LIKE CHEBYSHEV'S INEQUALITIES FOR CONVEX FUNCTIONS AND APPLICATIONS

Zinelâabidine Latreuch and Benharrat Belaïdi

Abstract. In this paper, we will show some new inequalities for convex functions, and we will also make a connection between them and Chebyshev's inequality, which implies the existence of new class of functions satisfying Chebyshev's inequality.

2000 *Mathematics Subject Classification*: 26D15.

Keywords: Chebyshev's inequality, Convex functions, Symmetric functions.

1. Introduction and main results

A classic result due to Chebyshev (1882-1883) [2, 5, 8, 9, 11] is stated in the following theorem.

Theorem 1. *Let $f, g : [a, b] \to \mathbb{R}$ and $p : [a, b] \to \mathbb{R}_+$ be integrable functions. If f and g are monotonic in the same direction, then*

$$\int_a^b p(x) \, dx \int_a^b p(x) f(x) g(x) \, dx \geq \int_a^b p(x) f(x) \, dx \int_a^b p(x) g(x) \, dx \qquad (1.1)$$

provided that the integrals exist. If f and g are monotonic in opposite directions, then the reverse of the inequality in (1.1) is valid. In both cases, equality in (1.1) holds if and only if either g or f is constant almost everywhere.

There exist several results which show that Chebyshev inequality is valid under weaker conditions, for example the condition that the functions be monotonic can be replaced by the condition that they be similarly ordered. In this case Theorem 1 is a simple consequence of the following identity:

$$\int_a^b p(x) \, dx \int_a^b p(x) f(x) g(x) \, dx - \int_a^b p(x) f(x) \, dx \int_a^b p(x) g(x) \, dx$$

$$= \frac{1}{2} \int_a^b \int_a^b p(x) p(y) (f(x) - f(y)) (g(x) - g(y)) \, dx dy. \qquad (1.2)$$

Note that the functions $f : I \to \mathbb{R}$ and $g : I \to \mathbb{R}$ are said to be similarly ordered if

$$(f(x) - f(y)) (g(x) - g(y)) \geq 0 \text{ for every } x, y \in I \qquad (1.3)$$

holds, and they are said to be oppositely ordered if the reverse inequality holds. Of course, the generalization of the identity in (1.2) for functions with several variables is also valid (see [1, 9]). Thus similar generalizations of Theorem 1 are valid for similarly ordered functions. The first such result is given by Hardy, Littlewood and Pólya ([5], p. 168).

Another modification of the conditions for Chebyshev inequality was given by Levin and Stečkin (see [6, 7]), and they proved the following result:

Theorem 2. *Let $p(x)$ be defined on $[0, 1]$ and satisfying the conditions:*
(i) $p(x)$ is increasing for $x \in \left[0, \frac{1}{2}\right]$;
(ii) $p(x) = p(1 - x)$ for $x \in [0, 1]$.
Then for every continuous convex function ϕ we have

$$\int_0^1 p(x) \phi(x) \, dx \leq \int_0^1 p(x) \, dx \int_0^1 \phi(x) \, dx.$$

The aim of this paper is to prove new type of inequalities for convex functions, and we put a link between these inequalities and Chebyshev's inequality.

Theorem 3. *Let $f, g : [a, b] \longrightarrow \mathbb{R}$ be convex (or concave) functions and $p : [a, b] \to \mathbb{R}_+$ be integrable symmetric function about $x = \frac{a+b}{2}$ (i.e., $p(a + b - x) = p(x)$, for all $x \in [a, b]$). Then*

$$\int_a^b p(x) f(x) g(x) \, dx + \int_a^b p(x) f(x) g(a + b - x) \, dx$$

$$\geq \frac{2}{\int_a^b p(x) \, dx} \int_a^b p(x) f(x) \, dx \int_a^b p(x) g(x) \, dx. \qquad (1.4)$$

If f is convex (or concave) and g is concave (or convex) functions, then the inequality (1.4) is reversed, equality in (1.4) holds if and only if either g or f is linear.

