is to find $\overline{x} \in K$ such that

$$[y - g(\overline{x}), \overline{x}] \geq F(g(\overline{x})) - F(y) + cf(\overline{x}), \quad \forall y \in K.$$

Remark.

- When $c = 0$, the extended (F-ICP) and extended (F-IVIP) reduce to (F-ICP) and (F-IVIP), respectively.

- When $c = 0$ and $g = I$, where I is the identity map, our definitions reduce to (F-CP) and (F-VIP).
- When $c = 0$, $g = I$ and $F = 0$, our definitions reduce to the classical complementarity and variational inequality problems in Banach spaces.

2. Main Results

We prove an equivalence theorem between the extended (F-ICP) and extended (F-IVIP).

Theorem 1. *If $\overline{x}$ is a solution of the extended (F-ICP), then $\overline{x}$ is also a solution of the extended (F-IVIP). Conversely, if $F : K \to \mathbb{R}$ is a positive homogeneous and convex function, $f : K \to \mathbb{R}$ is positive and $\overline{x}$ solves the extended (F-IVIP), then $\overline{x}$ solves the extended (F-ICP).*

Proof. Let $\overline{x}$ be a solution of the extended (F-ICP), then there exists $\overline{x} \in K$ such that

$$[g(\overline{x}), \overline{x}] + F(g(\overline{x})) + cf(\overline{x}) = 0, \quad [y, \overline{x}] + F(y) \geq 0, \quad \forall y \in K.$$

Now

$$\begin{aligned} [y - g(\overline{x}), \overline{x}] &= [y, \overline{x}] - [g(\overline{x}), \overline{x}] \\ &\geq F(g(\overline{x})) + cf(\overline{x}) - F(y), \quad \forall y \in K. \end{aligned}$$

Thus $\overline{x}$ is a solution of the extended (F-IVIP).
Conversely, let $\overline{x}$ be a solution of the extended (F-IVIP), then there exists $\overline{x} \in K$ such that

$$[y - g(\overline{x}), \overline{x}] \geq F(g(\overline{x})) - F(y) + cf(\overline{x}), \quad \forall y \in K. \tag{1}$$

Since $f : K \to \mathbb{R}$ is positive, $F : K \to \mathbb{R}$ is a positive homogeneous convex function and K is a convex cone, then taking $y = 2g(\overline{x})$ in (1), we get:

$$\begin{aligned} & [2g(\overline{x}) - g(\overline{x}), \overline{x}] \geq F(g(\overline{x})) - 2F(g(\overline{x})) + cf(\overline{x}). \\ \Rightarrow \quad & [g(\overline{x}), \overline{x}] \geq -F(g(\overline{x})) + cf(\overline{x}). \\ \Rightarrow \quad & [g(\overline{x}), \overline{x}] + F(g(\overline{x})) - cf(\overline{x}) \geq 0. \\ \Rightarrow \quad & [g(\overline{x}), \overline{x}] + F(g(\overline{x})) + cf(\overline{x}) \geq 0. \qquad (2) \end{aligned}$$

Again taking $y = \frac{1}{2}g(\overline{x})$, we get

$$\begin{aligned}
&[\frac{1}{2}g(\overline{x}) - g(\overline{x}), \overline{x}] \geq F(g(\overline{x})) - \frac{1}{2}F(g(\overline{x})) + cf(\overline{x}).\\
\Rightarrow\quad & -\frac{1}{2}[g(\overline{x}), \overline{x}] \geq \frac{1}{2}F(g(\overline{x})) + cf(\overline{x}).\\
\Rightarrow\quad & [g(\overline{x}), \overline{x}] + F(g(\overline{x})) + 2cf(\overline{x}) \leq 0.\\
\Rightarrow\quad & [g(\overline{x}), \overline{x}] + F(g(\overline{x})) + cf(\overline{x}) \leq 0. \qquad (3)
\end{aligned}$$

