Acta Universitatis Apulensis No. 35/2013
ISSN: 1582-5329 pp. 95-100

COINCIDENCE AND FIXED POINT THEOREMS IN TOPOLOGICAL ORDERED SPACES

M. Omidi, M.A. Omidi, R. Omidi, M. Fouzoni and A. Karamian

Abstract. This study concerns with employing the concept of $\triangle$-kkm maps, to verify coincidence and fixed point theorems in topological ordered spaces. The primary aim is substitute a new mild condition instead of the compactness of spaces.

2000 *Mathematics Subject Classification*: 39B82, 44B20, 46C05.

Keywords: Topological semilattice, Generalized $\triangle$-kkm mapping, Coincidence point, Fixed point, Partial ordering.

1. Introduction and preliminaries

A semilattice is a partially ordered set X, with the partial order " $\leq$ " for which any pair $(x, x^{'})$ of elements has a least upper bound denoted by $x * x^{'}$. It is easy to see that any nonempty finite subset A of X has a least upper bound denoted by, $\sup A$. In a partially ordered set $(X, \leq)$, two arbitrary elements x and $x^{'}$ are not necessary comparable, but in the case where $x \leq x^{'}$, the set $[x, x^{'}] = \{y \in X \; x \leq y \leq x^{'}\}$ is called an order interval.

Now assume that $(X, \leq)$ is a semilattice and $A \subseteq X$ is a nonempty finite subset. Then the set $\triangle(A) = \bigcup_{a \in A} [a \;, \sup A]$ is well defined and it has the following properties:

- $A \subseteq \triangle(A)$,
- if $A \subseteq B$ then $\triangle(A) \subseteq \triangle(B) \cdot$

We say that a subset $E \subseteq X$ is $\triangle$-convex if for each nonempty finite subset A $\subseteq E$, $\triangle(A)$ is a subset of E.

A topological semilattice or more exactly a topological sup-semilattice, is a topological space X with a partial ordering $\leq$ for which it is a semilattice with a continuous sup operation, i.e., the function, $(x, x^{'}) \to x * x^{'}$, from $X \times X$ to X is continuous.

In the following example we give a $\triangle$-convex set.

Example 1. Let $X = \{\ (x,1) : 0 \le x < 1\} \bigcup \{\ (x,y)\ : 0 \le y \le 1\ ,\ x \ge 1\ ,\ y \ge x-1\}$ be a subset of $\mathbb{R}^2$. With the partial ordering on $\mathbb{R}^2$ defined by

$$(a,b) \le (c,d) \Leftrightarrow c-a \ge 0, d-b \ge 0 \text{ and } d-b \le c-a,$$

X is $\triangle$-convex.

For each $D \subseteq X$, $< D >$ denote the family of finite subset of D,

$$\triangle(D) = \bigcup_{A \in <D>} \triangle(A)\cdot$$

If X is a nonempty set, 2^X denote the family of all subset of X. Let X and Y be two topological spaces, for $R \subseteq X \times Y$ and $x \in X$ and $y \in Y$ we put,

$$R(x) = \{y \in Y\ (x,y) \in R\}, R^{-1}(y) = \{x \in X\ (x,y) \in R\}\cdot \tag{1}$$

We recall the following theorem from [3].

Theorem 2. *Let X be topological semilattice with path-connected intervals, $X_0 \subseteq X$ be a nonempty subset of X and $R \subseteq X_0 \times X$ be a binary relation such that*

- *For each $x \in X_0$, the set $R(x) = \{y \in Y\ : (x,y) \in R\}$ is not nonempty and closed in $R(X_0)$,*
- *There exists $x_0 \in X_0$ such that the set $R(x_0)$ is compact,*
- *For any nonempty finite subset $A \subseteq X_0$*

$$\bigcup_{x \in A} [x\ , \sup A] \subseteq \bigcup_{x \in A} R(x)\cdot \tag{2}$$

Then the set $\bigcap_{x \in X_0} R(x)$ is not nonempty.