Corollary 4. *Let* $f, g : [a, b] \longrightarrow \mathbb{R}$ *be convex (or concave) functions. If* g *is symmetric function about* $x = \frac{a+b}{2}$*, then*

$$\int_a^b f(x) g(x) \, dx \geq \frac{1}{b-a} \int_a^b f(x) \, dx \int_a^b g(x) \, dx. \tag{1.5}$$

If f *is convex (or concave) and* g *is concave (or convex) functions, then the inequality* (1.5) *is reversed, equality in* (1.5) *holds if and only if either* g *or* f *is linear.*

Theorem 5. *Let* $f, g : [a, b] \longrightarrow \mathbb{R}$ *be convex (or concave) functions.*
(i) If f *and* g *are similarly ordered, then*

$$\int_a^b f(x) g(x) \, dx \geq \frac{1}{2} \left(\int_a^b f(x) g(x) \, dx + \int_a^b f(x) g(a+b-x) \, dx \right)$$

$$\geq \frac{1}{b-a} \int_a^b f(x) \, dx \int_a^b g(x) \, dx. \tag{1.6}$$

(ii) If f *and* g *are oppositely ordered, then*

$$\int_a^b f(x) g(a+b-x) \, dx \geq \frac{1}{b-a} \left(\int_a^b f(x) \, dx \right) \left(\int_a^b g(x) \, dx \right)$$

$$\geq \int_a^b f(x) g(x) \, dx. \tag{1.7}$$

Theorem 6. *Let* $f, g : [a, b] \longrightarrow \mathbb{R}$*, where* f *is convex function and* g *decreasing on* $\left[a, \frac{a+b}{2}\right]$ *and increasing on* $\left[\frac{a+b}{2}, b\right]$*. Then the inequality* (1.4) *holds.*

2. Some lemmas

Lemma 7. *Let* $f : [a, b] \longrightarrow \mathbb{R}$ *be convex (or concave) function. Then*

$$F(x) = f(x) + f(a+b-x)$$

is decreasing (increasing) on $\left[a, \frac{a+b}{2}\right]$ *and increasing (decreasing) on* $\left[\frac{a+b}{2}, b\right]$*.*

Proof. Suppose that f is a convex function. Since f is a convex function, then either f is monotonic on $]a, b[$, or there exists an $\zeta \in]a, b[$ such that f is decreasing on the interval $[a, \zeta]$ and increasing on the interval $[\zeta, b]$, for more information see ([10], p. 26). Since F is convex function on $[a, b]$, and symmetric about $x = \dfrac{a+b}{2}$, then we can deduce easily that F must be decreasing on the interval $\left[a, \frac{a+b}{2}\right]$ and increasing on the interval $\left[\frac{a+b}{2}, b\right]$. Now, if f is a concave function, then by using the same proof as above we obtain the result.

Lemma 8. *Let $f, g : [a, b] \longrightarrow \mathbb{R}$ be two integrable functions. If f and g are similarly ordered, then*

$$\int\limits_a^b f(x)\, g(x)\, dx \geq \int\limits_a^b f(x)\, g(a+b-x)\, dx. \tag{2.1}$$

If f and g are oppositely ordered, then the inequality (2.1) is reversed.

Proof. Since f and g are similarly ordered, then we have for all $x \in [a, b]$

$$(f(x) - f(a+b-x))(g(x) - g(a+b-x)) \geq 0, \tag{2.2}$$

which implies that

$$f(x)\, g(x) + f(a+b-x)\, g(a+b-x) \geq f(x)\, g(a+b-x)$$

$$+ g(x)\, f(a+b-x). \tag{2.3}$$

By integrating both sides of inequality (2.3) over the segment $[a, b]$ we obtain (2.1).

3. Proof of the Theorems

Proof of Theorem 3.

Proof. First, we suppose that f and g are convex (or concave) functions and we denote by F and G the following functions

$$F(x) = f(x) + f(a+b-x), \quad G(x) = g(x) + g(a+b-x).$$

Since f and g are convex functions, then by using Lemma 7 we deduce that F and G have the same variation. By applying Chebyshev's inequality in $\left[a, \frac{a+b}{2}\right]$, we obtain

$$\int\limits_a^{\frac{a+b}{2}} p(x)\, F(x)\, G(x)\, dx \geq \frac{1}{\int\limits_a^{\frac{a+b}{2}} p(x)} \int\limits_a^{\frac{a+b}{2}} p(x)\, F(x)\, dx \int\limits_a^{\frac{a+b}{2}} p(x)\, G(x)\, dx, \tag{3.1}$$