Now, from (2) and (3), we get

$$[g(\overline{x}), \overline{x}] + F(g(\overline{x})) + cf(\overline{x}) = 0. \qquad (4)$$

Using (1) and (4), we obtain

$$\begin{aligned}
[y, \overline{x}] &= [y - g(\overline{x}), \overline{x}] + [g(\overline{x}), \overline{x}]\\
&\geq F(g(\overline{x})) - F(y) + cf(\overline{x}) - F(g(\overline{x})) - cf(\overline{x}).
\end{aligned}$$

$$\Rightarrow [y, \overline{x}] + F(y) \geq 0, \quad \forall y \in K.$$

Hence $\overline{x}$ is a solution of the extended (F-ICP).

To prove the main theorem of the extended F-implicit variational inequality problem, we first define KKM-map and state the famous KKM-lemma.

Definition 1. *Let K be a nonempty subset of a topological vector space X. A point-to-set mapping $T : K \to 2^X$ is called a Knaster-Kuratowski-Mazurkiewicz-mapping (KKM-mapping) if for every finite subset $\{x_1, x_2, ..., x_n\}$ of K, $conv\{x_1, x_2, ..., x_n\}$ is contained in $\bigcup_{i=1}^{n} T(x_i)$, where $conv\{x_1, x_2, ..., x_n\}$ denotes the convex hull of $\{x_1, x_2, ..., x_n\}$.*

Lemma 2. *(Fan [3]) Let K be a nonempty convex subset of a Housdorff topological vector space X. Let $G : K \to 2^K$ be a KKM-mapping such that for any $y \in K$, $G(y)$ is closed and $G(\overline{y})$ is compact for some $\overline{y} \in K$. Then there exists $\overline{x} \in K$ such that $\overline{x} \in G(y)$ for all $y \in K$. Hence $\cap_{y \in K} G(y)$ is nonempty.*

Now we prove the following theorem for the existence of solution of the extended (F-IVIP).

Theorem 3. *Let X be a continuous real semi-inner product space and K be a nonempty closed convex subset of X. Assume that*
(a) $F, f : K \to \mathbb{R}$ are lower semi-continuous functions, $g : K \to K$ is continuous,

and c is a positive real constant;
(b) there exists a function $h : K \times K \to \mathbb{R}$ such that
(i) $h(x,x) \geq 0, \quad \forall x \in K$
(ii) $h(x,y) - [y - g(x), x] \leq F(y) - F(g(x)) - cf(x), \quad \forall x, y \in K$
(iii) the set $\{y \in K : h(x,y) < 0\}$ is convex for all $x \in K$;
(c) there exists a non empty, compact and convex subset C of K such that for all $x \in K \setminus C$ (complement of C), there exists $y \in C$ such that $[y - g(x), x] < F(g(x)) - F(y) + cf(x)$.

Then the extended (F-IVIP) has a solution. Moreover, the solution set of the extended (F-IVIP) is compact.

Proof. Construct the sets $G(y) = \{x \in C : \ [y - g(x), x] \geq F(g(x)) - F(y) + cf(x)\}$ for all $y \in K$. For each $y \in K$, we claim that $G(y)$ is closed. Let $\{x_n\}$ be a sequence in $G(y)$ and $\{x_n\}$ converges to x. Then

$$[y - g(x_n), x_n] \geq F(g(x_n)) - F(y) + cf(x_n), \quad \forall n.$$

As $n \to \infty$, we have

$$\begin{aligned} [y - g(x), x] = \lim_{n\to\infty} [y - g(x_n), x_n] &\geq \lim_{n\to\infty} F(g(x_n)) - F(y) + c \lim_{n\to\infty} f(x_n) \\ &\geq F(g(x)) - F(y) + cf(x). \end{aligned}$$

Hence $x \in G(y)$ and $G(y)$ is closed. Since any element $\overline{x} \in \bigcap_{y\in K} G(y)$ is a solution of the extended (F-IVIP), we show that $\bigcap_{y \in K} G(y) \neq \phi$. Since C is compact, it is sufficient to prove that the family $\{G(y)\}_{y\in K}$ has the finite intersection property.