Definition 1. *Let X be a topological semilattice and $X_0 \subseteq X$ be a nonempty subset of X. Then a multivalued $F : X_0 \to 2^X$ is a $\triangle$-kkm map if $\triangle(A) \subseteq \bigcup_{x \in A} F(x)$ for each $A \in < X_0 >$.*

Let X be a topological semilattice and $X_0 \subseteq$ X be a nonempty subset. Also let Y be a nonempty set and $F : X \to 2^Y$, $H : X_0 \to 2^Y$ be two mappings. We say that H is a generalized $\triangle$-kkm mapping w.r.t. F, if for each $A \in < X_0 >$, $F(\triangle(A)) \subseteq H(A)$.

For a mapping $F : X \to 2^Y$ which X and Y are arbitrary sets, we define $F^-, F_c^- : Y \to 2^X$ by

$$F^-(y) = \{x \in X : y \in F(x)\}$$

and

$$F_c^-(y) = X - F^-(y) = \{x \in X : y \notin F(x)\} \cdot$$

If X be a topological semilattice, Y be a nonempty set and $K : Y \to 2^X$ be a mapping, then the mapping $\triangle - K : Y \to 2^X$ is defined by

$$(\triangle - K)(y) = \bigcup_{A \in < K(y) >} \triangle(A) \cdot$$

2. Main results

We start this section with the following theorem that is one of our main results.

Theorem 3. *Let X be a topological semilattice, $X_0 \subseteq$ X be a nonempty, Y be a nonempty set and $F : X \to 2^Y$, $H : X_0 \to 2^Y$ be two mappings. Then the following conditions are equivalent:*

a) *H is generalized $\triangle$-kkm mapping w.r.t. $F\cdot$*

b) *$(\triangle - H_c^-)(y) \subseteq F_c^-(y)$ for each $y \in Y \cdot$*

c) *$\triangle - (H_c^- oF)$ has no fixed point$\cdot$*

Proof. (a)⇒(b) Let H be a generalized $\triangle$-kkm mapping w.r.t. F. Suppose that there exists $y \in Y$ such that $(\triangle - H_c^-)(y) \nsubseteq F_c^-(y)$. Then there exists $D \in < H_c^-(y) >$ and $x \in \triangle(D)$ such that $x \notin F_c^-(y)$. Therefore $y \notin H(D)$ and $y \in F(x)$.
Therefore $F(\triangle(D)) \nsubseteq H(A)$, which is in contradiction with (a).

(b) ⇒(c) Let there exists $x_0 \in X$ such that $x_0 \in (\triangle - (H_c^- oF))(x_0)$, then there exists $y \in F(x_0)$ and $D \in < H_c^-(y) >$ such that $x_0 \in \triangle(D)$. Since

$$x_0 \in \triangle(D) \subseteq \bigcup_{A \in < H_c^-(y) >} \triangle(A) = (\triangle - H_c^-)(y),$$

and $y \in F(x_0)$, $x_0 \notin F_c^-(y)$, this is in contradiction with (b).

(c) ⇒(a) Suppose that H is not a genralized $\triangle$-kkm mapping w.r.t. F. Then there exists $D \in < X_0 >$, $x_0 \in \triangle(D)$ and $y \in F(x_0) - H(D)$. Since $y \notin H(D)$, we have $D \subseteq H_c^-$ (y) and consequently

$$x_0 \in \triangle(D) \subseteq (\triangle - H_c^-)(y) \subseteq (\triangle - H_c^-)(F(x_0)) \cdot$$

Hence x_0 is a fixed point for $\triangle - (H_c^- oF)$.

Let X be a topological semilattice and Y be a topological space. A mapping $F : X \to 2^Y$ is said to have the $\triangle$-kkm property if for each closed-valued mapping $H : X_0 \subseteq X \to 2^Y$ generalized $\triangle$-kkm w.r.t. F, the family $\{H(x)\}_{x \in X_0}$ has the finite intersection property.

Theorem 4. *Let X be a topological semilattice, $X_0 \subseteq X$ be nonempty, Y be a topological space, $K : X_0 \to 2^Y$, $H : X \to 2^Y$ and $F : X \to 2^Y$. Suppose that*

a) *F has the $\triangle$-kkm property,*

b) *$F(X) - K(x)$ is closed for each $x \in X_0$,*

c) *for each $y \in F(X)$, $(\triangle - K^-)(y) \subseteq H^-(y)$,*

d) *$F(X) \subseteq K(A)$ for some $A \in < X_0 > \cdot$*

Then H and F have a coincidence point $x_0 \in X$.