where $p : [a, b] \rightarrow \mathbb{R}_+$ is integrable symmetric function. Then

$$\int\limits_a^{\frac{a+b}{2}} p(x) \left[f(x) g(x) + f(a+b-x) g(a+b-x) \right] dx$$

$$+ \int\limits_a^{\frac{a+b}{2}} p(x) \left[f(x) g(a+b-x) + f(a+b-x) g(x) \right] dx$$

$$\geq \left(\frac{1}{\int\limits_a^{\frac{a+b}{2}} p(x)} \right) \left(\int\limits_a^{\frac{a+b}{2}} p(x) \left[f(x) + f(a+b-x) \right] dx \right)$$

$$\times \left(\int\limits_a^{\frac{a+b}{2}} p(x) \left[g(x) + g(a+b-x) \right] dx \right). \tag{3.2}$$

Using the identities

$$\int\limits_a^{\frac{a+b}{2}} f(x) \, dx = \int\limits_{\frac{a+b}{2}}^b f(a+b-x) \, dx, \tag{3.3}$$

$$\int\limits_a^{\frac{a+b}{2}} p(x) \, dx = \frac{1}{2} \int\limits_a^b p(x) \, dx, \tag{3.4}$$

and

$$\int\limits_a^{\frac{a+b}{2}} f(x) g(a+b-x) \, dx = \int\limits_{\frac{a+b}{2}}^b f(a+b-x) g(x) \, dx, \tag{3.5}$$

we obtain

$$\int\limits_a^{\frac{a+b}{2}} p(x) f(x) g(x) \, dx + \int\limits_{\frac{a+b}{2}}^b p(x) f(x) g(x) \, dx$$

$$+ \int_a^{\frac{a+b}{2}} p(x) f(x) g(a+b-x) \, dx + \int_{\frac{a+b}{2}}^b p(x) f(x) g(a+b-x) \, dx$$

$$\geq \left(\frac{2}{\int_a^b p(x)} \right) \left(\int_a^{\frac{a+b}{2}} p(x) f(x) \, dx + \int_{\frac{a+b}{2}}^b p(x) f(x) \, dx \right)$$

$$\times \left(\int_a^{\frac{a+b}{2}} p(x) g(x) \, dx + \int_{\frac{a+b}{2}}^b p(x) g(x) \, dx \right). \tag{3.6}$$

From (3.6), we easily obtain (1.4). Now, if f is convex (or concave) and g is concave (or convex) functions, then by using the same proof as above we obtain the result.

Now, we prove that equality in (1.4) holds if and only if either g or f is linear. Without loss of generality, we suppose that f is linear and g is convex (or concave) function. Then

$$\begin{aligned} F(x) &= f(x) + f(a+b-x) \\ &= \alpha x + \beta + \alpha(a+b-x) + \beta = \alpha(a+b) + 2\beta \end{aligned}$$

and

$$G(x) = g(x) + g(a+b-x).$$

Since F is constant, G monotone on $\left[a, \frac{a+b}{2}\right]$ and $\left[\frac{a+b}{2}, b\right]$, then by applying Theorem 1, we obtain

$$\int_a^{\frac{a+b}{2}} p(x) F(x) G(x) \, dx = \frac{1}{\int_a^{\frac{a+b}{2}} p(x) \, dx} \int_a^{\frac{a+b}{2}} p(x) F(x) \, dx \int_a^{\frac{a+b}{2}} p(x) G(x) \, dx$$

which is equivalent to

$$\int_a^b p(x) f(x) g(x) \, dx + \int_a^b p(x) f(x) g(a+b-x) \, dx$$

$$= \frac{2}{\int_a^b p(x) \, dx} \int_a^b p(x) f(x) \, dx \int_a^b p(x) g(x) \, dx.$$

88

Proof of Theorem 5.