Let $\{y_1, y_2, ..., y_n\}$ be a finite subset of K and set $B = \overline{conv}(C \cup \{y_1, y_2, ..., y_n\})$, where $\overline{conv}$ denotes the closure of the convex hull. Then B is compact and convex subset of K. We construct the following point-to-set mappings $F_1, F_2 : B \to 2^B$ as follows :

$$\begin{aligned} F_1(y) &= \{x \in B : \ [y - g(x), x] \geq F(g(x)) - F(y) + cf(x)\} \\ F_2(y) &= \{x \in B : \ h(x,y) \geq 0\}, \quad \forall y \in B. \end{aligned}$$

We use the KKM-lemma to show the finite intersection property of the family $\{G(y)\}_{y\in K}$. To show this, we need the following steps:

(A) $F_1(y)$ is nonempty : From (b) (i) and (ii), we have $h(y,y) \geq 0$ and $h(y,y) - [y - g(y), y] \leq F(y) - F(g(y)) - cf(y)$. This implies that $[y - g(y), y] \geq F(g(y)) - F(y) + cf(y)$, and hence $F_1(y)$ is nonempty.

(B) $F_1(y)$ is closed: Let $\{x_n\}$ be sequence of elements in $F_1(y)$ and $\{x_n\}$ converges to x. We have $[y - g(x_n), x_n] \geq F(g(x_n)) - F(y) + cf(x_n)$. Letting $n \to \infty$, we get $[y - g(x), x] \geq F(g(x)) - F(y) + cf(x)$. Thus $x \in F_1(y)$, and hence $F_1(y)$ is closed.

(C) $F_1(y)$ is compact : $F_1(y)$ is compact since it is a closed subset of the compact set B.

(D) F_2 is a KKM-mapping : Assume that F_2 is not a KKM-mapping. Then there exists a finite subset $\{u_1, u_2, ..., u_n\}$ of B and $\lambda_i \geq 0,\ i = 1, 2, ..., n$ with $\sum_{i=1}^{n} \lambda_i = 1$ such that $u = \sum_{i=1}^{n} \lambda_i u_i \notin \bigcup_{j=1}^{n} F_2(u_j)$. Then $h(u, u_j) < 0$ for $j = 1, 2, ..., n$. From the assumption (b) (iii), we have $h\big(u, \sum_{i=1}^{n} \lambda_i u_i\big) < 0$. This implies that $h(u, u) < 0$, which contradicts the assumption (b) (i). Hence F_2 is a KKM-mapping.

(E) F_1 is a KKM-mapping : Let $x \in F_2(y)$. This implies that $h(x, y) \geq 0,\ \ \forall y \in B$. From (b) (ii), we have

$$\begin{aligned} & 0 - [y - g(x), x] \leq F(y) - F(g(x)) - cf(x) \\ \Rightarrow \quad & [y - g(x), x] \geq F(g(x)) - F(y) + cf(x). \end{aligned}$$

This implies that $x \in F_1(y)$ and $F_2(y) \subseteq F_1(y),\ \ \forall y \in B$. Thus F_1 is also a KKM-mapping.

From Lemma 2, there exists $\overline{x} \in B$ such that $\overline{x} \in F_1(y)$ for all $y \in B$. Hence we have

$$[y - g(\overline{x}), \overline{x}] \geq F(g(\overline{x})) - F(y) + cf(\overline{x}), \quad \forall y \in B.$$

By assumption (c), we get $\overline{x} \in C$ and moreover $\overline{x} \in G(y_i)$ for $i = 1, 2, ..., n$. So $\{G(y)\}_{y \in K}$ has the finite intersection property.