Proof. Define a mapping $T : X_0 \to 2^Y$ by $T(x) = F(X) - K(x)$. By condition (b), T has a closed-valued and

$$\bigcap_{x \in A} T(x) = \bigcap_{x \in A} (F(X) - K(x)) = F(X) - \bigcup_{x \in A} K(x) = \phi \cdot$$

Then family $\{T(x)\}_{x \in X_0}$ does not have the finite intersection property, but F has the $\triangle$-kkm property. Therefore, T is not a generalized $\triangle$-kkm mapping w.r.t F. By Theorem 3, there exists a $x_0 \in X$ such that $x_0 \in (\triangle - (T_c^- oF))(x_0)$. Hence there exists $y_0 \in F(x_0)$ and $E \in < T_c^-(y_0) >$ such that $x_0 \in \triangle(EE)$. Since $E \subseteq T_c^-(y_0)$, successively we have

$$y_0 \notin T(E), y_0 \in \bigcap_{x \in E} K(x), \text{then } E \subseteq K^-(y_0) \cdot$$

By (c) we get $x_0 \in \triangle E \subseteq H^-(y_0)$, where $y_0 \in H(x_0)$. So we have

$$y_0 \in H(x_0) \cap F(x_0) \cdot$$

Corollary 5. *Let X be a topological semilattice, $X_0 \subseteq X$ be a nonempty and $K : X_0 \to 2^X$, $H : X \to 2^X$ be two mappings such that:*

a) *K has open values*

b) *for each $y \in X$, $K^-(y)$ is nonempty and $(\triangle - K^-)(y) \subseteq H^-(y)$*

c) *there is $A \in < X_0 >$ such that $X = K(A) \cdot$*

Then H has a fixed point.

Proof. For each $x \in X$, define $F(x) = \{x\}$. Therefore F has the $\triangle$-kkm property, hence conditions of Theorem 4 are fulfilled. So there exists $x_0 \in$ X such that

$$H(x_0) \cap F(x_0) \neq \emptyset \cdot$$

Therefore, $x_0 \in H(x_0)$.

Acknowledgements. The first and fourth authors gratefully acknowledge the *kharazmi university of Tehran* for financial support.

References

[1] C. D. Horvath, *Some results on multivalued mappings and inequalities without convexity, in Nonlinear and convex Analysis*, B. L. Lin and S. Simons, Marcel Dekker, 1987, 96-106.

[2] C.D. Horvath, *contractibility and generalized convexity*, J .Math .Anal .Appl., 156 (1991), 341-357.

[3] C.D. Horvath and J.V. Llinares Ciscar, *Maximal elements and fixed points for binary relation on topological ordered spaces*, J. Math. Econom., 25 (1996), 291-306.

[4] M. Balaj and D. Erzse, *Fixed points coincidence points and generalized G-kkm mapping*, J. Fixed point theory, 6 (2005), 3-9.

[5] M. Bianchi and R. Pini, *A note on equilibrium problems with properly quasi-monotone bifunctions*, J. Global Optim. 20 (2001), 67-76.

[6] T.H. Chang and C.L. Yen, *KKM property and fixed point theorems*, J. Math. Anal. Appl., 203 (1996), 224-235.

[7] L.J. Lin, Q.H. Ansarin and J.Y. Wu, *Geometric properties and coincidence theorems with applications to generalized vector equilibrium problems*, J Optim . Theory Appl., 117 (2003), 121-137 .

Mashaalh Omidi
Department of Mathematics, Faculty of Computer and Mathematical Science,
Kharazmi university of Tehran
Tehran, Iran
email: *std_m.omidi@khu.ac.ir, m.omidi1978@gmail.com*,

Mohammad Amin Omidi
Department of Mathematics,
Razi University of Kermanshah,

Kermanshah, Iran
email: *amin.omid64@gmail.com*,

Roholah Omidi
Department of Mathematics,
Razi University of Kermanshah,
Kermanshah, Iran
email: *rh.omidi@gmail.com*,

Mohammad Fozouni
Department of Mathematics, Faculty of Computer and Mathematical Science,
Kharazmi university of Tehran
Tehran, Iran
email: *fozouni@khu.ac.ir, fozouni@hotmail*,

Ardeshir Karamian
Department of Mathematics, Faculty of Mathematics,
University of Sistan and Balochestan,
Zahedan, Iran
email: *ar_karamian1979@yahoo.com.*