Proof. (i) Since f and g are convex functions and similarly ordered, then by Lemma 8 we have

$$\int_a^b f(x)\, g(x)\, dx \geq \int_a^b f(x)\, g(a+b-x)\, dx, \qquad (3.7)$$

which we can write

$$2\int_a^b f(x)\, g(x)\, dx \geq \int_a^b f(x)\, g(a+b-x)\, dx + \int_a^b f(x)\, g(x)\, dx. \qquad (3.8)$$

By Theorem 3 and (3.8), we have

$$2\int_a^b f(x)\, g(x)\, dx \geq \int_a^b \left[f(x)\, g(a+b-x) + f(x)\, g(x) \right] dx$$

$$\geq \frac{2}{b-a} \left(\int_a^b f(x)\, dx \right) \left(\int_a^b g(x)\, dx \right).$$

So,

$$\int_a^b f(x)\, g(x)\, dx \geq \frac{1}{2}\int_a^b \left[f(x)\, g(a+b-x) + f(x)\, g(x) \right] dx$$

$$\geq \frac{1}{b-a} \left(\int_a^b f(x)\, dx \right) \left(\int_a^b g(x)\, dx \right). \qquad (3.9)$$

(ii) Since f and g are convex functions, then by Theorem 3

$$\int_a^b f(x)\, g(a+b-x)\, dx - \frac{1}{b-a} \left(\int_a^b f(x)\, dx \right) \left(\int_a^b g(x)\, dx \right)$$

$$\geq \frac{1}{b-a} \left(\int_a^b f(x)\, dx \right) \left(\int_a^b g(x)\, dx \right) - \int_a^b f(x)\, g(x)\, dx. \qquad (3.10)$$

On the other hand, we have

$$\frac{1}{b-a}\left(\int\limits_a^b f\left(x\right)dx\right)\left(\int\limits_a^b g\left(x\right)dx\right)\geq\int\limits_a^b f\left(x\right)g\left(x\right)dx, \tag{3.11}$$

because f and g are oppositely ordered. By (3.10) and (3.11), we get

$$\int\limits_a^b f\left(x\right)g\left(a+b-x\right)dx\geq\frac{1}{b-a}\left(\int\limits_a^b f\left(x\right)dx\right)\left(\int\limits_a^b g\left(x\right)dx\right)$$

$$\geq\int\limits_a^b f\left(x\right)g\left(x\right)dx, \tag{3.12}$$

and the proof of Theorem 5 is complete.

Proof of Theorem 6.

Proof. We denote by F and G the following functions

$$F\left(x\right)=f\left(x\right)+f\left(a+b-x\right),\ \ G\left(x\right)=g\left(x\right)+g\left(a+b-x\right).$$

Since f is convex function, then by Lemma 7, F is decreasing on $\left[a,\frac{a+b}{2}\right]$ and increasing on $\left[\frac{a+b}{2},b\right]$. In order to prove (1.4) we need to prove that G is decreasing on $\left[a,\frac{a+b}{2}\right]$ and increasing on $\left[\frac{a+b}{2},b\right]$. Let $x,y\in\left[a,\frac{a+b}{2}\right]$ and set $x^*=a+b-x$, $y^*=a+b-y$ where $x^*,y^*\in\left[\frac{a+b}{2},b\right]$. It's clear that if $x\leq y$, then $x^*\geq y^*$. Since g is decreasing on $\left[a,\frac{a+b}{2}\right]$ and increasing on $\left[\frac{a+b}{2},b\right]$, then we have

$$g\left(x\right)\geq g\left(y\right)$$

and

$$g\left(x^*\right)\geq g\left(y^*\right).$$

Then

$$G\left(x\right)=g\left(x\right)+g\left(x^*\right)\geq g\left(y\right)+g\left(y^*\right)=G\left(y\right),$$

which implies that G is decreasing on $\left[a,\frac{a+b}{2}\right]$. By the same method we can prove easily that G is increasing on $\left[\frac{a+b}{2},b\right]$. Then F and G have the same variation, and by applying Theorem 1 with $p:[a,b]\to\mathbb{R}_+$ is integrable symmetric function about $x=\frac{a+b}{2}$, we obtain inequality (1.4).

90

4. Applications

The following inequality is sharper (refinement) than Cauchy's inequality.

Theorem 9. *Let f be convex (or concave) function on $[a, b]$. Then we have*

$$\int_a^b f^2(x)\, dx \geq \int_a^b f(x) G(x)\, dx \geq \frac{1}{b-a}\left(\int_a^b f(x)\, dx\right)^2, \qquad (4.1)$$

where $G(x) = \frac{f(x)+f(a+b-x)}{2}$.