Since $g : K \to K$ is continuous, $F : K \to \mathbb{R}$ and $f : K \to \mathbb{R}$ are lower semi-continuous functions, then it is easy to see that the solutions set of extended (F-IVIP) is closed. From the assumption (c), any element outside the set C cannot be a solution of the extended (F-IVIP). Therefore the solution set of the extended (F-IVIP) must be contained in C. Since C is compact, the solution set of the extended (F-IVIP) is also compact.

Taking $f = 0$ in the above Theorem 3, we have the following corollary for the existence of the solution of the (F-IVIP).

Corollary 4. *Assume that*

(a) $g : K \to K$ *is continuous and* $F : K \to \mathbb{R}$ *is lower semi-continuous function;*

(b) there exists a function $h : K \times K \to \mathbb{R}$ *such that*

(i) $h(x, x) \geq 0, \quad \forall x \in K$

(ii) $h(x, y) - [y - g(x), x] \leq F(y) - F(g(x)), \quad \forall x, y \in K$

(iii) the set $\{y \in K : h(x,y) < 0\}$ *is convex for all* $x \in K$;
(c) there exists a non empty, compact and convex subset C *of* K *such that for all* $x \in K \setminus C$, *there exists* $y \in C$ *such that* $[y - g(x), x] < F(g(x)) - F(y)$.
Then the (F-IVIP) has a solution. Moreover, the solution set of (F-IVIP) is compact.

When $g = I$, we have the following corollary for the existence of the solution of the extended (F-VIP).

Corollary 5. *Assume that*
(a) $F, f : K \to \mathbb{R}$ *are lower semi-continuous functions;*
(b) there exists a function $h : K \times K \to \mathbb{R}$ *such that*
(i) $h(x,x) \geq 0, \quad \forall x \in K$
(ii) $h(x,y) - [y - x, x] \leq F(y) - F(x) - cf(x), \quad \forall x, y \in K$ *and* $0 < c \in \mathbb{R}$
(iii) the set $\{y \in K : h(x,y) < 0\}$ *is convex for all* $x \in K$;
(c) there exists a non empty, compact and convex subset C *of* K *such that for all* $x \in K \setminus C$, *there exists* $y \in C$ *such that* $[y - x, x] < F(x) - F(y) - cf(x)$.
Then the extended (F-VIP) has a solution. Moreover, the solution set of the extended (F-VIP) is compact.

Theorem 6. *Let* $g : K \to K$ *be a continuous function,* $f : K \to \mathbb{R}$ *be positive and lower semi-continuous function and* $F : K \to \mathbb{R}$ *be lower semi-continuous, positive, homogeneous and convex function. If the assumptions (b) and (c) in Theorem 3 hold, then the extended (F-ICP) has a solution. Moreover, the solution set of the extended (F-ICP) is compact.*

Proof. The proof follows from Theorem 1 and Theorem 3.

It is easy to see that assumptions (i) and (ii) of (b) in Theorem 3 imply that $[x - g(x), x] \geq F(g(x)) - F(x) + cf(x), \quad \forall x \in K$. Replacing the assumption (iii) of (b) in Theorem 3 by the convexity property of F and by applying Lemma 2, we obtain the following result for the existence of solution of the extended (F-IVIP).

Theorem 7. *Let* $F, f : K \to \mathbb{R}$ *be lower semi-continuous functions,* $g : K \to K$ *be continuous and* c *be a positive real constant. Assume that* $[x - g(x), x] \geq F(g(x)) - F(x) + cf(x), \ \forall x \in X$. *Moreover, if there exists a nonempty, compact and convex subset* C *of* K *such that for all* $x \in K \setminus C$, *there exists* $y \in C$ *such that* $[y - g(x), x] < F(g(x)) - F(y) + cf(x)$, *then the extended (F-IVIP) has a solution. Also the solution set of the extended (F-IVIP) is compact.*

Proof. Construct a set $A = \{(x,y) \in K \times K : \ [y - g(x), x] \geq F(g(x)) - F(y) + cf(x)\}$. The proof of the theorem consists of four steps:

Step-1 For each $x \in K$, we have $(x, x) \in A$ by our assumption $[x - g(x), x] \geq F(g(x)) - F(x) + cf(x),\ \forall x \in K$.

Step-2 Since $g : K \to K$ is continuous, $F : K \to \mathbb{R}$ is lower semi-continuous and $f : K \to \mathbb{R}$ is continuous, the sets $A_y = \{x \in K :\ (x, y) \in A\}$ are closed in K for all $y \in K$.

Step-3 We show that $A_x = \{y \in K :\ (x, y) \notin A\}$ is convex or empty for any given $x \in K$.
Suppose that $A_z \neq \phi$ for some $z \in K$. To show that A_z is convex. Let $y_t = ty_1 + (1 - t)y_2$ for any $y_1, y_2 \in A_z$ and $t \in [0, 1]$. Since $y_1, y_2 \in A_z$, we have $[y_i - g(z), z] < F(g(z)) - F(y_i) + cf(z)$ for $i = 1, 2$.
Again since $F : K \to \mathbb{R}$ is convex, we have

$$\begin{aligned}
[y_t - g(z), z] &= [ty_1 + (1-t)y_2 - g(z), z] \\
&= t[y_1 - g(z), z] + (1-t)[y_2 - g(z), z] \\
&< tF(g(z)) - tF(y_1) + tcf(z) \\
&\quad +(1-t)F(g(z)) - (1-t)F(y_2) + (1-t)cf(z) \\
&= F(g(z)) - \{tF(y_1) + (1-t)F(y_2)\} + cf(z) \\
&= F(g(z)) - F(y_t) + cf(z).
\end{aligned}$$

Therefore $y_t \in A_z$, and hence A_z is convex.

Step-4 Let $B = \{x \in K :\ (x, y) \in A,\ \forall y \in C\}$. We show that B is compact.
By assumption, for each $x \in K \setminus C$ there exists an element $y \in C$ such that $[y - g(x), x] < F(g(x)) - F(y) + cf(x)$. This implies that $(x, y) \notin A$, so that $x \notin B$. Hence we have $B \subseteq C$. Since $B = \bigcap\limits_{y \in C} A_y$ and A_y are closed, it follows that B is closed. Hence B is compact being a closed subset of the compact set C.

From the above four steps and Lemma 2.2, there exists $\overline{x} \in K$ such that $\overline{x} \times K \subseteq A$, that is $[y - g(\overline{x}), \overline{x}] \geq F(g(\overline{x})) - F(y) + cf(\overline{x}),\ \ \forall y \in K$.

As in Step-2, we can show that the solutions set of the extended (F-IVIP) is closed. From the assumptions, any element outside the set C cannot be a solution of the extended (F-IVIP). Therefore, the solutions set of the extended (F-IVIP) must be contained in C. The solutions set is compact being a closed subset of a compact set.

Theorem 8. *Assume that $g :\ K \to K$ is continuous, $f :\ K \to \mathbb{R}$ is continuous and positive, $F :\ K \to \mathbb{R}$ is positive, homogeneous, lower semi-continuous and convex function. If all assumptions of Theorem 7 hold, then the extended (F-ICP) has a solution. Moreover, the solutions set of (F-ICP) is compact.*

Proof. The proof follows from Theorem 1 and Theorem 7.

In the following example, we construct an extended (F-IVIP) and an extended (F-ICP) in a semi-inner product space. We find a solution of the extended (F-IVIP) and show that it is also a solution of the extended (F-ICP).