Proof. Putting $f(x) = g(x)$ in (1.6) we obtain (4.1).

The Riemann-Liouville fractional integral operator of order $\alpha \geq 0$, for a continuous function f is defined as [4, 12]

$$J^\alpha(f)(x) = \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1} f(t)\, dt \quad (\alpha > 0, x > 0), \quad J^0 f(x) = f(x),$$

where $\Gamma(\alpha) = \int_0^\infty e^{-u} u^{\alpha-1} du$ denotes the well-know Gamma function.

In the following application we prove that the Riemann-Liouville fractional integral operator satisfies new inequalities under some conditions.

Theorem 10. *Let f be convex function on $[0, +\infty)$.*
(i) If $\alpha \geq 2$, then

$$J^\alpha f(x) \geq \frac{1}{\Gamma(\alpha)}\left[2\frac{x^{\alpha-1}}{\alpha}\int_0^x f(t)\, dt - \int_0^x t^{\alpha-1} f(t)\, dt\right]. \qquad (4.2)$$

Furthermore, if f is increasing function, then we have

$$\frac{1}{\Gamma(\alpha)}\int_0^x t^{\alpha-1} f(t)\, dt \geq J^\alpha f(x) \geq \frac{1}{\Gamma(\alpha)}\left[2\frac{x^{\alpha-1}}{\alpha}\int_0^x f(t)\, dt - \int_0^x t^{\alpha-1} f(t)\, dt\right]. \quad (4.3)$$

(ii) If $0 < \alpha \leq 2$, then

$$J^\alpha f(x) \leq \frac{1}{\Gamma(\alpha)}\left[2\frac{x^{\alpha-1}}{\alpha}\int_0^x f(t)\, dt - \int_0^x t^{\alpha-1} f(t)\, dt\right]. \qquad (4.4)$$

Proof. (i) Let $g(x) = x^{\alpha-1}$, $\alpha > 0$. It's clear that g is increasing convex function on $[0, +\infty)$ for $\alpha \geq 2$ and concave for $0 < \alpha \leq 2$. By applying inequality (1.4) for f and g ($a = 0$, $b = x > 0$ and $p(x) = 1$) we obtain

$$\int_0^x t^{\alpha-1} f(t) \, dt + \int_0^x (x-t)^{\alpha-1} f(t) \, dt \geq \frac{2}{x} \left(\int_0^x t^{\alpha-1} dt \right) \left(\int_0^x f(t) \, dt \right),$$

which implies

$$\int_0^x (x-t)^{\alpha-1} f(t) \, dt \geq 2 \frac{x^{\alpha-1}}{\alpha} \left(\int_0^x f(t) \, dt \right) - \int_0^x t^{\alpha-1} f(t) \, dt. \tag{4.5}$$

Multiplying both sides of inequality (4.5) by $\frac{1}{\Gamma(\alpha)}$, we obtain

$$J^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t) \, dt$$

$$\geq \frac{1}{\Gamma(\alpha)} \left[2 \frac{x^{\alpha-1}}{\alpha} \left(\int_0^x f(t) \, dt \right) - \int_0^x t^{\alpha-1} f(t) \, dt \right].$$

Furthermore, if f is increasing on $[0, +\infty)$, then by applying (1.6) we obtain

$$2 \int_0^x t^{\alpha-1} f(t) \, dt \geq \int_0^x t^{\alpha-1} f(t) \, dt + \int_0^x (x-t)^{\alpha-1} f(t) \, dt$$

$$\geq \frac{2}{x} \left(\int_0^x t^{\alpha-1} dt \right) \left(\int_0^x f(t) \, dt \right),$$

which implies

$$\frac{1}{\Gamma(\alpha)} \int_0^x t^{\alpha-1} f(t) \, dt \geq J^\alpha f(x)$$

$$\geq \frac{1}{\Gamma(\alpha)} \left[2 \frac{x^{\alpha-1}}{\alpha} \left(\int_0^x f(t) \, dt \right) - \int_0^x t^{\alpha-1} f(t) \, dt \right].$$

(ii) It's clear that (1.4) is reversed if $g(x) = x^{\alpha-1}$ is concave function, i. e., for $0 < \alpha \leq 2$. By applying inequality (1.4) for f and g ($a = 0$, $b = x > 0$ and $p(x) = 1$) we obtain (4.4).