Example 1. Consider the real sequence space l^3. The semi-inner product in l^3 is defined as

$$[x,y] = \frac{1}{\|y\|}\sum_{n=1}^{\infty}|y_n|y_nx_n,\ \forall x=(x_n),\ y=(y_n)\in l^3.$$

Let $K = \{(x_1,x_2,0,0,....) : x_1,x_2 \geq 0\}$, which is a nonempty closed convex cone of l^3.

Let $C = \{(x_1,x_2,0,0,....) : 0 \leq x_1,x_2 \leq 1\}$ be the nonempty compact convex subset of K.

Let $g : K \to K$ be defined by $g(x) = (\frac{x_2}{2},\frac{x_1}{2},0,0,....)$, $F : K \to \mathbb{R}$ be defined by $F(x) = x_1$ and $f : K \to \mathbb{R}$ be defined by $f(x) = \|x\|$, where $x = (x_1,x_2,0,0,...) \in K$.

The extended (F-IVIP) is to find $\overline{x} \in K$ such that

$$[y-g(\overline{x}),\overline{x}] \geq F(g(\overline{x})) - F(y) + cf(\overline{x}), \forall y \in K,$$

where c is a positive real constant. Choosing any positive real constant c, we see that $\overline{x} = (0,0,0,0,..)$ is a solution to the proposed extended (F-IVIP). For any $y = (y_1,y_2,0,0,..) \in K$, we have

$$g(\overline{x}) = (0,0,0,..),\ [y-g(\overline{x}),\overline{x}] = 0,\ f(\overline{x}) = 0,\ F(g(\overline{x})) = 0,\ F(y) = y_1$$

and

$$[y-g(\overline{x}),\overline{x}] = 0 = F(g(\overline{x})) - F(y) + cf(\overline{x}), \forall y \in K.$$

Hence $\overline{x}$ is a solution to the proposed extended (F-IVIP). Again we see that

$$[g(\overline{x}),\overline{x}] + F(g(\overline{x})) + cf(\overline{x}) = 0 + 0 + \frac{1}{2}\|\overline{x}\| = 0$$

and

$$[y,\overline{x}] + F(y) = y_1 \geq 0, \forall y \in K.$$

Thus $\overline{x}$ is also a solution to the extended (F-ICP). Under the given assumptions one can see that both the extended (F-IVIP) and the extended (F-ICP) are equivalent by Theorem 1.

Note that Theorem 1 is not true if we remove the positivity conditions on the functions F and on f. We can see this in the following example.

Example 2. Consider the real sequence space l^3. The semi-inner product in l^3 is defined as

$$[x,y]=\frac{1}{\|y\|}\sum_{n=1}^{\infty}|y_n|y_nx_n,\ \forall x=(x_n),\ y=(y_n)\in l^3.$$

Let $K=\{(x_1,x_2,0,0,....):0\leq x_1,x_2\leq 1\}$, which is a nonempty closed convex subset of l^3. Let $g:K\to K$ be defined by $g(x)=(x_1,0,0,....)$, $F:K\to\mathbb{R}$ be defined by $F(x)=-x_1$, and $f:K\to\mathbb{R}$ be defined by $f(x)=-x_2$, where $x=(x_1,x_2,0,0,...)\in K$.

The extended (F-IVIP) is to find $\overline{x}\in K$ such that

$$[y-g(\overline{x}),\overline{x}]\geq F(g(\overline{x}))-F(y)+cf(\overline{x}),\forall y\in K.$$

Choosing $c=2$, we find that $\overline{x}=(1,1,0,0,..)$ is a solution to the proposed extended F-IVIP. For any $y=(y_1,y_2,0,0,..)\in K$, we calculate that
$[y-g(\overline{x}),\overline{x}]=\frac{1}{2^{\frac{1}{3}}}(y_1+y_2-1)$, $f(\overline{x})=-1$, $g(\overline{x})=(1,0,0,..)$, $F(g(\overline{x}))=-1$, $F(y)=-y_1$, $F(g(\overline{x}))-F(y)+2f(\overline{x})=y_1-3$.
It is clear that $\frac{1}{2^{\frac{1}{3}}}(y_1+y_2-1)\geq y_1-3$ for all $y=(y_1,y_2,0,0,..)\in K$, and hence $\overline{x}=(1,1,0,0,0,.,.)$ is a solution to the proposed extended (F-IVIP).
Again we see that

$$[g(\overline{x}),\overline{x}]+F(g(\overline{x}))+cf(\overline{x})=\frac{1}{2^{\frac{1}{3}}}-1-2\neq 0.$$

Thus $\overline{x}$ is not a solution of the extended (F-ICP).

Acknowledgements. The first author is thankful to Council of Scientific and Industrial Research (CSIR), Govt. of India, for their financial support.

References

[1] J. M. Borwein, M. A. H. Dempster, *The linear order complementarity problem*, Math. Oper. Res. 14, 3 (1989), 534-558.

[2] L. C. Ceng, J. C. Yao, *An extragradient-like approximation method for variational inequality problems and fixed point problems*, Appl. Math. Comput. 190 (2007), 205-215.

[3] K. Fan, *A generalization of Tychonoff's fixed point theorem*, Math. Ann. 142 (1961), 305-310.

[4] J. R. Giles, *Classes of semi-inner product spaces*, Trans. Amer. Math. Soc. 129 (1967), 436-446.

[5] Y. He, *Stable pseudomonotone variational inequality in reflexive Banach spaces*, J. Math. Anal. Appl. 330 (2007), 352-363.

[6] N. J. Huang, J. Li, *F-implicit complementarity problems in Banach spaces*, J. Anal. Appl. 23, 2 (2004), 293-302.

[7] Z. H. Huang, W. Z. Gu, *A smoothing-type algorithm for solving linear complementarity problems with strong convergence properties*, Appl. Math. Optim. 57 (2008), 17-29.

[8] P. Junlouchai, S. Plubtieng, *Existence of solutions for generalized variational inequality problems in Banach spaces*, Nonlinear Analysis 74 (2011), 999-1004.

[9] M. S. Khan, *An exsitence theorem for extended mildly nonlinear complementarity problem in semi-inner product spaces*, Comment. Math. Carolinae 36, 1 (1995), 25-31.

[10] T. C. Lin, *Convex sets, fixed points, variational and minimax inequalities*, Bull. Austral. Math. Soc. 34 (1986), 107-117.

[11] G. Lumer, *Semi-inner product spaces*, Trans. Amer. Math. Soc. 100 (1961), 29-43.

[12] G. Stampacchia, *Formes bilineaires coercivites sur les ensembles convexes*, Comptes Rendus de 1'Academie des Sciences, Paris 258 (1964), 4413-4416.

[13] M. M. Wong, Q. H. Ansari, J. C. Yao, *Existence of solutions of generalized variational inequalities in reflexive Banach spaces*, Appl. Math. Lett. 22 (2009), 197-201.

[14] H. Y. Yin, C. X. Xu, Z. X. Zhang, *The F-complementarity problem and its equivalence with the least element problem*, Acta Math. Sinica 44, 4 (2001), 679-686.

[15] J. Yu, H. Yang, *Existence of solutions for generalized variational inequality problems*, Nonlinear Analysis 71 (2009), 2327-2330.

[16] L. C. Zeng, J. C. Yao, *Existence theorems for variational inequalities in Banach spaces*, J. Opt. Th. Appl. 132, 2 (2007), 321-337.

Nabin Kumar Sahu, Chandal Nahak
Department of Mathematics
Indian Institute of Technology Kharagpur, Kharagpur-721302, India
email: *nabin@maths.iitkgp.ernet.in, cnahak@maths.iitkgp.ernet.in*

Sudarsan Nanda
KIIT University, Bhubaneswar - 751024, India
email: *snanda@kiit.ac.in*