Theorem 11. *(Refinement of Fejér inequality, [3]) Let f be convex function on $[a, b]$, and let p be a positive, convex and symmetric function about $x = \frac{a+b}{2}$. Then*

$$\int_a^b p(t) f(t) \, dt \geq \frac{1}{b-a} \int_a^b p(t) \, dt \int_a^b f(t) \, dt \geq f\left(\frac{a+b}{2}\right) \int_a^b p(t) \, dt. \qquad (4.6)$$

Proof. Since f is convex function on $[a, b]$ and p is convex symmetric about $x = \frac{a+b}{2}$, then by Corollary 4 we obtain

$$\int_a^b p(t) f(t) \, dt \geq \frac{1}{b-a} \int_a^b p(t) \, dt \int_a^b f(t) \, dt. \qquad (4.7)$$

On the other hand by Hermite-Hadamard inequality, we have

$$\frac{1}{b-a} \int_a^b f(t) \, dt \geq f\left(\frac{a+b}{2}\right). \qquad (4.8)$$

By using (4.8), we obtain from (4.7)

$$\int_a^b p(t) f(t) \, dt \geq \frac{1}{b-a} \int_a^b p(t) \, dt \int_a^b f(t) \, dt \geq f\left(\frac{a+b}{2}\right) \int_a^b p(t) \, dt$$

and the proof is complete.

REFERENCES

[1] O. S. Berljand, I. M. Nazarov and P. M. Svidskii, *On certain integral inequalities*, Doklady Akad. Nauk SSSR 173 (1967), 251-252.

[2] P. L. Čebyšev, *Polnoe Sobranie Sočineniĭ*,(Russian) [Complete Collected Works] Izdatelstvo Akademii Nauk SSSR, Moscow-Leningrad. 1946, 1947, 1948, vol. 1, 342 pp.; vol. 2, 520 pp.; vol. 3, 414 pp.; vol. 4, 255 pp.

[3] L. Fejér, *Über die Fourierreihen*, II., Math. Naturwiss Anz. Ungar. Akad. Wiss. Hungarian, vol. 24, 1906, 369-390.

[4] R. Gorenflo, F. Mainardi, *Fractional calculus: integral and differential equations of fractional order*, Springer, Vienna (1997), 223-276.

[5] G. H. Hardy, J. E. Littlwood and G. Pólya, *Inequalities*, 2d ed. Cambridge, at the University Press, 1952.

[6] S. J. Karlin and W. J. Studden, *Tchebycheff systems: With applications in analysis and statistics*, New York: Interscience Publishers, 1966.

[7] V. I. Levin and S. B. Stečkin, *Inequalities*, Amer. Math. Soc. Transl. (2) 14 (1960), 1–29.

[8] D. S. Mitrinović, J. E. Pečarić and A. M. Fink, *Classical and new inequalities in analysis. Mathematics and its Applications*, (East European Series), 61. Kluwer Academic Publishers Group, Dordrecht, 1993.

[9] D. S. Mitrinović, P. M. Vaisić, *History, variations and generalisations of the Čebyšev inequality and the question of some priorities*, Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No. 461–497 (1974), 1–30.

[10] C. P. Niculescu and L. E. Persson, *Convex functions and their applications*, A contemporary approach, CMC Books in Mathematics, Springer, New York, 2006.

[11] J. E. Pečarić, F. Proschan and Y. L. Tong, *Convex functions, partial orderings, and statistical applications*, Mathematics in Science and Engineering. 187. Boston, MA etc.: Academic Press, 1992.

[12] I. Podlubny, *Fractional differential equations*, Academic Press, Inc., San Diege, CA, 1999.

Zinelâabidine Latreuch and Benharrat Belaïdi
Department of Mathematics, Laboratory of Pure and Applied Mathematics,
University of Mostaganem (UMAB),
B. P. 227 Mostaganem-Algeria.
email: *z.latreuch@gmail.com; email: belaidi@univ-mosta.dz*